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Weyl, eigenfunction expansions and harmonic analysis on non-compact symmetric spaces

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1 Introduction

This text grew out of an attempt to understand a remark by Harish-Chandra in the introduction of [12]. In that paper and its sequel he determined the Plancherel decomposition for Riemannian symmetric spaces of the non-compact type. The associated Plancherel measure turned out to be related to the asymptotic behavior of the so-called zonal spherical functions, which are solutions to a system of invariant differential eigenequations. Harish-Chandra observed: 'this is reminiscent of a result of Weyl on ordinary differential equations', with reference to Hermann Weyl's 1910 paper, [29], on singular Sturm–Liouville operators and the associated expansions in eigenfunctions.

For Riemannian symmetric spaces of rank one the mentioned system of equations reduces to a single equation of the singular Sturm–Liouville type. Weyl's result indeed relates asymptotic behavior of eigenfunctions to the continuous spectral measure but his result is formulated in a setting that does not directly apply.

In [23], Kodaira combined Weyl's theory with the abstract Hilbert space theory that had been developed in the 1930's. This resulted in an efficient derivation of a formula for the spectral measure, previously obtained by Titchmarsh. In the same paper Kodaira discussed a class of examples that turns out to be general enough to cover all Riemannian symmetric spaces of rank 1.

It is the purpose of this text to explain the above, and to describe later developments in harmonic analysis on groups and symmetric spaces where Weyl's principle has played an important role.

2 Sturm–Liouville operators

A Sturm–Liouville operator is a second order ordinary differential operator of the form

$$L = -\frac{d}{dt}\, p\, \frac{d}{dt} + q, \tag{2.1}$$

defined on an open interval $]\,a,b\,[$, where $-\infty \leq a < b \leq +\infty$. Here p is assumed to be a C^1-function on $]\,a,b\,[$ with strictly positive real values; q is assumed to be a real valued continuous function on $]\,a,b\,[$.

The operator L is said to be regular at the boundary point a if a is finite, p extends to a C^1-function $[\,a,b\,[\; \to]\,0,\infty\,[$ and q extends to a continuous function on $[a,b\,[$. Regularity at the second boundary point b is defined similarly. The operator L is said to be regular if it is regular at both boundary points. In the singular case, no conditions are imposed on the behavior of the functions p and q towards the boundary points apart from those already mentioned.

The operator L is formally symmetric in the sense that

$$\langle Lf\,,\, g\rangle_{[a,b]} = \langle f\,,\, Lg\rangle_{[a,b]}$$

for all compactly supported C^2-functions f and g on $]\,a,b\,[$. Here we have denoted the standard L^2-inner product on $[a,b]$ by

$$\langle f\,,\, g\rangle_{[a,b]} = \int_a^b f(t)\,\overline{g(t)}\; dt.$$

For arbitrary C^2-functions f and g on $]\,a,b\,[$ it follows by partial integration that

$$\langle Lf\,,\, g\rangle_{[x,y]} - \langle f\,,\, Lg\rangle_{[x,y]} = [f,g]_y - [f,g]_x, \tag{2.2}$$

for all $a < x \leq y < b$. Here the sesquilinear form $[\,\cdot\,,\,\cdot\,]_t$ on $C^1(\,]\,a,b\,[\,)$ (for $a < t < b$) is defined by

$$[f,g]_t := p(t)\,[f(t)\,\overline{g'(t)} - f'(t)\,\overline{g(t)}]. \tag{2.3}$$

To better understand the nature of this form, let $\langle\,\cdot\,,\,\cdot\,\rangle$ denote the standard Hermitian inner product on $\mathbb{C}^2$, and define the (anti-symmetric) sesquilinear form $[\,\cdot\,,\,\cdot\,]$ on $\mathbb{C}^2$ by

$$[v,w] := \langle Jv\,,\, w\rangle, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.4}$$

Define the evaluation map $\varepsilon_t : C^1(\,]\,a,b\,[\,) \to \mathbb{C}^2$ by

$$\varepsilon_t(f) := (\, f(t)\,,\, p(t)f'(t)\,). \tag{2.5}$$

Then the form (2.3) is given by $[f, g]_t = [\,\varepsilon_t(f)\,,\,\varepsilon_t(g)\,]$. We now observe that for ξ a non-zero vector in $\mathbb{R}^2$,

$$\langle \varepsilon_t(f)\,,\,\xi\rangle = 0 \iff \varepsilon_t(f) \in \mathbb{C}\cdot J\xi. \tag{2.6}$$

Hence, if f, g are functions in $C^1(\,]\,a, b\,[\,)$, then by anti-symmetry of the form $[\,\cdot\,,\,\cdot\,]$ we see that

$$\langle \varepsilon_t(f)\,,\,\xi\rangle = \langle \varepsilon_t(g)\,,\,\xi\rangle = 0 \implies [f, g]_t = 0. \tag{2.7}$$

For a complex number $\lambda \in \mathbb{C}$ we denote by $\mathcal{E}_\lambda$ the space of complex valued C^2-functions f on $]\,a, b\,[$, satisfying the eigenequation $Lf = \lambda f$. This eigenequation is equivalent to a system of two linear first order equations for the function $\varepsilon(f) : t \mapsto \varepsilon_t(f)$. It follows that for every $a < c < b$ and every $v \in \mathbb{C}^2$ there is a unique function $s(\lambda, \cdot\,)v = s_c(\lambda, \cdot\,)v \in C^2(\,]\,a, b\,[\,)$ such that

$$s(\lambda, \cdot\,)v \in \mathcal{E}_\lambda, \qquad \text{and} \qquad \varepsilon_c(s(\lambda, \cdot\,)v) = v. \tag{2.8}$$

By uniqueness, $s(\lambda)v$ depends linearly on v, and by holomorphic parameter dependence of the system, the map $\lambda \mapsto s(\lambda)v$ is entire holomorphic from $\mathbb{C}$ to $C^2(\,]\,a, b\,[\,)$.

3 The case of a regular operator

After these preliminaries, we recall the theory of eigenfunction expansions for a regular Sturm–Liouville operator L on $[a, b]$. Let ξ_a, ξ_b be two non-zero vectors in $\mathbb{R}^2$. We consider the linear space $C^2_\xi([a, b])$ of C^2-functions $f : [a, b] \to \mathbb{C}$ satisfying the homogeneous boundary conditions

$$\langle \varepsilon_a(f)\,,\,\xi_a\rangle = 0, \qquad \langle \varepsilon_b(f)\,,\,\xi_b\rangle = 0. \tag{3.1}$$

For all functions f and g in this space, and for $t = a, b$, we now have the conclusion of (2.7). In view of (2.2), this implies that L is symmetric on the domain $C^2_\xi([a, b])$, i.e.,

$$\langle Lf\,,\,g\rangle_{[a,b]} = \langle f\,,\,Lg\rangle_{[a,b]},$$

for all $f, g \in C^2_\xi([a, b])$. In this setting we have the following result on eigenfunction expansions. Let $\sigma(L, \xi)$ be the set of $\lambda \in \mathbb{C}$ for which the intersection $\mathcal{E}_{\lambda,\xi} := \mathcal{E}_\lambda \cap C^2_\xi([a, b])$ is non-trivial.

Theorem 3.1 *The set $\sigma(L,\xi)$ is a discrete subset of $\mathbb{R}$ without accumulation points. For each $\lambda \in \sigma(L,\xi)$ the space $\mathcal{E}_{\lambda,\xi}$ is one dimensional. Finally,*

$$L^2([a,b]) = \widehat{\bigoplus}_{\lambda\in\sigma(L,\xi)} \ \mathcal{E}_{\lambda,\xi} \qquad (\textit{orthogonal direct sum}). \tag{3.2}$$

We will sketch the proof of this result; this allows us to describe what was known about the spectral decomposition associated with a Sturm–Liouville operator when Weyl entered the scene.

For $\lambda \in \mathbb{C}$, let φ_λ be the function in $\mathcal{E}_\lambda$ determined by $\varepsilon_a(\varphi_\lambda) = J\xi_a$. Then $\langle \varepsilon_a(\varphi_\lambda)\,,\,\xi_a\rangle = 0$, hence $[\varphi_\lambda, \varphi_\lambda]_a = 0$. The function $\lambda \mapsto \varphi_\lambda$ is entire holomorphic with values in $C^2([a,b])$. We observe that $\mathcal{E}_{\lambda,\xi} \neq 0$ if and only if φ_λ belongs to $\mathcal{E}_{\lambda,\xi}$, in which case $\mathcal{E}_{\xi,\lambda} = \mathbb{C}\varphi_\lambda$. We thus see that the condition $\lambda \in \sigma(L,\xi)$ is equivalent to the condition $\langle \varepsilon_b(\varphi_\lambda)\,,\,\xi_b\rangle = 0$.

The function $\chi : \lambda \mapsto \langle \varepsilon_b(\varphi_\lambda)\,,\,\xi_b\rangle$ is holomorphic with values in $\mathbb{C}$, and from (2.2) we deduce that

$$(\lambda - \overline{\lambda})\ \langle \varphi_\lambda\,,\,\varphi_\lambda\rangle_{[a,b]} = [\varphi_\lambda, \varphi_\lambda]_b.$$

In view of (2.7) we now see that the function χ does not vanish for $\operatorname{Im}\lambda \neq 0$. Its set of zeros, which equals $\sigma(L,\xi)$, is therefore a discrete subset of $\mathbb{R}$ without accumulation points. Replacing L by a translate $L+\mu$ with $-\mu \in \mathbb{R}\setminus\sigma(L,\xi)$ if necessary, we see that without loss of generality we may assume that $0 \notin \sigma(L,\xi)$. This implies that L is injective on $C^2_\xi([a,b])$. Let $g \in C([a,b])$ and consider the equation $Lf = g$. Writing this equation as a system of first order equations in terms of $\varepsilon(f)$, using a fundamental system for the associated homogeneous equation, and applying variation of the constant one finds a unique solution $f \in C^2_\xi([a,b])$ to the equation. It is expressed in terms of g by an integral transform $\mathcal{G}$ of the form

$$\mathcal{G}g(t) = \int_a^b G(t,\tau)\, g(\tau)\ d\tau,$$

with integral kernel $G \in C([a,b]\times[a,b])$, called Green's function. The operator $\mathcal{G}$ turns out to be a two-sided inverse to the operator $L : C^2_\xi(\,]\,a,b\,[\,) \to C([a,b])$.

It follows from D. Hilbert's work on integral equations, [21], that the map $(f,g) \mapsto \langle f\,,\,\mathcal{G}g\rangle_{[a,b]}$ may be viewed as a non-degenerate Hermitian form in infinite dimensions, which allows a diagonalization over an orthonormal basis φ_k of $L^2([a,b])$, with associated non-zero diagonal elements λ_k, for $k \in \mathbb{N}$. In today's terminology we would say that the operator $\mathcal{G}$ is symmetric and completely continuous, or compact, and

Hilbert's result has evolved into the spectral theorem for such operators. From this the result follows with $\sigma(L,\xi) = \{\lambda_k^{-1} \mid k \in \mathbb{N}\}$.

4 The singular Sturm–Liouville operator

We now turn to the more general case of a (possibly) singular operator L on $]\,a, b\,[$. Weyl had written a thesis with Hilbert, leading to the paper [28], generalizing the theory of integral equations to 'singular kernels.' It was a natural idea to apply this work to singular Sturm–Liouville operators. At the time Weyl started his research it was understood that the regular cases involved discrete spectrum. On the other hand, from his work on singular integral equations it had become clear that continuous spectrum had to be expected.

Also, if one considers the example with $a = 0,\ b = \infty$, and $p = 1, q = 0$, then $L = -d^2/dt^2$ is regular at 0 and singular at ∞. Fix the boundary datum $\xi_0 = (0,1)$. Then one obtains the eigenfunctions $\cos\sqrt{\lambda}t$ of L, with eigenvalue $\lambda \geq 0$. In this case a function $f \in C_c^2([0,\infty\,[\,)$, satisfying the boundary condition $\langle \varepsilon_0(f)\,,\,\xi_0\rangle = 0$ admits the decomposition

$$f(t) = \int_0^\infty a(\sqrt{\lambda})\,\cos(\sqrt{\lambda}t)\,\frac{d\lambda}{\pi\sqrt{\lambda}}$$

involving the continuous spectral measure $\frac{d\lambda}{\pi\sqrt{\lambda}}$. Here of course, the function a is given by the cosine transform

$$a(\sqrt{\lambda}) = \int_0^\infty f(t)\cos\sqrt{\lambda}t\;dt.$$

Thus, no boundary condition needs to be imposed at infinity. At the time, Weyl faced the task to unify these phenomena, where both discrete and continuous spectrum (in his terminology 'Punktspektrum' and 'Streckenspektrum') could occur, and to clarify the role of the boundary conditions. Finally, the question arose what could be said of the spectral measure.

In [29], Weyl had the important idea to construct a Green operator for the eigenvalue problem $Lf = \lambda f$ with λ a *non-real* eigenvalue. He fixed boundary conditions for the Green kernel depending on a beautiful geometric classification of the situation at the boundary points which we will now describe. We will essentially follow Weyl's argument, but in order to postpone choosing bases, we prefer to use the language of projective space rather than refer to affine coordinates as Weyl did in [29], p. 226. The reader may consult the appendix for a quick review of the description of circles in one dimensional complex projective space in terms of Hermitian forms of signature type $(1,1)$.

Returning to the singular Sturm–Liouville problem, we make the following observation about real boundary data at a point $x \in]\,a,b\,[$.

Lemma 4.1 *Let $\lambda \in \mathbb{C}$ and let $f \in \mathcal{E}_\lambda \setminus \{0\}$. Then the following assertions are equivalent.*

(a) $\exists\, \xi \in \mathbb{R}^2 \setminus \{0\}: \quad \langle \varepsilon_x(f)\,,\, \xi \rangle = 0;$
(b) $[\varepsilon_x(f)] \in \mathbb{P}^1(\mathbb{R});$
(c) $[f,f]_x = 0.$

Proof As $[f,f]_x = [\varepsilon_x(f), \varepsilon_x(f)]$, these are basically assertions about $\mathbb{C}^2$, which are readily checked. □

It follows that the zero set

$$C_{\lambda,x} := \{f \in \mathcal{E}_\lambda \mid [f,f]_x = 0\}$$

defines a circle in the projective space $\mathbb{P}(\mathcal{E}_\lambda)$. Indeed, let $\overline{\varepsilon}_x : \mathbb{P}(\mathcal{E}_\lambda) \to \mathbb{P}^1(\mathbb{C})$ be the projective isomorphism induced by the evaluation map (2.5), then $\overline{\varepsilon}_x(C_{\lambda,x}) = \mathbb{P}^1(\mathbb{R})$.

The following important observation is made in Weyl's paper [29], Satz 1.

Proposition 4.1 *Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then the circle $C_{\lambda,x}$ in $\mathbb{P}(\mathcal{E}_\lambda)$ depends on $x \in]\,a,b\,[$ in a continuous and strictly monotonic fashion. Moreover, if $x \to b$ then $C_{\lambda,x}$ tends to either a circle or a point. A similar statement holds for $x \to a$.*

We shall denote by $C_{\lambda,b}$ the limit of the set $C_{\lambda,x}$ for $x \to b$. The notation $C_{\lambda,a}$ is introduced in a similar fashion. The proof of the above result is both elegant and simple.

Proof We fix a point $c \in]\,a,b\,[$. For $x \in]\,c,b\,[$ we define the Hermitian inner product $\langle \cdot\,,\,\cdot \rangle_x$ on $\mathcal{E}_\lambda$ by $\langle f\,,\,g \rangle_x = \langle f\,,\,g \rangle_{[c,x]}$. It follows from (2.2) that

$$[f,f]_x = [f,f]_c + 2i\,\mathrm{Im}\,(\lambda)\langle f\,,\,f \rangle_x,$$

for $f \in \mathcal{E}_\lambda$ and $x \in]\,c,b\,[$. Without loss of generality, let $\mathrm{Im}\,(\lambda) > 0$. Then it follows that $x \mapsto -i[f,f]_x$ is a real valued, strictly increasing continuous function. All results follow from this. □

In his paper [29], Weyl uses a basis $f_1, f_2 \in \mathcal{E}_\lambda$ such that $\varepsilon_c(f_2), \varepsilon_c(f_1)$ is the standard basis of $\mathbb{C}^2$. Then $[f_1, f_2]_c = 1$. In the affine chart determined by f_1, f_2 the circle $C_{\lambda,c}$ equals the real line. The circles $C_{\lambda,x}$ therefore form a decreasing family of circles which are either all contained in the upper half plane or in the lower half plane. The form $i[\,\cdot\,,\,\cdot\,]_x$ is with respect to the basis f_1, f_2 given by the Hermitian matrix $H_{kl} = i[f_k, f_l]_x$. It follows that the center of $C_{\lambda,x}$ is given by $i[f_1, f_2]_x/(-i[f_1, f_1]_x)$, see (11.2). If $\operatorname{Im}\lambda > 0$ then the denominator of this expression is positive for $t > c$ whereas the numerator has limit i for $x \downarrow c$. It follows that in the affine coordinate z parametrizing $f_2 + zf_1$ the circles $C_{\lambda,x}$ lie in the upper half plane. Likewise, for $\operatorname{Im}\lambda < 0$ all circles lie in the lower half plane.

The limit of $C_{\lambda,x}$ as x tends to one of the boundary points is closely related to the L^2-behavior of functions from $\mathcal{E}_\lambda$ at that boundary point.

Lemma 4.2 *Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and let $f \in \mathcal{E}_\lambda \setminus \{0\}$ be such that $\mathbb{C}f \in C_{\lambda,b}$. Then $f \in L^2([c, b\,[\,)$ for all $c \in \,]\,a, b\,[$.*

Proof We may fix a basis f_1, f_2 of $\mathcal{E}_\lambda$ such that in the associated affine chart, $C_{\lambda,c}$ corresponds to the real line. Then for every $x \neq c$ the circle $C_{\lambda,x}$ is entirely contained in the associated affine chart. There exists a sequence of points $x_n \in \,]\,c, b\,[$ and $F_n \in C_{\lambda,x_n}$ such that $x_n \to b$ and $F_n \to F := \mathbb{C}f$.

We agree to write $f_z = zf_1 + f_2$. Then there exist unique $z_n \in \mathbb{C}$ such that $F_n = \mathbb{C}f_{z_n}$. Now z_n converges to a point z_∞ and $F = \mathbb{C}f_{z_\infty}$. For $m < n$ we have

$$\langle f_{z_n}\,,\, f_{z_n}\rangle_{x_m} \leq \langle f_{z_n}\,,\, f_{z_n}\rangle_{x_n} = -(\lambda - \overline{\lambda})^{-1}\,[f_{z_n}, f_{z_n}]_c.$$

The expression on the right-hand side has a limit L for $n \to \infty$. It follows that

$$\langle f_{z_\infty}\,,\, f_{z_\infty}\rangle_{x_m} \leq L.$$

This is valid for any m. Taking the limit for $x_m \to b$ we conclude that $f_{z_\infty} \in L^2([c, b\,[\,)$. □

Lemma 4.3 *Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and assume that $\mathcal{E}_\lambda|_{[c,b\,[} \subset L^2([c, b\,[\,)$.*

(a) *The Hermitian form $h_{\lambda,x} := i[\,\cdot\,,\,\cdot\,]_x|_{\mathcal{E}_\lambda}$ has a limit $h_{\lambda,b} = i[\,\cdot\,,\,\cdot\,]_{\lambda,b}$, for $x \to b$.*

(b) *The form $h_{\lambda,b}$ is Hermitian and non-degenerate of signature* $(1, 1)$.

(c) *The limit set $C_{\lambda,b}$ is the circle given by $[f, f]_{\lambda,b} = 0$.*

(d) *In the space* $\mathrm{Hom}(\bar{\mathcal{E}}_\lambda^*, \mathcal{E}_\lambda)$ *the inverse* $h_{\lambda,x}^{-1}$ *converges to* $h_{\lambda,b}^{-1}$ *as* $x \to b$.

Proof (a) From (2.2) it follows by taking the limit for $x \to b$ that

$$[f, g]_{\lambda,b} = \lim_{x \to b} [f, g]_x = [f, g]_c + (\lambda - \bar{\lambda})\langle f\, ,\, g\rangle_{[c,b]}$$

for all $f, g \in \mathcal{E}_\lambda$. This establishes the existence of the limit $h_{\lambda,b}$. As $h_{\lambda,x}$ is a Hermitian form for every $x \in]\,a, b\,[$, the limit is Hermitian as well.

(b, d) Fix a basis f_1, f_2 of $\mathcal{E}_\lambda$ and write $f(z) := z_1 f_1 + z_2 f_2$, for $z \in \mathbb{C}^2$. For $x \in [c, b]$ we define the Hermitian matrix H_x by $i[f(z), f(w)]_x = \langle H_x z\, ,\, w\rangle$. Then $H_x \to H_b$ as $x \to b$. We will finish the proof by showing that $\det H_x$ is a constant function of $x \in [c, b\,[$, so that $\det H_b = \det H_c < 0$ and moreover $H_x^{-1} \to H_b^{-1}$.

Write $\varepsilon_x(f)$ for the linear endomorphism of $\mathbb{C}^2$ given by $z \mapsto \varepsilon_x(f(z))$. Then

$$[f(z), f(w)]_x = \langle J\, \varepsilon_x(f) z\, ,\, \varepsilon_x(f) w\rangle = \langle \varepsilon_x(f)^* J\, \varepsilon_x(f) z\, ,\, w\rangle,$$

so that $H_x = i\, \varepsilon_x(f)^* J \varepsilon_x(f)$. It follows that $\det H_x = -|\det \varepsilon_x(f)|^2$. By a straightforward calculation one sees that $\det \varepsilon_x(f) = [f_1, \bar{f}_2]_x$. Now $L\bar{f}_2 = \bar{\lambda} f_2$, so that from (2.2) it follows that $[f_1, \bar{f}_2]_x - [f_1, \bar{f}_2]_c = 0$ for all $x \in [c, b\,[$. Hence $\det \varepsilon_x(f) = \det \varepsilon_c(f)$ for all $x \geq c$.

Finally, the proof of (c) is straightforward. □

Combining Lemmas 4.2 and 4.3 we obtain the following corollary.

Corollary 4.4 *Let* $\lambda \in \mathbb{C} \setminus \mathbb{R}$. *Then precisely one of the following statements is valid.*

(a) *The limit set* $C_{\lambda,b}$ *is a circle. For any* $c \in]\,a, b\,[$ *the space* $\mathcal{E}_\lambda|_{[c,b\,[}$ *is contained in* $L^2([c, b\,[)$.

(b) *The limit set* $C_{\lambda,b}$ *consists of a single point. For any* $c \in]\,a, b\,[$ *the intersection of* $\mathcal{E}_\lambda|_{[c,b\,[}$ *with* $L^2([c, b\,[)$ *is one dimensional.*

At a later stage in his paper, [29], Satz 5, Weyl used spectral considerations to conclude that if (a) holds for a particular non-real eigenvalue $\lambda \in \mathbb{C}$, then $\mathcal{E}_\lambda|_{[c,b\,[}$ consists of square integrable functions for any eigenvalue λ. We will return to this in the next section, see Lemma 6.2. It follows that the validity of (a), and hence the validity of the alternative (b), is independent of the particular choice of the non-real eigenvalue λ.

If (a) holds, the operator L is said to be of *limit circle type* at b ('Grenzkreistypus'), and if (b) holds, L is said to be of *limit point type* at b ('Grenzpunkttypus'). With obvious modifications, similar results and terminology apply to the other boundary point, a. We note that a regular Sturm–Liouville operator is of the limit circle type at both boundary points.

Weyl observed that for each boundary point, the type of L determines whether boundary conditions should be imposed or not. Indeed, if L is of the limit point type at the boundary point, then no boundary condition is needed there. On the other hand, if L is of the limit circle type at a boundary point, then a boundary condition is required to ensure self-adjointness. Following Weyl, we shall now describe how boundary conditions can be imposed in the limit circle case.

The idea is to fix a non-real eigenvalue $\lambda \in \mathbb{C}$ and to construct a Green function for the operator $L - \lambda I$. Weyl did this for the particular value $\lambda = i$, but observed that the method works for any choice of non-real λ, see [29], text above Satz 5. In the mentioned paper Weyl considers the case $a = 0, b = \infty$, and L regular at a, but the method works in general. In what follows, our treatment will deviate from Weyl's with regard to technical details. However, in spirit we will stay close to his original method.

We define $\mathcal{D}$ to be the space of functions $f \in C^1(\,]\,a, b\,[\,)$ such that f' is locally absolutely continuous (so that Lf is locally integrable). Moreover, we define $\mathcal{D}_b$ to be the subspace of functions $f \in \mathcal{D}$ such that both f and Lf are square integrable on $[c, b\,[$ for some (hence any) $c \in \,]\,a, b\,[$. The subspace $\mathcal{D}_a$ is defined in a similar fashion.

Given two functions $f, g \in \mathcal{D}_b$, it follows by application of (2.2) that

$$[f, g]_b := \lim_{x \to b} [f, g]_x$$

exists. If $\chi \in \mathcal{D}_b$, then we denote by $\mathcal{D}_b(\chi)$ the space of functions $f \in \mathcal{D}_b$ such that $[f, \chi]_b = 0$. We now select a non-zero function $\varphi_{b,\lambda} \in \mathcal{E}_\lambda$ such that the associated point $\mathbb{C}\varphi_{b,\lambda} \in \mathbb{P}(\mathcal{E}_\lambda)$ belongs to the limit set $C_{\lambda,b}$. It is possible to characterize the function $\varphi_{b,\lambda}$ by its limit behavior towards b.

Lemma 4.5

(a) *If L is of limit point type at b, then $\mathcal{E}_\lambda \cap \mathcal{D}_b = \mathbb{C}\varphi_{b,\lambda}$.*

(b) *If L is of limit circle type at b, then there exists a function $\chi_b \in \mathcal{D}_b$ such that $\mathcal{E}_\lambda \cap \mathcal{D}_b(\chi_b) = \mathbb{C}\varphi_{b,\lambda}$.*

Proof (a) follows from Corollary 4.4. For (b), assume that L is of limit circle type at b. Then $\mathcal{E}_\lambda \subset \mathcal{D}_b$. Take $\chi_b = \varphi_{b,\lambda}$. Then the space on the left-hand side of the equality equals the space of $f \in \mathcal{E}_\lambda$ with $[f, \chi_b]_c = 0$. The latter space is one dimensional since $[\,\cdot\,,\,\cdot\,]_b$ is non-degenerate on $\mathcal{E}_\lambda$. On the other hand, $\varphi_{b,\lambda}$ belongs to it, by Lemma 4.3, and the result follows. $\square$

To make the treatment as uniform as possible, we agree to always use the dummy boundary datum $\chi_b = 0$ in case L is of limit point type at b. In the limit circle case we select χ_b as in Lemma 4.5 (b). Then we always have

$$\mathcal{E}_\lambda \cap \mathcal{D}_b(\chi_b) = \mathbb{C}\varphi_{b,\lambda}.$$

We follow the similar convention for a choice of boundary datum $\chi_a \in \mathcal{D}_a$, so that $\mathcal{D}_a(\chi_a) \cap \mathcal{E}_\lambda$ is a line representing a point of the limit set $C_{\lambda,a}$. Moreover, we choose a non-zero eigenfunction $\varphi_{a,\lambda}$ spanning this line. Before proceeding we observe that it follows from Proposition 4.1 that

$$C_{\lambda,a} \cap C_{\lambda,b} = \emptyset.$$

This implies that $\varphi_{a,\lambda}$ and $\varphi_{b,\lambda}$ form a basis of $\mathcal{E}_\lambda$. The above choices having been made, we put

$$\mathcal{D}_\chi = \mathcal{D}_a(\chi_a) \cap \mathcal{D}_b(\chi_b).$$

Then $\mathcal{D}_\chi$ is a subspace of $L^2(\,]\,a, b\,[\,)$. It contains $C_c^2(\,]\,a, b\,[\,)$, hence is dense. Moreover, it follows from the above that

$$\mathcal{D}_\chi \cap \mathcal{E}_\lambda = 0.$$

Still under the assumption that $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we now consider the differential equation

$$(L - \lambda)f = g \tag{4.1}$$

where g is a given square integrable function on $\,]\,a, b\,[\,$. The equation may be rewritten as a first order equation for the $\mathbb{C}^2$-valued function $\varepsilon(f)$. The matrix with columns $\varepsilon(\varphi_{a,\lambda})$ and $\varepsilon(\varphi_{b,\lambda})$ is a fundamental matrix for this system. By variation of the constant one finds a function $f \in \mathcal{D}$, satisfying (4.1). If g has compact support then f can be uniquely fixed by imposing the boundary conditions

$$\lim_{x\to a}[f, \chi_a]_x = 0, \qquad \lim_{x\to b}[f, \chi_b]_x = 0.$$

This function is expressed in terms of g by means of an integral operator,

$$f(t) = \mathcal{G}_\lambda g(t) := \int_a^b G_\lambda(t,\tau)\, g(\tau)\, d\tau, \qquad (4.2)$$

whose integral kernel, called the Green function, is given by

$$G_\lambda(t,\tau) = w(\lambda)^{-1}\, \varphi_{a,\lambda}(t)\varphi_{b,\lambda}(\tau), \qquad (t \leq \tau), \qquad (4.3)$$

and $G_\lambda(t,\tau) = G_\lambda(\tau,t)$ for $t \geq \tau$. Here $w(\lambda)$ is the Wronskian, defined by

$$w(\lambda) = [\varphi_{b,\lambda}, \overline{\varphi}_{a,\lambda}]_c,$$

for a fixed $c \in]\,a,b\,[$; note that the expression on the right-hand side is independent of c, by (2.2). It is an easy matter to show that $\mathcal{G}_\lambda$ is well defined on $L^2(\,]\,a,b\,[\,)$, with values in $\mathcal{D}$. Moreover, $(L-\lambda I)\mathcal{G}_\lambda = I$. Finally, $\mathcal{G}_\lambda$ maps functions with compact support into $\mathcal{D}_\chi$.

At this point Weyl essentially proves the following result. He specializes to $\lambda = i$ and splits G_λ into real and imaginary part, but the crucial idea is to approximate the Green kernel by Green kernels associated to a regular Sturm–Liouville problem on smaller compact intervals, where the spectral decomposition of Theorem 3.1 is applied.

Theorem 4.2 *The Green operator $\mathcal{G}_\lambda$ is a bounded linear endomorphism of $L^2(\,]\,a,b\,[\,)$, with operator norm at most* $|\mathrm{Im}\,\lambda|^{-1}$.

Proof For $z \in \mathbb{C}$ we consider the eigenfunction $\varphi_b^z = \varphi_{b,\lambda} + z\varphi_{a,\lambda}$. As $\mathbb{C}\varphi_{b,\lambda}$ is contained in the limit set $C_{\lambda,b}$, there exists a sequence of points $b_n \in]\,a,b\,[$ and $z_n \in \mathbb{C}$ such that $b_n \nearrow b$, $z_n \to 0$ and $\varphi_b^n := \varphi_b^{z_n}$ represents a point of the circle C_{λ,b_n}. Similarly, there is a sequence of points $a_n \in]\,a,b\,[$, $w_n \in \mathbb{C}$ such that $a_n \searrow a$, $w_n \to 0$ and $\varphi_a^n := \varphi_{a,\lambda} + w_n\varphi_{b,\lambda}$ represents a point of C_{λ,a_n}. Define G_λ^n as in (4.3), but with $\varphi_{a,\lambda}$ and $\varphi_{b,\lambda}$ replaced by φ_a^n and φ_b^n respectively. Then it is readily seen that $G_\lambda^n \to G_\lambda$, locally uniformly on $]\,a,b\,[\times]\,a,b\,[$. Apply the spectral decomposition associated with the regular Sturm–Liouville problem for L on $[a_n,b_n]$ with boundary data φ_a^n and φ_b^n. Then the operator $\mathcal{G}_\lambda^n : L^2([a_n,b_n]) \to L^2([a_n,b_n])$ satisfies $(L-\lambda)\circ\mathcal{G}_\lambda^n = I$, hence diagonalizes with eigenvalues $(\nu-\lambda)^{-1}$, $\nu \in \mathbb{R}$. All of these eigenvalues have length at most $|\mathrm{Im}\,\lambda|^{-1}$, so that $\|\mathcal{G}_\lambda^n\| \leq |\mathrm{Im}\,\lambda|^{-1}$. It now follows by taking limits that for all $f,g \in C_c(\,]\,a,b\,[\,)$,

$$|\langle f\,,\,\mathcal{G}_\lambda g\rangle| = \lim_{n\to\infty} |\langle f\,,\,\mathcal{G}_\lambda^n g\rangle\| \leq |\mathrm{Im}\,\lambda|.$$

This implies the result. □

Corollary 4.6 *The operator $\mathcal{G}_\lambda$ is a bounded linear endomorphism of the space $L^2(\,]\,a,b\,[\,)$ with image equal to $\mathcal{D}_\chi$. Moreover, $\mathcal{G}_\lambda$ is a two-sided inverse to the operator $L-\lambda I : \mathcal{D}_\chi \to L^2(\,]\,a,b\,[\,)$.*

Proof It is easy to check that $\mathcal{G}_\lambda$ is continuous as a map $L^2(\,]\,a,b\,[\,) \to C^1(\,]\,a,b\,[\,)$. Using (2.2) and Theorem 4.2 it is then easy to check that $g \mapsto [\mathcal{G}_\lambda g, \chi_b]_b$ is continuous on $L^2(\,]\,a,b\,[\,)$. As this functional vanishes on functions with compact support, it follows that $\mathcal{G}_\lambda$ maps $L^2(\,]\,a,b\,[\,)$ into $\mathcal{D}_b(\chi_b)$. By a similar argument at the other boundary point we conclude that $\mathcal{G}_\lambda$ maps into $\mathcal{D}_\chi$.

We observed already that $(L-\lambda I)\mathcal{G}_\lambda = I$ on $L^2(\,]\,a,b\,[\,)$. It follows that $(L-\lambda I)\circ[\mathcal{G}_\lambda(L-\lambda I) - I] = 0$ on $\mathcal{D}_\chi$. As $\mathcal{G}_\lambda$ maps into $\mathcal{D}_\chi$, on which $L-\lambda I$ is injective, it follows that $\mathcal{G}_\lambda(L-\lambda I) - I$ on $\mathcal{D}_\chi$. All assertions follow. □

Looking at Weyl's result from a modern perspective, it is now possible to show that the densely defined operator L with domain $\mathcal{D}_\chi$ is self-adjoint. To prepare for this, we need a better understanding of the boundary conditions.

In what follows we will assume that L is of limit circle type at b, so that $\mathcal{E}_\lambda \subset \mathcal{D}_b$. For $x \in\,]\,a,b\,[$, the map $\varepsilon_x : \mathcal{E}_\lambda \to \mathbb{C}^2$ is a linear isomorphism. We define the map $\beta_{\lambda,x} : \mathcal{D}_b \to \mathcal{E}_\lambda$ by $\varepsilon_x \circ \beta_x(f) = \varepsilon_x(f)$, for $f \in \mathcal{D}_b$. Then $\beta_{\lambda,x}$ may be viewed as a projection onto $\mathcal{E}_\lambda$. For $f, g \in \mathcal{D}_b$,

$$[\beta_{\lambda,x}(f), \beta_{\lambda,x}(g)]_x = [f,g]_x. \tag{4.4}$$

This implies that

$$[\beta_{\lambda,x}(f),\,\cdot\,]_x = [f,\,\cdot\,]_x \quad \text{on } \mathcal{E}_\lambda. \tag{4.5}$$

As the form $[\,\cdot\,,\,\cdot\,]_b$ is non-degenerate on $\mathcal{E}_\lambda$, we may define a linear map $\beta_{\lambda,b} : \mathcal{D}_b \to \mathcal{E}_\lambda$ by (4.5) with $x = b$. Then again $\beta_{\lambda,b} = I$ on $\mathcal{E}_\lambda$, so that $\beta_{\lambda,b}$ may be viewed as a projection onto $\mathcal{E}_\lambda$.

Lemma 4.7 *Let $f \in \mathcal{D}_b$. Then $\beta_{\lambda,x}(f) \to \beta_{\lambda,b}(f)$ in $\mathcal{E}_\lambda$, as $x \to b$.*

Proof Let γ_x denote the sesquilinear form $[\,\cdot\,,\,\cdot\,]_x$ on $\mathcal{E}_\lambda$, for $a < x \le b$. Then for $x \to b$, we have the limit behavior $\gamma_x \to \gamma_b$ in $\mathrm{Hom}(\mathcal{E}_\lambda, \overline{\mathcal{E}}_\lambda^*)$ and $\gamma_x^{-1} \to \gamma_b^{-1}$ in the space $\mathrm{Hom}(\overline{\mathcal{E}}_\lambda^*, \mathcal{E}_\lambda)$, see Lemma 4.3.

From (4.5) we deduce that $\gamma_x(\beta_{\lambda,x}(f)) = [f,\,\cdot\,]_x$, for all $f \in \mathcal{D}_b$. It follows that $\gamma_x(\beta_{\lambda,x}(f)) \to [f,\,\cdot\,]_b = \gamma_b(\beta_{\lambda,b})(f)$, hence

$$\beta_{\lambda,x}(f) = \gamma_x^{-1}\gamma_x\beta_{\lambda,x}(f) \to \beta_{\lambda,b}(f),$$

for $x \to b$. □

Corollary 4.8 *For all $f, g \in \mathcal{D}_b$ we have $[f,g]_b = [\beta_{\lambda,b}(f), \beta_{\lambda,b}(g)]_b$.*

Proof This follows from (4.4) by passing to the limit for $x \to b$. □

The following immediate corollary clarifies the nature of the boundary datum χ_b.

Corollary 4.9 *Let $\chi_b \in \mathcal{D}_b$. Then $\mathcal{D}_b(\chi_b)$ depends on χ_b through its image $\beta_{\lambda,b}(\chi_b)$ in $\mathcal{E}_\lambda$.*

It follows that in the present setting (L of limit circle type at b), the equality of Lemma 4.5 (b) is equivalent to $\mathbb{C}\beta_{\lambda,b}(\chi_b) = \mathbb{C}\varphi_{b,\lambda}$. In other words, let $\widetilde{C}_{\lambda,b}$ denote the preimage in $\mathcal{E}_\lambda \setminus \{0\}$ of the limit circle $C_{\lambda,b}$. Then functions from $\beta_{\lambda,b}^{-1}(\widetilde{C}_{\lambda,b})$ provide appropriate boundary data at b.

The following result is needed to determine the adjoint of the Green operator $\mathcal{G}_\lambda$. As $\mathcal{E}_\lambda \subset \mathcal{D}_b$ it follows that $\mathcal{E}_{\bar\lambda} = \overline{\mathcal{E}_\lambda}$ is contained in $\mathcal{D}_b$ as well. We put $\varphi_{b,\bar\lambda} := \overline{\varphi}_{b,\lambda}$; then the above definitions are valid with $\bar\lambda$ instead of λ. It is immediate from the definitions that

$$\beta_{\lambda,b}(\bar f) = \overline{\beta_{\bar\lambda,b}(f)}, \qquad (f \in \mathcal{D}_b). \tag{4.6}$$

Lemma 4.10

(a) $\beta_{\bar\lambda,b} \circ \beta_{\lambda,b} = \beta_{\bar\lambda,b}$;
(b) $\beta_{\bar\lambda,b}(\varphi_{b,\lambda}) = c\,\overline{\varphi}_{b,\lambda}$, *with c a non-zero complex scalar;*
(c) $\beta_{\lambda,b}(\bar\chi_b) = \bar c\,\varphi_{b,\lambda}$;
(d) $\mathcal{D}_b(\overline{\chi}_b) = \mathcal{D}_b(\chi_b)$.

Proof (a) We check that $\beta_{\bar\lambda,x} = \beta_{\bar\lambda,x}\beta_{\lambda,x}$ by applying ε_x on the left. Now use Lemma 4.7.

(b) Let ψ be an eigenfunction in $\mathcal{E}_{\bar\lambda}$. Then $[\varphi_{\lambda,b}, \psi]_x$ is constant as a function of x, by (2.2). Hence, $[\beta_{\bar\lambda,b}(\varphi_{\lambda,b}), \psi]_b = [\varphi_{\lambda,b}, \psi]_x$. It follows that $\beta_{\bar\lambda,b}(\varphi_{\lambda,b})$ is non-zero, whereas $[\beta_{\bar\lambda,b}(\varphi_{\lambda,b}), \varphi_{\bar\lambda,b}]_b = [\varphi_{\lambda,b}, \varphi_{\bar\lambda,x}]_x = 0$. The assertion follows.

(c) Applying (4.6), (a) and (b) we obtain: $\overline{\beta_{\lambda,b}(\bar\chi_b)} = \beta_{\bar\lambda,b}(\chi_b) =$

$\beta_{\bar\lambda,b}\beta_{\lambda,b}(\chi_b) = \beta_{\bar\lambda}(\varphi_{b,\lambda}) = c\overline{\varphi}_{b,\lambda}$. Finally, (d) is an immediate consequence of (c). $\square$

We now return to the situation of a general singular Sturm–Liouville operator L.

Theorem 4.3 *The operator L with domain $\mathcal{D}_\chi$ is self-adjoint.*

Proof From Lemma 4.10 it follows that $\mathcal{D}_\chi = \mathcal{D}_{\overline{\chi}}$. The adjoint $\mathcal{G}^*$ of the operator $\mathcal{G} = \mathcal{G}_\lambda$ has integral kernel $G^*_\lambda(t,\tau) := \overline{G_\lambda(\tau,t)}$. This is precisely the Green kernel associated with the eigenvalue $\bar\lambda$ and the boundary data $\bar\chi_a, \bar\chi_b$. As its image $\mathcal{D}_{\bar\chi}$ equals $\mathcal{D}_\chi$, it follows that $\mathcal{G}^*$ is the two-sided inverse of the bijection $L - \bar\lambda I : \mathcal{D}_\chi \to L^2(\,]\,a,b\,[\,)$. These facts imply that the adjoint L^* equals L. $\square$

At this point one can prove the following generalization of Theorem 3.1, due to Weyl, [29], Satz 4. Let $\sigma(L,\chi)$ be the set of $\lambda \in \mathbb{C}$ for which $\mathcal{E}_\lambda \cap \mathcal{D}_\chi \neq 0$.

Theorem 4.4 (Weyl 1910) *Let L be of limit circle type at both end points. Then $\sigma(L,\chi)$ is a discrete subset of $\mathbb{R}$, without accumulation points. Moreover, $L^2(\,]\,a,b\,[\,)$ is the orthogonal direct sum of the spaces $\mathcal{E}_\lambda \cap \mathcal{D}_\chi$.*

Proof Weyl proved this by using the Green operator $\mathcal{G}$ corresponding to the eigenvalue i. Let G_2 be the imaginary part of its kernel. Then G_2 is real valued, symmetric and square integrable, hence admits a diagonalization. In today's terminology, the associated integral operator $\mathcal{G}_2$, which equals $(2i)^{-1}[\mathcal{G} - \mathcal{G}^*]$, is self-adjoint and Hilbert-Schmidt, hence compact. All its eigenspaces are finite dimensional, and contained in $\mathcal{D}_\chi$, since $\mathcal{D}_\chi = \mathcal{D}_{\bar\chi}$. Moreover, each of them is invariant under the symmetric operator L. $\square$

The regular case may be viewed as a special case of the above. Indeed, if ξ_a, ξ_b are the boundary data of Theorem 3.1, let $\mu \in \mathbb{C}\backslash\mathbb{R}$ be arbitrary, and for $x = a, b$, let χ_x be the constant function with value $J\xi_x$. Then $\mathcal{E}_{\lambda,\xi} = \mathcal{E}_\lambda \cap \mathcal{D}_\chi$, for all $\lambda \in \mathbb{C}$.

5 Weyl's spectral theorem

Using Green's function G_λ for non-real λ and his earlier work on singular integral equations, [28], Weyl was able to establish the existence of a spectral decomposition of $L^2(\,]\,a,b\,[\,)$ in terms of eigenfunctions of L with real eigenvalue. In [29] he considers the case $a=0, b=\infty$, and assumes that L is regular (hence of circle limit type) at 0. Let $\xi_a \in \mathbb{R}^2 \setminus \{0\}$ be a boundary datum at a, and fix a unit vector $\eta \in \mathbb{R}^2$ perpendicular to ξ. Let χ_0 be the constant function with value η and let χ_∞ be a boundary datum at ∞ (if L is of limit point type at ∞, we take the dummy boundary datum $\chi_\infty = 0$). For each $\lambda \in \mathbb{C}$ let φ_λ be the unique eigenfunction of L with eigenvalue λ and $\varphi_\lambda(a) = \eta$. Then according to Weyl, [29], Satz 5,7, there exists a right-continuous monotonically increasing function ρ such that each function $f \in C^2(\,]\,0,\infty\,[\,) \cap \mathcal{D}_\chi$ admits a decomposition of the form

$$f(x) = \int_{\mathbb{R}} \varphi_\lambda(x)\, dF(\lambda) \tag{5.1}$$

with uniformly and absolutely converging integral; here $dF(\lambda)$ is a regular Borel measure, defined by

$$dF(\Delta) = \int_0^\infty f(t) \int_\Delta \varphi_\lambda(t)\, d\rho(\lambda)\, dt. \tag{5.2}$$

In the above, $d\rho$ denotes the regular Borel measure determined by the formula $d\rho(\,]\,\mu,\nu]) = \rho(\nu) - \rho(\mu)$, for all $\mu < \nu$.

Actually, Weyl's original formulation was different and involved a discrete and a continuous part. His formulation follows from the one above by the observation that ρ admits a unique decomposition $\rho = \rho_d + \rho_c$ with ρ_c a continuous monotonically increasing function with $\rho_c(0) = 0$, and with ρ_d a right-continuous monotonically increasing function which is constant on each interval where it is continuous.

In case L is of the limit circle type at infinity, the decomposition is discrete by Theorem 4.4, so that $\rho_c = 0$, so that the above gives rise to a discrete decomposition. In case L is of the limit point type at ∞, the decomposition is of mixed discrete and continuous type.

It has now become customary to write

$$dF(\lambda) = \mathcal{F}f(\lambda)\, d\rho(\lambda), \qquad \mathcal{F}f(\lambda) = \int_0^\infty f(t)\, \varphi_\lambda(t)\, dt, \tag{5.3}$$

with the interpretation that the integral converges as an integral with values in $L^2(\mathbb{R}, d\rho)$.

We will call ρ the spectral function associated with the operator L, the boundary data χ_0, χ_∞, and the choice of eigenfunctions φ_λ. In [29], Weyl also addressed the natural problem to determine its continuous part ρ_c.

Theorem 5.1 (Weyl 1910) *Assume L is a Sturm–Liouville operator of the form* (2.1) *on* $[0, \infty[$, *regular at* 0. *Assume moreover that the coefficients p and q satisfy the conditions*

(a) $\lim_{t\to\infty} t|p(t)-1| = 0, \qquad \lim_{t\to\infty} t\,q(t) = 0,$
(b) $\int_0^\infty t|p(t)-1|\,dt < \infty, \qquad \int_0^\infty t|q(t)| < \infty.$

Then L is of the limit point type at ∞. Let ξ_a, η, χ_a and φ_λ be defined as above and let ρ be the associated spectral function. Then the support of $d\rho_d$ is finite and contained in the open negative real half line $]-\infty, 0[$. The support of $d\rho_c$ is contained in the closed positive real half line $[0, \infty[$. There exist uniquely determined continuous functions $a, b :]0, \infty[\to \mathbb{R}$ such that

$$\varphi_\lambda(t) = a(\lambda)\cos(t\sqrt{\lambda}) + b(\lambda)\sin(t\sqrt{\lambda}) + o(t). \tag{5.4}$$

In terms of these coefficients, the spectral measure $d\rho_c$ is given by

$$d\rho_c(\lambda) = \frac{1}{a(\lambda)^2 + b(\lambda)^2}\,\frac{d\lambda}{\pi\sqrt{\lambda}}.$$

Here we note that by (5.4) and the condition on f, the integral (5.3) is absolutely convergent. If $p = 1$ and $q = 0$, then of course one has $a(\lambda) = \eta_1$ and $b(\lambda) = \eta_2$, and one retrieves the continuous measure $d\rho_c(\lambda) = (\pi\sqrt{\lambda})^{-1}d\lambda$.

Let $c(\sqrt{\lambda}) := \frac{1}{2}(a(\lambda) - ib(\lambda))$. Then

$$\varphi_\lambda(t) = c(\lambda)e^{it\sqrt{\lambda}} + \overline{c(\lambda)}e^{-it\sqrt{\lambda}} + o(t)$$

and the spectral measure is given by

$$d\rho(\lambda) = \frac{d\sqrt{\lambda}}{2\pi|c(\sqrt{\lambda})|^2} \tag{5.5}$$

We may view the operator L as a perturbation of the operator $-d^2/dt^2$. At infinity the eigenfunction φ_λ behaves asymptotically as a linear combination of the exponential eigenfunctions for the unperturbed problem, with amplitudes of equal modulus $|c(\sqrt{\lambda})|$. The spectral measure of the perturbed problem is obtained from the spectral measure of the unperturbed problem by dividing through $|c(\sqrt{\lambda})|^2$. As we will see later, this principle is omnipresent in the theory of harmonic analysis of non-compact Riemannian symmetric spaces, of non-compact real semisimple Lie groups, and of their common generalization, the so-called semisimple symmetric spaces.

6 Dependence on the eigenvalue parameter

In this section we will prove holomorphic dependence of the Green function G_λ on the parameter λ. This is not obvious from the definition (4.3). Indeed, in the limit circle case at b, the particular normalization of $\varphi_{b,\lambda}$ chosen only guarantees real analytic dependence on the parameter λ (this fairly easy result will not be needed in the sequel). In the limit point case, only the line $\mathbb{C}\varphi_{b,\lambda}$ does not depend on the choices made, but the dependence of $\varphi_{b,\lambda}$ on λ may be arbitrary. The following result suggests to look for differently normalized eigenfunctions, which do depend holomorphically on λ.

Lemma 6.1 *The Green kernel G_λ defined by* (4.2) *depends on $\varphi_{a,\lambda}$ and $\varphi_{b,\lambda}$ through their images in $\mathbb{P}(\mathcal{E}_\lambda)$.*

Proof This is caused by the division by the Wronskian $w(\lambda) = [\varphi_{b,\lambda}, \overline{\varphi}_{a,\lambda}]$. □

The following result follows by application of the method of variation of the constant as explained in [7], Thm. 2.1, p. 225. The assertion about holomorphic dependence is not given there, but follows by the same method of proof.

Lemma 6.2 *Let $a < c < b$ and assume that for some $\lambda_0 \in \mathbb{C}$ the eigenspace $\mathcal{E}_{\lambda_0}|_{[c,b[}$ is contained in $L^2([c,b[)$. Then for each eigenvalue $\lambda \in \mathbb{C}$ the associated eigenspace $\mathcal{E}_\lambda|_{[c,b[}$ is contained in $L^2([c,b[)$.*

Moreover, for each $c \in]a,b[$ and all $v \in \mathbb{C}^2$, the function $\lambda \mapsto s_c(\lambda,\,\cdot\,)v|_{[c,b[}$ (see (2.8)) *is entire holomorphic as a function with values in $L^2([c,b[)$.*

In the following, we assume that L is of limit circle type at b. From the text below (4.5) we recall the definition of the map $\beta_{\lambda,b} : \mathcal{D}_b \to \mathcal{E}_\lambda$, for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Lemma 6.3 *Let L be of the limit circle type at b. Then for all $\mu, \lambda \in \mathbb{C} \setminus \mathbb{R}$,*

(a) $\beta_{\mu,b} \circ \beta_{\lambda,b} = \beta_{\mu,b}$;
(b) *the restriction $\beta_{\mu,b}|_{\mathcal{E}_\lambda}$ is a linear isomorphism onto $\mathcal{E}_\mu$;*
(c) *the restriction $\beta_{\mu,b}|_{\mathcal{E}_\lambda}$ induces a projective isomorphism $\mathbb{P}(\mathcal{E}_\lambda) \to \mathbb{P}(\mathcal{E}_\mu)$, mapping the limit circle $C_{\lambda,b}$ onto the limit circle $C_{\mu,b}$.*

Proof Assertion (a) is proved in the same fashion as assertion (a) of Lemma 4.10. Since $[\,\cdot\,,\,\cdot\,]_b$ is non-degenerate on both $\mathcal{E}_\lambda$ and $\mathcal{E}_\mu$, assertion (b) follows by application of Corollary 4.8. Finally, (c) follows from the identity of Corollary 4.8, in view of Lemma 4.3. $\square$

The following result suggests the modification of the eigenfunctions in (4.2) that we are looking for.

Lemma 6.4 *Let L be of the limit circle type at b. Let $\chi_b \in \mathcal{D}_b$ and assume that for some $\mu \in \mathbb{C} \setminus \mathbb{R}$ the function $\beta_{\mu,b}(\chi_b)$ is non-zero and represents a point of the limit circle $C_{\mu,b}$. Then*

(a) *for each $\lambda \in \mathbb{C}\setminus\mathbb{R}$ the function $\beta_{\lambda,b}(\chi)$ is a non-zero eigenfunction in $\mathcal{E}_\lambda$ which represents a point of the limit circle $C_{\lambda,b}$;*

(b) *for each $c \in]\,a,b\,[$, the map $\lambda \mapsto [\varepsilon_c(\beta_{\lambda,b}(f))]$ is holomorphic from $\mathbb{C} \setminus \mathbb{R}$ to $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$.*

Proof By the first assertion of Lemma 6.3, $\beta_{\lambda,b}(\chi) = \beta_{\lambda,b}\beta_{\mu,b}(\chi) = \beta_{\lambda,b}(\varphi_{b,\mu})$. Assertion (a) follows by application of the remaining assertions of the mentioned lemma.

We now turn to (b). We will prove the holomorphy in a neighborhood of the fixed point $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$. As $\beta_{\lambda,b}(\chi) = \beta_{\lambda,b}(\beta_{\lambda_0,b}(\chi))$ we may as well assume that $\chi \in \mathcal{E}_{\lambda_0}$ and that $[\chi] \in C_{\lambda_0,b}$. Select a sequence x_n in $]\,a,b\,[$ converging to b, and for each n a point $p_n \in C_{\lambda_0,x_n}$ such that $p_n \to [\chi]$. There exist $\chi_n \in \mathcal{E}_{\lambda_0}$ such that $[\chi_n] = p_n$ and $\chi_n \to \chi$ in $\mathcal{E}_{\lambda_0}$. We define $\varphi_{\lambda,n} \in \mathcal{E}_\lambda$ by $\varepsilon_{x_n}\varphi_{n,\lambda} = \varepsilon_{x_n}\chi_n$. Then in the notation of (4.5), $\varphi_n(\lambda)$ equals $\beta_{\lambda,x_n}(\chi_n)$ and represents a point of C_{λ,x_n}. For each fixed λ the sequence $\beta_{\lambda,x_n}|_{\mathcal{E}_{\lambda_0}}$ in $\mathrm{Hom}(\mathcal{E}_{\lambda_0}, \mathcal{E}_\lambda)$ has limit $\beta_{\lambda,b}$. Hence $\varphi_n(\lambda) \to \beta_{\lambda,b}(\chi)$, pointwise in λ.

Let $c \in]\,a,b\,[$. Passing to a subsequence we may assume that $x_n > c$ for all $n \geq 1$. The map $\overline{\varepsilon}_c : \mathbb{P}(\mathcal{E}_\lambda) \to \mathbb{P}^1(\mathbb{C})$ maps the circle $C_{\lambda,c}$ onto $\mathbb{P}^1(\mathbb{R})$. Let Ω be a connected open neighborhood of λ_0. Then it follows by application of Proposition 4.1 that all circles $\overline{\varepsilon}_c(C_{\lambda,x_n})$ are contained in one particular connected component U of $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$. This implies that $\psi_n(\lambda) := \overline{\varepsilon}_c\beta_{x_n,\lambda}(f) \in U$ for every $\lambda \in \Omega$. By using an affine chart containing the compact closure of U we see that the sequence ψ_n has a subsequence converging locally uniformly to a holomorphic limit function $\psi : \Omega \to U$. By pointwise convergence, $\psi(\lambda) = [\varepsilon_c\beta_{\lambda,b}(\chi)]$, and (b) follows. $\square$

Corollary 6.5 *Let L be of the limit circle type at b and let $\chi_b \in \mathcal{D}_b$ be as in the above lemma. There exists a family of functions $\varphi_{\lambda,b} \in C^2(\,]\,a,b\,[\,)$ depending holomorphically on $\lambda \in \mathbb{C} \setminus \mathbb{R}$ such that for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$*

(a) $\varphi_{b,\lambda} \in \mathcal{E}_\lambda \setminus \{0\}$;
(b) *$\varphi_{b,\lambda}$ represents the point $[\beta_{\lambda,b}(\chi_b)]$ of the limit circle $C_{\lambda,b}$.*

The following analogous result in the limit point case can be proved using a similar method, see [7], Thm. 2.3, p. 229, for details.

Lemma 6.6 *Let L be of the limit point type at b. Then there exists a family of functions $\varphi_{b,\lambda} \in C^2(\,]\,a,b\,[\,)$, depending holomorphically on the parameter $\lambda \in \mathbb{C} \setminus \mathbb{R}$, such that for each $\lambda \in \mathbb{C} \setminus \mathbb{R}$,*

(a) $\varphi_{b,\lambda} \in \mathcal{E}_\lambda \setminus \{0\}$;
(b) *the function $\varphi_{b,\lambda}$ represents the limit point in $\mathbb{P}(\mathcal{E}_\lambda)$.*

Let L be arbitrary again. We fix boundary data χ_a, χ_a as indicated in the previous section, so that $L : \mathcal{D}_\chi \to L^2(\,]\,a,b\,[\,)$ is self-adjoint. Accordingly, we fix holomorphic families of eigenfunctions $\varphi_{a,\lambda}, \varphi_{b,\lambda} \in \mathcal{E}_\lambda$ in the manner indicated in Corollary 6.5 and Lemma 6.6.

Finally, we define the Green function G_λ by means of the formula (4.3). The functions $\varphi_{a,\lambda}, \varphi_{b,\lambda}$ used here are renormalizations of those used in Section 4. By Lemma 6.1 this does not affect the definition of the Green kernel.

Corollary 6.7 *The Green kernel $G_\lambda \in C(\,]\,a,b\,[\,\times\,]\,a,b\,[\,)$ depends holomorphically on the parameter $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*

This result of course realizes the resolvent $(L-\lambda I)^{-1}$ of the self-adjoint operator L with domain $\mathcal{D}_\chi$ explicitly as an integral operator with kernel depending holomorphically on λ.

7 A paper of Kodaira

For the general singular Sturm–Liouville problem, there exists a spectral decomposition similar to (5.1), but with a spectral matrix instead of the spectral function ρ. Weyl observed this in [30]. The spectral matrix was later determined by E.C. Titchmarsh who used involved direct computations using the calculus of residues, see [27].

Independently, K. Kodaira [23] rediscovered the result by a very elegant method, combining Weyl's construction of the Green function with the general spectral theory for self-adjoint unbounded operators on Hilbert space, as developed in the 1930's by J. von Neumann and M. Stone. Weyl was very content with this work of Kodaira, as becomes clear from the following quote from the Gibbs lecture delivered in 1948, [31], p. 124: 'The formula (7.5) was rediscovered by Kunihiko Kodaira (who of course had been cut off from our Western mathematical literature since the end of 1941); his construction of ρ and his proofs for (7.5) and the expansion formula [...], still unpublished, seem to clinch the issue. It is remarkable that forty years had to pass before such a thoroughly satisfactory direct treatment emerged; the fact is a reflection on the degree to which mathematicians during this period got absorbed in abstract generalizations and lost sight of their task of finishing up some of the more concrete problems of undeniable importance.'

We will now describe the spectral decomposition essentially as presented by Kodaira [23]. Fix boundary data χ_a and χ_b as in Theorem 4.3. We use the notation $\mathcal{H} := L^2(\,]\,a, b\,[\,)$. Then the operator L with domain $\mathcal{D}_\chi$ is a self-adjoint operator in the Hilbert space $\mathcal{H}$; it therefore has a spectral resolution dE.

To obtain a suitable parametrization of the space of eigenfunctions for L, fix $c \in \,]\,a, b\,[$ and recall that the map $\varepsilon_c : f \mapsto (f(c), p(c)f'(c))$ is a linear isomorphism from $\mathcal{E}_\lambda$ onto $\mathbb{C}^2$, for each $\lambda \in \mathbb{C}$. We define the function $s(\lambda) = s_c(\lambda, \cdot\,) : \,]\,a, b\,[\, \to \mathrm{Hom}(\mathbb{C}^2, \mathbb{C})$ as in (2.6). Then $\lambda \mapsto s(\lambda)$ may be viewed as an entire holomorphic map with values in $C^2(\,]\,a, b\,[\,) \otimes \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}))$. Moreover, for each $\lambda \in \mathbb{C}$ the map $v \mapsto s(\lambda)v$ is a linear isomorphism from $\mathbb{C}^2$ onto $\mathcal{E}_\lambda$.

For $f \in C_c(\,]\,a, b\,[\,)$ and $\lambda \in \mathbb{R}$ we define the Fourier transform

$$\mathcal{F}f(\lambda) = \int_a^b s(\lambda, x)^* f(x)\, dx, \tag{7.1}$$

where $s(\lambda, x)^* \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2)$ is the adjoint of $s(\lambda, x)$ with respect to the standard Hermitian inner products on $\mathbb{C}^2$ and $\mathbb{C}$.

By a spectral matrix we shall mean a function $P : \mathbb{R} \to \mathrm{End}(\mathbb{C}^2)$ with the following properties

(a) $P(x)^* = P(x)$, i.e., $P(x)$ is Hermitian with respect to the standard inner product, for all $x \in \mathbb{R}$;
(b) P is continuous from the right;
(c) $P(0) = 0$ and $P(y) - P(x)$ is positive semi-definite for all $x \leq y$.

Associated with a spectral matrix as above there is a unique regular Borel measure dP on $\mathbb{R}$, with values in the space of positive semi-definite Hermitian endomorphisms of $\mathbb{C}^2$, such that $dP(\,]\,\mu, \nu]) = P(\nu) - P(\mu)$

for all $\mu \leq \nu$. Conversely, a measure with these properties comes from a unique spectral matrix P. Given a spectral matrix P, we define $\mathcal{M}_2 = \mathcal{M}_{2,P}$ to be the space of Borel measurable functions $\varphi : \mathbb{R} \to \mathbb{C}^2$ with

$$\langle \varphi \, , \, \varphi \rangle_P := \int_{\mathbb{R}} \langle \varphi(\nu) \, , \, dP(\nu)\varphi(\nu) \rangle < \infty.$$

Moreover, we define $\mathfrak{H} = \mathfrak{H}_P$ to be the Hilbert space completion of the quotient $\mathcal{M}_2 / \mathcal{M}_2^{\perp}$.

Let $T_\lambda := \varepsilon_c \circ \mathcal{G}_\lambda$. Then $T_\lambda : \mathcal{H} \to \mathbb{C}^2$ is a continuous linear map. We denote its adjoint by T_λ^*. Kodaira uses the elements $\gamma_1(\lambda), \gamma_2(\lambda)$ of $\mathcal{H}$ determined by $\mathrm{pr}_j \circ T_\lambda = \langle \,\cdot\, , \, \gamma_j(\lambda) \rangle$.

Theorem 7.1 (Kodaira 1949) *The spectral function P determined by*

$$dP(\nu) = |\nu - \lambda|^2 \, T_\lambda \circ dE(\nu) \circ T_\lambda^* \tag{7.2}$$

is independent of $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Moreover, it has the following properties.

(a) *The Fourier transform extends to an isometry from the Hilbert space $\mathcal{H} = L^2(\,]\, a, b\, [\,)$ onto the Hilbert space $\mathfrak{H} = \mathfrak{H}_P$.*

(b) *The spectral resolution $dE(\nu)$ of the self-adjoint operator L with domain $\mathcal{D}_\chi$ is given by*

$$\mathcal{F} \circ dE(S) = 1_S \circ \mathcal{F},$$

for every Borel measurable set $S \subset \mathbb{R}$; here 1_S denotes the map induced by multiplication with the characteristic function of S.

For the proof of Theorem 7.1, which involves ideas of Weyl [29], we refer the reader to Kodaira's paper [23]. In addition to the above, Kodaira proves more precise statements about the nature of the convergence of the integrals in the associated inversion formula.

After having introduced the spectral matrix, Kodaira gives an ingenious short proof of an expression for the spectral matrix which had been found earlier by Titchmarsh. We observe that the $\mathbb{C}^2$-valued functions $F_a(\lambda) = \varepsilon_c(\varphi_{a,\lambda})$ and $F_b(\lambda) = \varepsilon_c(\varphi_{b,\lambda})$ are holomorphic functions of $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The matrix $F(\lambda)$ with columns $F_a(\lambda)$ and $F_b(\lambda)$ is invertible for $\lambda \in \mathbb{C} \setminus \mathbb{R}$. By the above definitions,

$$\varphi_{a,\lambda}(t) = s(\lambda, t) F_a(\lambda), \qquad \varphi_{b,\lambda}(t) = s(\lambda, t) F_b(\lambda). \tag{7.3}$$

We now define the 2×2 matrix $M(\lambda)$, the so-called characteristic matrix, by $M(\lambda) = -(\det F)^{-1} F_a F_b^{\mathrm{T}}$, i.e.,

$$M(\lambda) = -\det F(\lambda)^{-1} \begin{pmatrix} F_{a1}F_{b1} & F_{a1}F_{b2} \\ F_{a2}F_{b1} & F_{a2}F_{b2} \end{pmatrix}_\lambda \tag{7.4}$$

Actually, Kodaira uses the symmetric matrix $M(\lambda) - \frac{1}{2}J$, which has the same imaginary part. The matrix $M(\lambda)$ depends holomorphically on the parameter $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Theorem 7.2 (Titchmarsh, Kodaira) *The spectral matrix P is given by the following limit:*

$$P(\nu) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{[\delta, \nu+\delta]+i\varepsilon} \operatorname{Im} M(\lambda)\, d\lambda. \tag{7.5}$$

Proof Multiplying both sides of (7.2) with $|\nu - \lambda|^{-2}$ and integrating over $\mathbb{R}$, we find that

$$\int_{\mathbb{R}} |\nu - \lambda|^{-2}\, dP(\nu) = T_\lambda T_\lambda^*.$$

By a straightforward, but somewhat tedious calculation, using (2.2) and $[\varphi_{a,\lambda}, \varphi_{a,\lambda}]_a = [\varphi_{b,\lambda}, \varphi_{b,\lambda}]_b = 0$, it follows that

$$\operatorname{Im} \lambda\, T_\lambda T_\lambda^* = \frac{1}{2i} \frac{1}{|[F_b, \overline{F}_a]|^2} (\, [F_a, F_a] F_b \overline{F}_b^{\mathrm{T}} - [F_b, F_b]\, F_a \overline{F}_a^{\mathrm{T}}\,).$$

This in turn implies that $\operatorname{Im} \lambda \cdot T_\lambda T_\lambda^* = \operatorname{Im} M(\lambda)$. Hence,

$$\int_{\mathbb{R}} |\nu - \lambda|^{-2} \operatorname{Im} \lambda\, dP(\nu) = \operatorname{Im} M(\lambda).$$

From this (7.5) follows by a straightforward argument. $\square$

After this, Kodaira shows that the above result can be extended to a more general basis of eigenfunctions. A fundamental system for L is a linear map $s(\lambda) : \mathbb{C}^2 \to \mathcal{E}_\lambda$, depending entire holomorphically on $\lambda \in \mathbb{C}$ as a $C^2(\,]\,a, b\,[\,)$-valued function, such that the following conditions are fulfilled for all $\lambda \in \mathbb{C}$:

(a) $\overline{s(\lambda) v} = s(\bar{\lambda})(\bar{v}), \quad (v \in \mathbb{C}^2)$;
(b) $\det(\varepsilon_x \circ s(\lambda)) = 1, \quad (x \in \,]\,a, b\,[\,)$.

Put $s_j(\lambda) = s(\lambda) e_j$, then condition (b) means precisely that the Wronskian $[s_1(\lambda), \overline{s_2(\lambda)}]_x$ equals 1. Write $\psi(\lambda) = \varepsilon_c \circ s(\lambda) \in \operatorname{End}(\mathbb{C}^2)$. Then it follows that ψ entire holomorphic, and that $\det \psi(\lambda) = 1$. Moreover,

$$s(\lambda, x) = s_c(\lambda, x) \psi(\lambda).$$

We may define the Fourier transform associated with s by the identity (7.1). The associated spectral function P is expressed in terms of the spectral matrix P_c for s_c by the equation

$$dP(\lambda) = \psi(\lambda)^{-1}\, dP_c(\lambda)\, \psi(\lambda)^{*-1}.$$

We define the matrix F for s by the identity (7.3). Then the associated matrix M, defined by (7.4) is given by

$$M(\lambda) = \psi(\lambda)^{-1}\, M_c(\lambda)\, \psi(\lambda)^{\mathrm{T}-1}.$$

Kodaira shows that with these definitions, the identity (7.5) is still valid.

8 A special equation

In the second half of the paper [23], Kodaira applies the above results to the time independent one dimensional Schrödinger operator

$$L = -\frac{d^2}{dt^2} + m(m+1)t^{-2} + V(t),$$

with $m \geq -\frac{1}{2}$ and $tV(t)$ a real valued real analytic function on an open neighborhood of $[0, \infty[$, such that

$$tV(t) = \mathcal{O}(t^{-\varepsilon}), \quad \text{for } t \to \infty,$$

with $\varepsilon > 0$. Actually, Kodaira considers a more general problem with weaker requirements both at infinity and zero, but we shall not need this. It is in fact not clear that his condition on the behavior of V at 0 is strong enough for the subsequent argument to be valid, as was pointed out by [22], p. 206. Kodaira's argumentation, which we shall now present, is valid under the hypotheses stated above, as they imply that the eigenequation $Lf = \lambda f$ has a regular singularity at zero. Because of this, the asymptotic behavior of the eigenfunctions towards zero is completely understood. Indeed, the associated indicial equation has solutions $m+1$ and $-m$, where $m+1 \geq -m$. Let c_0 be any non-zero real constant. Then there exists a unique eigenfunction $s_1(\lambda) \in \mathcal{E}_\lambda$ such that

$$s_1(\lambda, t) = c_0\, t^{m+1} \varphi(\lambda, t)$$

with $\varphi(\lambda, \cdot\,)$ real analytic in an open neighborhood of 0 and $\varphi(\lambda, 0) = 1$. It can be shown that $\varphi(\lambda, t)$ is entire holomorphic in λ and real valued for real λ. Kodaira claims that there exists a second eigenfunction $s_2(\lambda) \in \mathcal{E}_\lambda$, depending holomorphically on λ, such that s_1, s_2 form a fundamental system fulfilling the requirements (a) and (b) stated below Theorem 7.2. Using the theory of second order differential equations with a regular singularity this can indeed be proved along the following lines.

If $k := (m+1) - (-m) = 2m+1$ is strictly positive, there exists a second eigenfunction $s_2(\lambda) \in \mathcal{E}_\lambda$ with

$$s_2(\lambda, t) = -c_0^{-1}(2m+1)^{-1}\, t^{-m} + \mathcal{O}(t^{-m+\varepsilon}), \qquad (t \to 0).$$

If k is not an integer, this eigenfunction is unique. If k is an integer, then $s_2(\lambda)$ has a series expansion in terms of t^{-m+r} and $t^{m+1+s}\log t$, $(r, s \in \mathbb{N})$, and is uniquely determined by the requirement that the coefficient of $t^{-m+k} = t^{m+1}$ is zero. Finally, if $k = 0$, i.e., $m = -\frac{1}{2}$, then there exists a unique second eigenfunction $s_2(\lambda, t)$ with

$$s_2(\lambda, t) = c_0^{-1} t^{1/2} \log t + \mathcal{O}(t^{1/2+\varepsilon}), \qquad (t \to 0).$$

In all cases, by arguments involving monodromy for t around zero it can be shown that $s_2(\lambda, t)$ is entire holomorphic in λ and real valued for real λ. Finally, from the series expansions for these functions and their derivatives, it follows that the Wronskian $[s_1(\lambda), \overline{s_2(\lambda)}]_t$ behaves like $1 + \mathcal{O}(t^\varepsilon)$ for $t \to 0$. Since the Wronskian is constant, this implies that s_1, s_2 is a fundamental system.

From the asymptotic behavior of s_1, s_2 it is seen that at the boundary point 0, the operator L is of limit circle type if and only if $m > \frac{1}{2}$. It is of limit point type if $-\frac{1}{2} \leq m \leq \frac{1}{2}$. In the first case we fix the boundary datum $\chi_0 = s_1(0, \cdot\,)$ at 0 and in the second case we fix the (dummy) boundary datum $\chi_0 = 0$. In all cases s_1 is square integrable on $]\,0, 1]$, so that $\varphi_{0\lambda} = s_1(\lambda)$ and $F_0(\lambda) = (1, 0)^{\mathrm{T}}$, in the notation of (7.3).

We now turn to the asymptotic behavior at ∞. Kodaira first shows that for every ν with $\operatorname{Im}\nu \geq 0$, $\nu \neq 0$, there is a unique solution Φ_ν to the equation $Lf = \nu^2 f$ such that

$$\Phi_\nu(t) \sim e^{i\nu t}, \qquad (t \to \infty),$$

the asymptotics being preserved if the expressions on both sides are differentiated once with respect to t. Moreover, both $\Phi_\nu(t)$ and $\Phi_\nu'(t)$ are continuous in (t, ν) and holomorphic in ν for $\operatorname{Im}\nu > 0$.

For $\operatorname{Im}\nu < 0$ the function $\Psi_\nu = \overline{\Phi_{\overline{\nu}}}$ belongs to $\mathcal{E}_{\nu^2}$ and $\Psi_\nu(t) \sim e^{-i\nu t}$ for $t \to \infty$. This shows that Ψ_ν is not square integrable towards infinity, so that L is of limit point type at infinity. We may therefore take

$$\varphi_{\infty\lambda} = \Phi_\nu, \qquad (\operatorname{Im}\lambda > 0,\ \operatorname{Im}\nu > 0,\ \nu^2 = \lambda).$$

It follows from the above that $\Phi_\nu(t) = a(\nu)\, s_1(\nu^2, t) + b(\nu)\, s_2(\nu^2, t)$, with

a, b continuous on $\operatorname{Im}\nu \geq 0,\ \nu \neq 0$, and holomorphic on $\operatorname{Im}\nu > 0$. We note that $F_\infty(\lambda) = (a(\nu), b(\nu))^{\mathrm{T}}$. Using the similar expression for $\Phi_{-\bar{\nu}}$ it follows that

$$\overline{a(\nu)} = a(-\bar{\nu}), \quad \overline{b(\nu)} = b(-\bar{\nu}). \tag{8.1}$$

If ν is real and non-zero, then Φ_ν and $\Phi_{-\nu}$ form a basis of $\mathcal{E}_{\nu^2}$ and from the asymptotic behavior of the (constant) Wronskian $[\Phi_\nu, \overline{\Phi}_{-\nu}]_t$ one reads off that

$$b(\nu)\, a(-\nu) - a(\nu)\, b(-\nu) = 2i\nu, \qquad (\nu \in \mathbb{R} \setminus \{0\}). \tag{8.2}$$

From (8.1) and (8.2) it follows that

$$\operatorname{Im}\ a(\nu)\overline{b(\nu)} = -\nu, \qquad (\nu \in \mathbb{R} \setminus \{0\}). \tag{8.3}$$

In particular, a and b do not vanish anywhere on $\mathbb{R} \setminus \{0\}$.

We can now determine the spectral matrix for this problem. Indeed, for $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\nu > 0,\ \nu^2 = \lambda$,

$$F(\lambda) = \begin{pmatrix} 1 & a(\nu) \\ 0 & b(\nu) \end{pmatrix},$$

so that

$$\operatorname{Im} M(\lambda) = -\operatorname{Im} \begin{pmatrix} a(\nu)b(\nu)^{-1} & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\operatorname{Im} \frac{a(\nu)}{b(\nu)} & 0 \\ 0 & 0 \end{pmatrix}$$

by (7.4). From this we conclude that the spectral matrix $P(\lambda)$ has zero entries except for the one in the upper left corner, which we denote by $\rho(\lambda)$. The second component of $\mathcal{F}f$ now plays no role in the Plancherel formula. Indeed, define

$$\mathcal{F}_1 f(\lambda) = \int_0^\infty f(t)\, s_1(\lambda, t)\, dt,$$

then we have the following.

Corollary 8.1 *$\mathcal{F}_1$ extends to an isometry from the space $L^2(\,]\,0, \infty\,[\,)$ onto $L^2(\mathbb{R}, d\rho)$. The spectral function ρ is given by*

$$\rho(\lambda) = -\frac{1}{\pi} \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \int_{[\delta, \lambda+\delta]+i\varepsilon} \operatorname{Im} \frac{a(\sqrt{\mu})}{b(\sqrt{\mu})}\, d\mu, \tag{8.4}$$

where the square root $\sqrt{\mu}$ with positive imaginary part should be taken.

Since a and b are holomorphic in the upper half plane, $a(\sqrt{\mu})\, b(\sqrt{\mu})^{-1}$ is meromorphic over the interval $]-\infty, 0\,[$, so that on $]-\infty, 0\,[$, the measure $d\rho$ is a countable sum of point measures. Indeed, let S be the (discrete) subset of zeros for a on the positive imaginary axis $i\,]\,0,\infty\,[$. Then

$$d\rho|_{\,]\,-\infty,0\,[} = \sum_{\sigma\in S} 2\mathrm{Res}_{\nu=\sigma}\,\frac{\nu\, a(\nu)}{b(\nu)}\cdot\delta_{[\sigma^2]}$$

On the other hand, for $\lambda>0$, if $\mu\to\lambda$, then the integrand of (8.4) tends to $\mathrm{Im}\, a(\sqrt{\lambda})b(\sqrt{\lambda})^{-1}$, with local uniformity in λ. In view of (8.3) it now follows that

$$d\rho(\lambda)|_{\,]\,0,\infty\,[} = -\frac{1}{\pi}\mathrm{Im}\,\frac{a(\sqrt{\lambda})}{b(\sqrt{\lambda})} = \frac{1}{\pi}\frac{\sqrt{\lambda}\,d\lambda}{|b(\sqrt{\lambda})|^2} = \frac{2}{\pi}\frac{\lambda\, d\sqrt{\lambda}}{|b(\sqrt{\lambda})|^2}.$$

Finally, if $s_1(0)$ is not square integrable at infinity, then $\rho_0 := d\rho(\{0\}) = 0$. On the other hand, if it is, then $\rho_0 := d\rho(\{0\})$ equals the squared L^2-norm of $s_1(0)$.

Finally, since $s_1(0,\lambda)$ is real valued for λ real, whereas $\overline{\Phi_{-\nu}} = \Phi_\nu$ for real ν, there exists a real analytic function $c:\mathbb{R}\setminus\{0\}\to\mathbb{C}$ such that

$$s_1(0,\nu^2) = c(\nu)\Phi_\nu + \overline{c(-\nu)}\Phi_{-\nu}$$

for all $\nu\in\mathbb{R}\setminus\{0\}$. This gives rise to the equations

$$\begin{cases} a(\nu)c(\nu) + a(-\nu)\overline{c(-\nu)} &= 1\\ b(\nu)c(\nu) + b(-\nu)\overline{c(-\nu)} &= 0.\end{cases}$$

Using (8.2) we now deduce that

$$c(\nu) = -b(\nu)/2i\nu, \qquad (\nu\in\mathbb{R}\setminus\{0\}). \tag{8.5}$$

Therefore,

$$d\rho(\lambda)|_{\,]\,0,\infty\,[} = \frac{1}{2\pi}\frac{d\sqrt{\lambda}}{|c(\sqrt{\lambda})|^2}.$$

We thus see that the principle formulated below (5.5) still holds in this setting.

9 Riemannian symmetric spaces

A Riemannian symmetric space is a connected Riemannian manifold X with the property that the local geodesic reflection at each point extends to a global isometry of X. Up to covering, each such space allows a decomposition into a product of three types of symmetric space. Those with zero sectional curvature (the Euclidean spaces), those with positive sectional curvature (among which the Euclidean spheres) and those with negative sectional curvature (among which the hyperbolic spaces). It follows from the work of E. Cartan, that the spaces of negative sectional curvature are precisely those given by $X = G/K$, where G is a connected real semisimple Lie group of non-compact type, with finite center, and where K is a maximal compact subgroup of G. The Killing form of G naturally induces a G-invariant Riemannian metric on G/K. The group K is the fixed point group of a Cartan involution θ of G; this involution induces the geodesic reflection in the origin $\bar{e} = eK$ of X.

A typical example of a symmetric space of this type is the space X of positive definite symmetric $n \times n$-matrices on which $G = \mathrm{SL}(n, \mathbb{R})$ acts by $(g, h) \mapsto ghg^{\mathrm{T}}$. The stabilizer of the identity matrix equals $\mathrm{SO}(n)$ and the associated Cartan involution $\theta : G \to G$ is given by $g \mapsto (g^{\mathrm{T}})^{-1}$. The geodesic reflection in the identity matrix I is given by $h \mapsto h^{-1}$.

In the general setting, the derivative of the Cartan involution at the identity element of G induces an involution θ_* of the Lie algebra $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ decomposes as a direct sum of vector spaces

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ_*, respectively. It can be shown that the map

$$(X, k) \mapsto \exp Xk, \quad \mathfrak{p} \times K \to G \tag{9.1}$$

is an analytic diffeomorphism onto G. In particular, this implies that the exponential map induces a diffeomorphism $\mathrm{Exp} : X \mapsto \exp XK$, $\mathfrak{p} \to G/K$. Let $\mathfrak{a}$ be a subspace of $\mathfrak{p}$, maximal subject to the condition that it is abelian for the Lie bracket of $\mathfrak{g}$. Every other such subspace is K-conjugate to $\mathfrak{a}$. The dimension r of $\mathfrak{a}$ is called the rank of the symmetric space G/K.

In the example $G = \mathrm{SL}(n, \mathbb{R})$, the Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ consists of all traceless $n \times n$-matrices, and θ_* is given by $X \mapsto -X^{\mathrm{T}}$. The Cartan decomposition (9.1) is given by the decomposition of a matrix in terms of a positive definite symmetric one times an orthogonal one. The algebra $\mathfrak{a}$ now consists of the traceless diagonal matrices, so that the rank of $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ equals $n - 1$. For $n = 2$ the space is isomorphic to the hyperbolic upper half plane, equipped with the action of $\mathrm{SL}(2, \mathbb{R})$ through fractional linear transformations.

By a result of Harish-Chandra, the algebra $\mathbb{D}(G/K)$ of G-invariant linear partial differential operators on G/K is a polynomial algebra of rank r. More precisely, let M be the centralizer of $\mathfrak{a}$ in K, and let

$W := N_K(\mathfrak{a})/M$, the normalizer modulo the centralizer of $\mathfrak{a}$ in K. As a subgroup of $\mathrm{GL}(\mathfrak{a})$, this group is the reflection group associated with the roots of $\mathfrak{a}$ in $\mathfrak{g}$. It is therefore called the Weyl group of the pair $(\mathfrak{g}, \mathfrak{a})$. There exists a canonical isomorphism γ from $\mathbb{D}(G/K)$ onto $P(\mathfrak{a}_{\mathbb{C}}^*)^W$, the algebra of W-invariants in the polynomial algebra of the complexified dual space $\mathfrak{a}_{\mathbb{C}}^*$ (equipped with the dualized Weyl group action). By a result of C. Chevalley, the algebra $P(\mathfrak{a}_{\mathbb{C}}^*)^W$ is known to be polynomial of rank r.

In the example $\mathrm{SL}(n, \mathbb{R})$, the Weyl group is given by the natural action of the permutation group S_n on the space $\mathfrak{a}$ of traceless diagonal matrices. Here the algebra $P(\mathfrak{a}_{\mathbb{C}}^*)^W$ corresponds to the algebra of S_n-invariants in $\mathbb{C}[T_1, \dots, T_n]/(T_1 + \cdots + T_n)$, which is of course well known to be a polynomial algebra of $n-1$ generators of its own right. We note that in the case of rank 1, the algebra $\mathbb{D}(G/K)$ consists of all polynomials in the Laplace-Beltrami operator.

In the papers [12],[13], Harish-Chandra created a beautiful theory of harmonic analysis for left K-invariant functions on the symmetric space G/K, culminating in a Plancherel formula for $L^2(G/K)^K$, the space of left-K-invariant functions on G/K, square integrable with respect to the Riemannian volume form. We will now give a brief outline of the main results.

For $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, we consider the following system of simultaneous eigenequations on G/K :

$$Df = \gamma(D, i\nu)f, \qquad (D \in \mathbb{D}(G/K)). \tag{9.2}$$

For $r = 1$, this system is equivalent to a single eigenequation for the Laplace operator. Each eigenfunction is analytic, by ellipticity of the Laplace operator. The space of K-invariant functions satisfying (9.2) is one dimensional and spanned by the so-called elementary spherical function φ_ν, normalized by $\varphi_\nu(eK) = 1$. This function can be constructed as a matrix coefficient $x \mapsto \langle 1_K\,,\, \pi_\nu(x) 1_K \rangle$, with 1_K a K-fixed vector in a suitable continuous representation of G in an infinite dimensional Hilbert space, obtained by the process of induction. By Weyl invariance of the polynomials $\gamma(D)$ it follows that $\varphi_{w\nu} = \varphi_\nu$, for all $w \in W$.

In terms of the elementary spherical functions one may define the so-called Fourier transform of a function $f \in C_c^\infty(G/K)^K$ by

$$\mathcal{F}_{G/K} f(\nu) = \int_{G/K} f(x)\, \varphi_{-\nu}(x) dx, \qquad (\nu \in \mathfrak{a}^*),$$

with dx the G-invariant volume measure on G/K.

By analyzing the system of differential equations (9.2) it is possible to obtain rather detailed information on the asymptotic behavior of the elementary spherical functions φ_ν towards infinity. It can be shown that the map $K/M \times \mathfrak{a} \to G/K$, $(kM, X) \mapsto k \exp XK$ is surjective. For obvious reasons, the associated decomposition

$$G/K = K \exp \mathfrak{a} \cdot \bar{e} \tag{9.3}$$

is called the polar decomposition of G/K. In it, the $\mathfrak{a}$-part of an element is uniquely determined modulo the action of W. Let $\mathfrak{a}^+$ be a choice of positive Weyl chamber relative to W, then it follows that $G/K = K \exp \overline{\mathfrak{a}^+} \cdot \bar{e}$ with uniquely determined $\overline{\mathfrak{a}^+}$-part. Moreover, the map $K/M \times \mathfrak{a}^+ \to G/K$ is an analytic diffeomorphism onto an open dense subset of G/K.

Accordingly, each elementary spherical function is completely determined by its restriction to $A^+ := \exp(\mathfrak{a}^+)$. Moreover, the restricted function $\varphi_\nu|_{A^+}$ satisfies the system of equations arising from (9.2) by taking radial parts with respect to the polar decomposition (9.3). Using a characterization of $\mathbb{D}(G/K)$ in terms of the universal algebra $U(\mathfrak{g})$, Harish-Chandra was able to analyze these radial differential equations in great detail. This allowed him to show that, for generic $\nu \in \mathbb{C}$, the behavior of the function φ_ν towards infinity is described by

$$\varphi_\nu(k \exp X) = \sum_{w \in W} c(w\nu) e^{(iw\nu - \rho)(X)}[1 + R_{w\nu}(X)], \tag{9.4}$$

for $k \in K$ and $X \in \mathfrak{a}^+$. Here $\rho \in \mathfrak{a}^*$ is half the sum of the positive roots, counted with multiplicities, $c(\nu)$, the so-called c-function, is a certain meromorphic function of the parameter $\nu \in \mathfrak{a}^*_{\mathbb{C}}$, and $R_\nu(X)$ is a certain analytic function of X, depending meromorphically on the parameter ν. Moreover, the asymptotic behavior of R_ν is described by

$$R_\nu(tX) = \mathcal{O}(e^{-tm(X)}), \quad (X \in \mathfrak{a}^+,\, t \to \infty), \tag{9.5}$$

with $m(X)$ a positive constant, depending on X in a locally uniform way. Each of the summands in (9.4) is an eigenfunction of the radial system of differential equations of its own right.

Theorem 9.1 (Harish-Chandra's Plancherel formula). *The function c has no zeros on $\mathfrak{a}^*$. Moreover, let*

$$dm(\nu) := \frac{d\nu}{|c(\nu)|^2}, \tag{9.6}$$

with $d\nu$ a suitable normalization of Lebesgue measure on $\mathfrak{a}^*$ (see further *down). Then $dm(\nu)$ is Weyl-group invariant and the Fourier transform $\mathcal{F}_{G/K}$ extends to an isometry from $L^2(G/K)^K$ onto $L^2(\mathfrak{a}^*, dm(\nu))^W$.*

In footnote 3), p. 242, to the introduction of his paper [12], Harish-Chandra mentions: 'This is reminiscent of a result of Weyl [[29], p. 266] on ordinary differential equations.' It seems that Harish-Chandra was actually very inspired by Weyl's paper. In [5], p. 38, A. Borel writes: '[...] less obviously maybe, Weyl was also of help via his work on differential equations [29], which gave Harish-Chandra a crucial hint in his quest for an explicit form of the Plancherel measure. [...] It was the reading of [29] which suggested to Harish-Chandra that the measure should be the inverse of the square modulus of a function in λ describing the asymptotic behavior of the eigenfunctions [...] and I remember well from seminar lectures and conversations that he never lost sight of that principle, which is confirmed by his results in the general case as well.'

It is the purpose of the rest of this section to show that for the rank one case Theorem 9.1 is in fact a rather direct consequence of Kodaira's generalization of Weyl's result, described in Section 8.

Before we proceed it should be mentioned that in [12] and [13] Theorem 9.1 was completely proved for spaces of rank 1. Moreover, for these spaces the c-function was explicitly determined as a certain quotient of Gamma factors.

For spaces of arbitrary rank Theorem 9.1 was proved modulo two conjectures. The first of these concerned the injectivity of the Fourier transform and the second certain estimates for the c-function. The first conjecture was proved by Harish-Chandra himself, in his work on the so-called discrete series of representations for G, [14]. The validity of the second conjecture followed from the work of S. Gindikin and F. Karpelevic, [11], where a product decomposition of the c-function in terms of rank one c-functions was established. Simpler proofs of Theorem 9.1 were later found through the contributions of [20], [10], [26].

The precise normalization of the Lebesgue measure $d\nu$ may be given as follows. The polar decomposition (9.3) gives rise to an integral formula

$$\int_{G/K} f(x)\, dx = \int_K \int_{\mathfrak{a}^+} f(k \exp X) J(X)\, dX\, dk, \tag{9.7}$$

with dk normalized Haar measure on K, J a suitable Jacobian, and dX suitably normalized Lebesgue measure on $\mathfrak{a}$. The Jacobian J and the measure dX are uniquely determined by the above formula and the requirement that $J(tX)$ behaves asymptotically as $e^{t2\rho(X)}$, for $X \in \mathfrak{a}^+$

and $t \to \infty$. Let $d\xi$ denote the dual Lebesgue measure on $\mathfrak{a}^*$. Then

$$d\nu = \frac{1}{|W|} \frac{d\xi}{(2\pi)^n},$$

with $|W|$ the number of elements of the Weyl group.

We now turn to the setting of a space of rank 1. A typical example of such a space is the n-dimensional hyperbolic space X_n, which may be realized as the submanifold of $\mathbb{R}^{n+1}$ given by the equation $x_1^2 - (x_2^2 + \cdots + x_n^2) = 1,\ x_1 > 0$. Its Riemannian metric is induced by the indefinite standard inner product of signature $(1, n)$ on $\mathbb{R}^{n+1}$. As a homogeneous space $X_n \simeq \mathrm{SO}(1, n)/\mathrm{SO}(n)$.

More generally, as $\mathfrak{a}$ is one dimensional, all roots in R are proportional. Let α be the simple root associated with the choice of positive chamber $\mathfrak{a}^+$. Then $-\alpha$ is a root as well, and possibly $\pm 2\alpha$ are roots as well. No other multiples of α occur. We fix the unique element $H \in \mathfrak{a}$ with $\alpha(H) = 1$.

Via the map $tH \mapsto t$ we identify $\mathfrak{a}$ with $\mathbb{R}$; likewise, via the map $t\alpha \mapsto t$ we identify $\mathfrak{a}^*$ with $\mathbb{R} \simeq \mathbb{R}^*$. Then $\mathfrak{a}^+ =]\,0, \infty\,[$. Rescaling the Riemannian metric if necessary we may as well assume that under these identifications, both dX and $d\xi$ correspond to the standard Lebesgue measure on $\mathbb{R}$.

Let m_1, m_2 denote the root multiplicities of $\alpha, 2\alpha$, i.e., m_j is the dimension of the eigenspace of $\mathrm{ad}(H)$ in $\mathfrak{g}$ with eigenvalue j. Then with the above identifications,

$$\rho = \frac{1}{2}(m_1 + 2m_2).$$

The Laplace operator Δ satisfies $\gamma(\Delta, i\nu) = (-\|\nu\|^2 - \|\rho\|^2)$ with $\|\cdot\|$ the norm on $\mathfrak{a}^*$ dual to the norm on $\mathfrak{a}$ induced by the Riemannian inner product on $\mathfrak{g}/\mathfrak{k} \simeq \mathfrak{p}$. Multiplying Δ with a suitable negative constant, we obtain an operator L_0 with $\gamma(L_0, i\nu) = \nu^2 + \rho^2$. Let $\widetilde{L} = L_0 - \rho^2$, then $\tilde{L}\varphi_\nu = \nu^2 \varphi_\nu$.

The Jacobian J mentioned above is given by the formula

$$J(t) = (e^t - e^{-t})^{m_1} (e^{2t} - e^{-2t})^{m_2}.$$

Let $L := J^{1/2} \circ \mathrm{rad}\,(\widetilde{L}) \circ J^{-1/2}$ be the conjugate of the radial part of $\widetilde{L}$ by multiplication with $J^{1/2}$. Put

$$s(\nu, t) := J^{1/2}(t)\, \varphi_\nu(\exp tH). \tag{9.8}$$

Then the system of equations (9.2) is equivalent to the single eigenequation

$$Ls(\nu, \cdot\,) = \nu^2 s(\nu, \cdot\,)$$

By a straightforward calculation, see [10], p. 156, it follows that

$$L = -\frac{d^2}{dt^2} + q(t), \qquad q(t) = \frac{1}{2}J^{-1}\frac{d^2}{dt^2}J - \frac{1}{4}J^{-2}(\frac{d}{dt}J)^2 - \rho^2.$$

By using the Taylor series of $J(t)$ at 0 we see that there exists a real analytic function V on $\mathbb{R}$ such that

$$q(t) = m(m+1)t^{-2} + V(t), \qquad (t > 0),$$

where

$$m = \frac{1}{2}(m_1 + m_2) - 1 \geq -\frac{1}{2}.$$

On the other hand, at infinity, $J(t)$ equals $e^{2t\rho}$ times a power series in terms of powers of e^{-2t} with constant term 1. From this we see that $q(t) = \mathcal{O}(e^{-2t})$, so that $V(t) = \mathcal{O}(t^{-2})$ as $t \to \infty$. It follows that our operator L satisfies all requirements of Section 8.

We now observe that $\varphi_\nu(e) = 1$ and $J(t)^{1/2} \sim 2^\rho t^{m+1}$ $(t \to 0)$. Let $c_0 := 2^\rho$ and let $s_1(\lambda, t)$ be defined as in Section 8, for $\lambda \in \mathbb{C}$. Then it follows that $s(\nu, t) = s_1(\nu^2, t)$ for all $\nu \in \mathbb{C}$. Moreover, it follows from (9.4) that the function Φ_ν of Section 8 is given by

$$\Phi_\nu(t) = e^{-t\rho}J(t)^{1/2}\, c(\nu)\, e^{i\nu}(1 + R_\nu(t)).$$

In particular, it depends meromorphically on the parameter ν. From this it follows that the functions $\nu \mapsto a(\nu), b(\nu)$ are meromorphic. By analytic continuation it now follows that the identity (8.5) extends to an identity of meromorphic functions. From its explicitly known form as a quotient of Gamma factors, it follows that the function $\nu \mapsto c(\nu)$ has no zeros on $i\,]\,0, \infty\,[$. Moreover, it has a zero of order 1 at 0. Using (8.5) we now see that b has no zeros on $i[0, \infty\,]$, so that the spectral measure $d\rho$ has no discrete part. Hence,

$$d\rho(\lambda) = \left.\frac{d\sqrt{\lambda}}{2\pi|c(\sqrt{\lambda})|^2}\right|_{]\,0,\infty\,[}.$$

Let $\mathcal{F}_1$ be the Fourier transform defined in terms of $s_1(\lambda, t) = s(\sqrt{\lambda}, t)$, see Section 8. Then it follows by application of (9.7) and (9.8) that

$$\mathcal{F}_{G/K}f(\nu) = \mathcal{F}_1(J^{1/2}f \circ \exp|_{\mathfrak{a}^+})(\nu^2), \quad (\nu \in \mathbb{R}), \tag{9.9}$$

for every $f \in C_c(G/K)$.

By Corollary 8.1 the Fourier transform $\mathcal{F}_1$ is an isometry from the space $L^2(\mathfrak{a}^+, dX)$ onto the space $L^2(\,]\,0,\infty\,[\,, d\rho)$. Moreover, the map $f \mapsto J^{1/2} f \circ \exp|_{\mathfrak{a}^+}$ is an isometry from $L^2(G/K)^K$ onto $L^2(\mathfrak{a}^+, dX)$.

Finally, since $W = \{\pm I\}$, whereas the function $\nu \mapsto |c(\nu)|^2$ is even by (8.5) and (8.1), pull-back by the map $\nu \mapsto \nu^2$ defines an isometry

$$L^2(\,]\,0,\infty\,[\,, d\rho) \xrightarrow{\simeq} L^2(\mathfrak{a}^*, \frac{1}{2}\frac{d\xi}{2\pi|c(\nu)|^2})^W. \tag{9.10}$$

By (9.9) $\mathcal{F}_{G/K}$ is the composition of the three mentioned isometries. The assertion of Theorem 9.1 follows.

10 Analysis on groups and symmetric spaces

After his work on the Riemannian symmetric spaces, Harish-Chandra continued to work on a theory of harmonic analysis for real semisimple Lie groups in the 1960's. His objective was to obtain an explicit Plancherel decomposition for $L^2(G)$, the space of square integrable functions with respect to a fixed choice of (bi-invariant) Haar measure on G.

In the case of a compact group, the Plancherel formula is described in terms of representation theory and consists of the Peter–Weyl decomposition combined with the Schur-orthogonality relations.

In the more general case of a real semisimple Lie group, the situation is far more complicated. If G is simple and non-compact, then the non-trivial irreducible unitary representations of G are infinite dimensional. Moreover, there is a mixture of discrete and continuous spectrum.

An irreducible unitary representation is said to be of the discrete series if it contributes discretely to $L^2(G)$, i.e., it is embeddable as a closed invariant subspace for the left regular representation. Equivalently, this means that its matrix coefficients are square integrable. An irreducible unitary representation has a character, which is naturally defined as a conjugation invariant distribution on G. A deep theorem of Harish-Chandra in the beginning of the 1960's asserts that in fact all such characters are locally integrable. Moreover, they are analytic on the open dense subset of regular elements.

In [14] and [15], Harish-Chandra gave a complete classification of the discrete series. First of all, G has discrete series if and only if it has a compact Cartan subgroup. Moreover, the representations of the discrete series are completely determined by the restriction of their characters to this compact Cartan subgroup. Harish-Chandra achieved their classification and established a character formula on the compact Cartan which shows remarkable resemblance with Weyl's character formula.

In the early 1970's, Harish-Chandra, [16], [17], [18], completed his work on the Plancherel decomposition. The orthocomplement of the discrete part of $L^2(G)$ is decomposed in terms of representations of the so-called generalized principal series. These are induced representations of the form

$$\pi_{P,\xi,\lambda} = \mathrm{Ind}_P^G(\xi \otimes e^{i\lambda} \otimes 1),$$

where P is a (cuspidal) parabolic subgroup of G, with a so-called Langlands decomposition $P = M_P A_P N_P$. Moreover, ξ is a discrete series representation of M_P and $e^{i\lambda}$ is a unitary character of the vectorial group A_P. The space $L^2(G)$ splits into a finite orthogonal direct sum of closed subspaces $L^2(G)_{[P]}$, each summand corresponding to an equivalence class of parabolic subgroups with K-conjugate A_P-part. Here G counts for a parabolic subgroup, and $L^2(G)_{[G]}$ denotes the discrete part of $L^2(G)$.

Each summand $L^2(G)_{[P]}$ decomposes discretely into a countable orthogonal direct sum of spaces $L^2(G)_{[P],\xi}$ parametrized by (equivalence classes of) discrete series representations of M_P. Finally, each of the spaces $L^2(G)_{[P],\xi}$ has a continuous decomposition parametrized by $\lambda \in \mathfrak{a}_P^*$. Harish-Chandra achieved this continuous decomposition by reduction to the space of functions transforming finitely under the action of the maximal compact subgroup K.

Let δ_L, δ_R be two irreducible representations of K and let $L^2(G)_{[P],\xi,\delta}$ be the part of $L^2(G)_{[P],\xi}$ consisting of bi-K-finite functions of left K-type δ_L and right K-type δ_R. The decomposition of this space is described in terms of *Eisenstein integrals.* These are essentially $K \times K$-finite matrix coefficients of type (δ_L, δ_R) of the induced representation involved. The Eisenstein integrals $E([P], \xi, \lambda, \psi)$ are functions on G which depend analytically on the parameter $\lambda \in \mathfrak{a}^*$. In addition, they depend linearly on a certain parameter ψ, which ranges over a certain finite dimensional Hilbert space $\mathcal{A}_2(M_P, \xi, \delta)$ of functions $M \times K \times K \to \mathbb{C}$. The Eisenstein integrals satisfy eigenequations coming from the bi-G-invariant differential operators on G. As in the previous section these equations can be analyzed in detail, and it can be shown that the integrals behave asymptotically like

$$\begin{aligned} &E([P], \xi, \lambda, \psi)(k_1\, m \exp X\, k_2) \\ &\qquad \sim \sum_{w \in W(\mathfrak{a}_Q|\mathfrak{a}_P)} e^{(iw\lambda - \rho_Q)(X)}\, [c_{Q|P,\xi}(w, \lambda)\psi](m, k_1, k_2) \end{aligned}$$

for $m \in M_Q$, $k_1, k_2 \in K$, and as X tends to infinity in $\mathfrak{a}_Q^+$; here Q is a parabolic subgroup in the same equivalence class as P and $W(\mathfrak{a}_Q|\mathfrak{a}_P)$ denotes the finite set of isomorphisms $\mathfrak{a}_P \to \mathfrak{a}_Q$ induced by the adjoint action of K. Each coefficient $c_{Q|P,\xi}(w, \lambda)$ is an isomorphism from the finite dimensional Hilbert space $\mathcal{A}_2(M_P, \xi, \delta)$ onto the similar space $\mathcal{A}_2(M_Q, w\xi, \delta)$. It can be shown that

$$c_{Q|P,\xi}(w, \lambda)^* c_{Q|P,\xi}(w, \lambda) = \eta(P, \xi, \lambda)\ I$$

with $\eta(P, \xi, \lambda)$ a strictly positive scalar, independent of Q, w, δ and depending real analytically on $\lambda \in \mathfrak{a}_P^*$. Finally, the measure for the Plancherel decomposition of $L^2(G)_{[P],\xi}$ is given by

$$\frac{d\lambda}{\eta(P, \xi, \lambda)}. \tag{10.1}$$

In this sense, Weyl's principle is valid for all continuous spectral parameters in the Plancherel decomposition for G.

In the 1980's and 1990's, much progress was made in harmonic analysis on general semisimple symmetric spaces. These are pseudo-Riemannian symmetric spaces of the form G/H, with G a real semisimple Lie group and H (an open subgroup of) the group of fixed points for an involution σ of G. This class of spaces contains both the Riemannian symmetric spaces and the semisimple groups. Indeed the group G is a homogeneous space for the action of $G \times G$ given by $(x, y) \cdot g = xgy^{-1}$. The stabilizer of the identity element e_G equals the diagonal H of $G \times G$, which is the group of fixed points for the involution $\sigma : (x, y) \mapsto (y, x)$. As a decomposition for the left times right regular action of $G \times G$ on $L^2(G)$ the Plancherel decomposition becomes multiplicity free. This is analogous to what happens for the Peter-Weyl decomposition for compact groups.

Another interesting class of semisimple symmetric spaces is formed by the pseudo-Riemannian hyperbolic spaces $\mathrm{SO}_e(p, q)/\mathrm{SO}_e(p-1, q)$, $p > 1$.

For general semisimple symmetric spaces, M. Flensted-Jensen, [9], gave the first construction of discrete series assuming the analogue of Harish-Chandra's rank condition. The full classification of the discrete series was then given by T. Oshima and T. Matsuki [25].

In [2], E.P. van den Ban and H. Schlichtkrull gave a description of the most continuous part of the Plancherel decomposition. Here, a new phenomenon is that the Plancherel decomposition may have finite multiplicities. Nevertheless, the multiplicities can be parametrized in such a way that Weyl's principle generalizes to this context. Then, P. Delorme, partly in collaboration with J. Carmona, determined the full Plancherel decomposition for G/H, [6], [8]. Around the same time this was also achieved by E.P. van den Ban and H. Schlichtkrull, [3],[4], with a completely different proof. In all these works, the appropriate analogue of (10.1) goes through. For more information, we refer the reader to the survey articles in [1].

Parallel to the developments sketched above, G. Heckman and E. Opdam [19] developed a theory of hypergeometric functions, generalizing the elementary spherical functions of the Riemannian symmetric spaces. For these spaces, the algebra of radial components of invariant differential operators is entirely determined by a root system and root multiplicities. The generalization is obtained by allowing these multiplicities to vary in a continuous fashion. In the associated Plancherel decomposition, established by Opdam, [24], Weyl's principle holds through the analogue of (9.6).

11 Appendix: circles in $\mathbb{P}^1(\mathbb{C})$

If V is a two dimensional complex linear space, then by $\mathbb{P}(V)$ we denote the 1-dimensional projective space of lines $\mathbb{C}v$, with $v \in V \setminus \{0\}$. In a natural way we will identify subsets of $\mathbb{P}(V)$ with $\mathbb{C}$-homogeneous subsets of V containing 0. In particular, the empty set is identified with $\{0\}$. The group $\mathrm{GL}(V)$ of invertible complex linear transformations of V naturally acts on $\mathbb{P}(V)$.

Let β be Hermitian form on V, i.e., $\beta : V \times V \to \mathbb{C}$ is linear in the first and conjugate linear in the second component, and $\beta(v,w) = \overline{\beta(w,v)}$ for all $v, w \in V$. By symmetry, $\beta(v,v) \in \mathbb{R}$ for all $v \in V$. We denote by $\mathcal{B}$ the space of Hermitian forms β on V for which the function $v \mapsto \beta(v,v)$ has image $\mathbb{R}$. Equivalently, this means that there exists a basis v_1, v_2 of V such that $\beta(v_1,v_1) = 1$ and $\beta(v_2,v_2) = -1$. It follows from this that the group $\mathrm{GL}(V)$ acts transitively on $\mathcal{B}$ by $g \cdot \beta(v,w) = \beta(g^{-1}v, g^{-1}w)$.

We note that for any Hermitian form β on V the map $v \mapsto \beta(v,\,\cdot\,)$ induces a linear map from V to the conjugate linear dual space $\overline{V}^*$. This map is an isomorphism if and only if β is non-degenerate. Let γ be any choice of positive definite Hermitian form on V. Then $H_\beta = \gamma^{-1} \circ \beta$ is a linear endomorphism of V; from $\beta(v,w) = \gamma(H_\beta v, w)$ for $v, w \in V$ we see that H_β is symmetric with respect to the inner product γ. The condition that $\beta \in \mathcal{B}$ is equivalent to the condition that H_β has both a strictly positive and a strictly negative eigenvalue, which in turn is equivalent to the condition that $\det H_\beta < 0$. For obvious reasons we will call $\mathcal{B}$ the space of Hermitian forms of signature $(1,1)$.

By a circle in $\mathbb{P}(V)$ we mean a set of the form

$$C_\beta := \{v \in V \mid \beta(v,v) = 0\}$$

with $\beta \in \mathcal{B}$. For $g \in \mathrm{GL}(V)$ we have $g(C_\beta) = C_{g\cdot\beta}$ so that the natural action of $\mathrm{GL}(V)$ on the collection of circles is transitive.

We now turn to the case of $\mathbb{C}^2$ equipped with the standard Hermitian inner product. Accordingly, any form $\beta \in \mathcal{B}$ is represented by a unique Hermitian matrix H of strictly negative determinant. We will use the standard embedding $\mathbb{C} \hookrightarrow \mathbb{P}^1(\mathbb{C}) := \mathbb{P}(\mathbb{C}^2)$ given by $z \mapsto \mathbb{C}(z,1)$. The complement of the image of this embedding consists of the single point $\infty_\mathbb{C} := \mathbb{C}(1,0)$. The inverse map $\chi : \mathbb{P}^1(\mathbb{C}) \setminus \{\infty_\mathbb{C}\} \to \mathbb{C}$ is called the standard affine chart. It is straightforwardly verified that $\infty_\mathbb{C}$ belongs to C_β if and only if the entry H_{11} equals zero. In this case the intersection of C_β with the standard affine chart is given by $2\mathrm{Re}\,(H_{21}z) = -H_{22}$, which is the straight line $-H_{21}^{-1}(\frac{1}{2}H_{22} + i\mathbb{R})$. In particular, the form

$$i[z,w] = i(z_1\overline{w}_2 - z_2\overline{w}_1) \tag{11.1}$$

is represented by the Hermitian matrix iJ (see (2.4)), and the associated circle in $\mathbb{P}^1(\mathbb{C})$ equals the closure $\mathbb{P}^1(\mathbb{R}) := \mathbb{C}\mathbb{R}^2 = \mathbb{R} \cup \{\infty_\mathbb{C}\}$ of the real line.

In the remaining case the circle C_β is completely contained in the standard affine chart, and in the affine coordinate it equals a circle with respect to the standard Euclidean metric on $\mathbb{C} \simeq \mathbb{R}^2$. The radius r and the center α are given by

$$r^2 = -\frac{\det H}{|H_{11}|^2}, \qquad \alpha = -\frac{H_{12}}{H_{11}}. \tag{11.2}$$

The preimage under χ of the interior of this circle is the subset of $\mathbb{P}^1(\mathbb{C})$ given by the inequality

$$\mathrm{sign}(H_{11})\beta(z,z) < 0.$$

We note that all circles and straight lines in $\mathbb{C}$ are representable in the above fashion. In the standard affine coordinate, the action of the group $\mathrm{GL}(2,\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$ is represented by the action through fractional linear transformations on $\mathbb{C}$. Accordingly, we retrieve the well-known fact that this action preserves the set of circles and straight lines.

More generally, let v_1, v_2 be a complex basis of V. Then the natural map $z \mapsto z_1v_1 + z_2v_2$ induces a diffeomorphism $\mathrm{v} : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}(V)$. The map $\chi_v := \chi \circ \mathrm{v}^{-1} : \mathbb{P}(V) \setminus \mathbb{C}v_1 \to \mathbb{C}$ is said to be the affine chart determined by v_1, v_2. Note that $z = \chi_v(\mathbb{C}(zv_1 + v_2))$, for $z \in \mathbb{C}$. The general linear group $\mathrm{GL}(V)$ acts on the set of affine charts by $(g, \psi) \mapsto \psi \circ g^{-1}$, so that $g \cdot \chi_v = \chi_{gv}$. Clearly, the action is transitive. It follows that the transition map between any pair of affine charts is given by a fractional linear transformation.

From the above considerations it follows that a circle C in $\mathbb{P}(V)$ corresponds to a circle in the affine chart χ_v if and only if $\mathbb{C}v_1$ does not lie on C. Otherwise, the circle is represented by a straight line in χ_v.

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