Harmonic analysis of non-Riemannian symmetric spaces

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Mini-course

Methods in representation theory and operator algebras

CIRM 6–10 January, 2025.

Lecture 1: Basic structure

Setting

G real semisimple Lie gp, connected, $\# Z(G) < \infty$

σ involution of G, i.e. $\sigma \in \mathrm{Aut}(G)$, $\sigma^2 = \mathrm{id}_G$

$G^\sigma := \{ g \in G \mid \sigma g = g \} < G$, closed sub gp

$H < G^\sigma$ open subgroup of G^σ ($\Leftrightarrow (G^\sigma)_e < H < G^\sigma$)

$X = G/H$: semisimple symmetric space

$\mathfrak{g} = \mathrm{Lie}(G)$, $\sigma_* := d\sigma(e) : \mathfrak{g} \to \mathfrak{g}$ inf invol^n

abbr: σ

Geometry

$\mathfrak{g} = \underset{+1}{\mathfrak{h}} \oplus \underset{-1}{\mathfrak{q}}$ eigenspaces for. σ

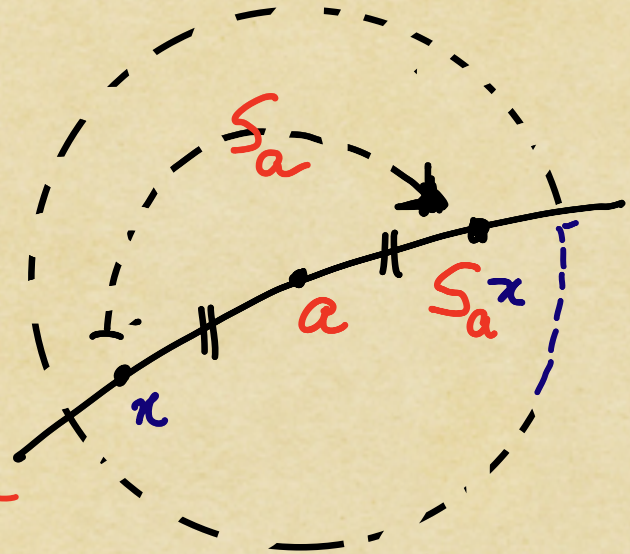
Exerc: $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$, $[\mathfrak{q}.\mathfrak{q}] \subset \mathfrak{h}$. $[\mathfrak{h}.\mathfrak{q}] \subset \mathfrak{q}$

$T_e(X) \simeq \mathfrak{g}/\mathfrak{h} \xleftarrow{\simeq} \mathfrak{q}$, B Killing form on $\mathfrak{g}$

- $B|_{\mathfrak{q}\times\mathfrak{q}} \rightsquigarrow$ G-inv't pseudo-Riemannian str on G/H

- $\forall a \in X$ S_a local geodesic reflection in a extends to global isometry $X \to X$.

make X a ps-Riemannian symm space



Examples

- Riemannian case

  σ a Cartan involution

  $(B|_{\mathfrak{h}\times\mathfrak{h}} > 0, \ B|_{\mathfrak{q}\times\mathfrak{q}} > 0)$

  $H = K, \quad \mathfrak{q} = \mathfrak{p}.$

- group case

  $G = {}'G \times {}'G, \quad H = \mathrm{diag}({}'G \times {}'G)$

  $G \curvearrowright {}'G, \quad (x,y)\cdot g = xgy^{-1}$

  $H = G_e; \quad {}'G \simeq G/H$

- $G = SL(p+q, \mathbb{R})$ $\quad H = S(GL(p) \times GL(q)) \quad \sigma : g \mapsto J g J$

  $$J = J_{p,q} = \begin{array}{cc} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} & \begin{array}{c} p \\ q \end{array} \end{array}$$

• Hyperbolic case $\quad G = SO_e(p,q), \quad H = SO_e(p-1,q)$

$$\sigma : \quad g \longmapsto J_{p-1}\, g\, J_{p-1} .$$

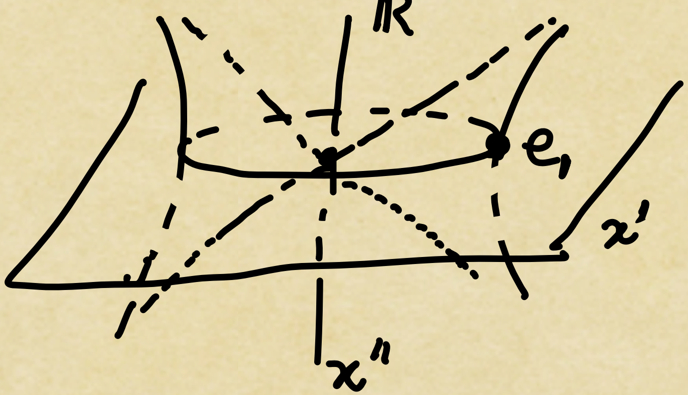
Model $\quad \mathbb{R}^p \times \mathbb{R}^q \ni x = (x', x'')$

metric $\quad \langle x, y \rangle := \langle x', y' \rangle_{\mathbb{R}^p} - \langle x'', y'' \rangle_{\mathbb{R}^q}$ on $\mathbb{R}^{p+q}$

invariant under $G = SO_e(p,q) \circlearrowright \mathbb{R}^{p+q}$

$X := \{ x \in \mathbb{R}^{p+q} \mid \langle x, x \rangle = 1 \}$

equipped with restr. metric



$G \circlearrowright X$ transitively

$G_{e_1} = SO_e(p-1,q) = H$

signature restr metric $(\underset{+1}{p-1}, \underset{-1}{q})$

($p \geq 2$: ps. Riemannian, $p = 1$: Riemannian

General setting

G, σ, $G_e^\sigma < H < G^\sigma$, $\underset{\sigma}{\mathfrak{g}} = \underset{+1}{\mathfrak{h}} \oplus \underset{-1}{\mathfrak{q}}$, $Ad(H): \mathfrak{q} \to \mathfrak{q}$

Lemma $\exists \left(\begin{array}{l} \theta : \mathfrak{g} \to \mathfrak{g} \\ \text{Cartan inv} \end{array} \right)$: $\theta \circ \sigma = \sigma \circ \theta$.

($\Rightarrow$ $\theta\sigma$ is invol^{n})

Cor: $\underset{\sigma}{\mathfrak{g}} = \underset{+1}{\mathfrak{h}} \oplus \underset{-1}{\mathfrak{q}} = \underset{\theta \; +1,\,-1}{\mathfrak{k} \oplus \mathfrak{p}} = \underbrace{\mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{p} \cap \mathfrak{q}}_{\mathfrak{g}_+} \oplus \underbrace{\mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{p} \cap \mathfrak{h}}_{\mathfrak{g}_-}$. $\sigma\theta$

Thm: The map $K \times (\mathfrak{p} \cap \mathfrak{q}) \times (\mathfrak{p} \cap \mathfrak{h}) \longrightarrow G$,

$(k, X, Y) \longmapsto k \exp X \exp Y$ is a diffeo onto

Cor: The map $K \times (\mathfrak{p} \cap \mathfrak{q}) \longrightarrow G$, $(k, X) \longmapsto k \exp X$ induces

$$G/H \simeq K \times_{K \cap H} (\mathfrak{p} \cap \mathfrak{q})$$

vector bundle over $K/K \cap H$ with fiber $\mathfrak{p} \cap \mathfrak{q}$

Pf: Exerc

Special case: $\sigma = \Theta$: $G/K \simeq K \times_K \mathfrak{p} \simeq \{*\} \times \mathfrak{p}$.

Exerc. $\mathfrak{sl}(2, \mathbb{R}) = \mathbb{R}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbb{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Show that $SL(2, \mathbb{R}) / SO(1,1) \simeq SO(2) \times \mathbb{R}$ (as mfds).

Def Cartan subspace of $\mathfrak{q}$ is a subspace $\mathfrak{b} \subset \mathfrak{q}$ which is maximal subject to conditions

1) $\mathfrak{b}$ abelian

2) $\forall X \in \mathfrak{b}$ $\operatorname{ad} X : \mathfrak{g} \to \mathfrak{g}$ is semisimple

Exerc. check what this means for the group case

Thm There are finitely many $\operatorname{Ad}(H)$ — conjugacy classes of Cartan subspaces of $\mathfrak{q}$. They all have same dimension, called the rank of G/H.

Next: the analogue of $\mathfrak{a} \subset \mathfrak{p}$

Fix: $\mathfrak{a}_{\mathfrak{q}} \subset \mathfrak{p} \cap \mathfrak{q}$ maximal abelian subspace

Lemma $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}_{\mathfrak{q}}) = \{ \alpha \in \mathfrak{a}_{\mathfrak{q}}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0 \}$ is a possibly non-reduced root system.

Fix: Σ^+ positive system, Δ simple roots

Def $W = W(\mathfrak{a}_{\mathfrak{q}})$ associated Weyl gp.

Lemma $N_K(\mathfrak{a}_{\mathfrak{q}}) \longrightarrow GL(\mathfrak{a}_{\mathfrak{q}})$, $k \longmapsto \mathrm{Ad}(k)|_{\mathfrak{a}_{\mathfrak{q}}}$ induces a gp isom $N_K(\mathfrak{a}_{\mathfrak{q}}) / Z_K(\mathfrak{a}_{\mathfrak{q}}) \xrightarrow{\simeq} W$.

Def $W_{K \cap H} = \mathrm{image}\left(N_{K \cap H}(\mathfrak{a}_{\mathfrak{q}})\right)$.

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Rem. $\sigma\theta = I$ on $\mathfrak{a}_q$. So $\forall_{\alpha\in\Sigma}$: $\sigma\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_\alpha$

Put $\mathfrak{g}_{\alpha\pm} := \mathfrak{g}_\alpha \cap \mathfrak{g}_\pm$, $m_\alpha^\pm = \dim \mathfrak{g}_{\alpha\pm}$.

$G_+ := G^{\sigma\theta}$ is reductive, Cartan deco $= (K\cap H)\exp(\mathfrak{p}\cap\mathfrak{q})$

$\Sigma_+ := \{\alpha\in\Sigma \mid \mathfrak{g}_{\alpha,+} \neq 0\}$ root system of $(\mathfrak{g}_+, \mathfrak{a}_q)$

$\Sigma_+^+ = \Sigma_+ \cap \Sigma^+$, $W_+ := W(\Sigma_+)$.

Rem $W_+ \subset W_{K\cap H}$ Equality $\iff$: H essentially connected (assumed from now on)

Def $\mathfrak{a}_{\mathfrak{q}}^{reg} := \{ X \in \mathfrak{a}_{\mathfrak{q}} \mid \forall_{\alpha \in \Sigma^+} \ \alpha(X) \neq 0 \} = W \cdot \mathfrak{a}_{\mathfrak{q}}^{+}$

$\mathfrak{a}_{\mathfrak{q}+}^{reg} := \text{\_\_\_\_\_\_\_\_} \ \alpha \in \Sigma_+^+ \ \text{\_\_\_\_\_\_\_\_} = W_{K\cap H}\, \mathfrak{a}_{\mathfrak{q}+}^{+}$

Obv: $\mathfrak{a}_{\mathfrak{q}}^{reg\,(+)} \subset \mathfrak{a}_{\mathfrak{q}+}^{reg\,(+)}$ Put $A_{\mathfrak{q}}^{reg} = \exp \mathfrak{a}_{\mathfrak{q}}^{reg}$, etc

Lemma $G = K\overline{A_{\mathfrak{q}+}^{+}}H$, with unique $\overline{A_{\mathfrak{q}+}^{+}}$-part.

Proof Since $G_+ = (K\cap H)\overline{A_{\mathfrak{q}+}^{+}}(K\cap H)$ (Riemannian case)

$$G = K \exp(\mathfrak{p}\cap\mathfrak{q}) \exp(\mathfrak{p}\cap\mathfrak{h}) = K\,\overline{A_{\mathfrak{q}+}^{+}}(K\cap H)\exp(\mathfrak{p}\cap\mathfrak{h})$$
$$= K\,\overline{A_{\mathfrak{q}+}^{+}}\,H. \quad \boxtimes$$

Cor $X_+ := K A_{\mathfrak{q}}^{reg} H$ open and dense in X.

<u>Exerc</u> Suppose $\mathcal{W} \subset N_K(\mathfrak{a}_{\mathfrak{q}})$ finite. Then

$$X_+ = \coprod_{v \in \mathcal{W}} K A_{\mathfrak{q}}^+ v H \iff \mathcal{W} \xrightarrow{1-1} W / W_{K \cap H}$$

Dual Riemannian space

$$\mathfrak{g} = \underbrace{(\mathfrak{k} \cap \mathfrak{h}) \oplus (\mathfrak{p} \cap \mathfrak{q})}_{\mathfrak{g}_+} \oplus \underbrace{(\mathfrak{k} \cap \mathfrak{q}) \oplus (\mathfrak{p} \cap \mathfrak{h})}_{\mathfrak{g}_-} \leftarrow \sigma\theta$$

<u>Def</u> $\mathfrak{g}^d < \mathfrak{g}_{\mathbb{C}}$ by $\mathfrak{g}^d := \mathfrak{g}_+ \oplus i\mathfrak{g}_-$ (real form of $\mathfrak{g}_{\mathbb{C}}$) of

<u>Put</u> $\mathfrak{k}^d := \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{g}^d, \quad \mathfrak{p}^d := \mathfrak{q}_{\mathbb{C}} \cap \mathfrak{g}^d$.

$\mathfrak{g}^d = \mathfrak{k}^d \oplus \mathfrak{p}^d$ is Cartan deco, $\theta^d = \sigma_{\mathbb{C}}|_{\mathfrak{g}^d}$

1-12

Put $\sigma^d := \theta_c|_{\mathfrak{g}^d}$. **Note:** $\theta^d \circ \sigma^d = \sigma^d \circ \theta^d$

$\mathfrak{h}^d := \mathfrak{k}_c \cap \mathfrak{g}^d$, $\mathfrak{q}^d := \mathfrak{p}_c \cap \mathfrak{g}^d$ (σ^d)

Duality $(\mathfrak{g}, \sigma, \theta) \longleftrightarrow (\mathfrak{g}^d, \sigma^d, \theta^d)$

Rem^k $\mathfrak{p} \cap \mathfrak{q} = \mathfrak{p}^d \cap \mathfrak{q}^d$, $\mathfrak{a}_{\mathfrak{q}^d} := \mathfrak{a}_{\mathfrak{q}}$.

Application: structure of $\mathbb{D}(G/H)$

Def $\mathcal{D}(G/H) := \{$ linear PDO's $C^\infty(G/H) \circlearrowleft \}$

action of G by $g.D = L_g \circ D \circ L_g^{-1}$.

$\mathbb{D}(G/H) := \mathcal{D}(G/H)^G$.

Not^n for $X \in \mathfrak{g}$, $R_X : C^\infty(G) \circlearrowleft$ def^d by $R_X f(g) = \frac{d}{dt} f(g \exp tX)\Big|_{t=0}$

Def $U(\mathfrak{g})$ is univ^l envel algebra of $\mathfrak{g}_{\mathbb{C}}$

Rem $R : \mathfrak{g} \to \mathrm{End}(C^\infty(G))$ is Lie algebra homomorphism

$\rightsquigarrow R : U(\mathfrak{g}) \to \mathrm{End}_{\mathbb{C}}(C^\infty(G)) \rightsquigarrow r : U(\mathfrak{g})^H \to \mathrm{End}(C^\infty(G))$.

Lemma r maps $U(\mathfrak{g})^H$ onto $\mathbb{D}(G/H)$ and factors through isomorphism $\bar{r} : U(\mathfrak{g})^H / U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h} \xrightarrow{\simeq} \mathbb{D}(G/H)$.

Rem in particular true for $H = K$.

Rem Since H is ess. conn^d, $U(\mathfrak{g})^{\mathfrak{h}} \supset U(\mathfrak{g})^H$ induces.

$$U(\mathfrak{g})^{\mathfrak{h}} / U(\mathfrak{g})^{\mathfrak{h}} \cap U(\mathfrak{g})\mathfrak{h} \simeq U(\mathfrak{g})^H / U(\mathfrak{g})^H \cap U(\mathfrak{g})\mathfrak{h}$$

Rem The map $\mathfrak{g}^d \longrightarrow \mathfrak{g}_{\mathbb{C}}$ induces an isomorphism of complex algebras $U(\mathfrak{g}^d) \xrightarrow{\simeq} U(\mathfrak{g})$, via which we identify. Furthermore

$$U(\mathfrak{g})^{\mathfrak{h}} = U(\mathfrak{g}^d)^{\mathfrak{h}_{\mathbb{C}}} = U(\mathfrak{g})^{\mathfrak{k}^d}$$

Lemma Suppose $G < G_{\mathbb{C}}$, and let G^d, K^d be the analytic subgroups of $G_{\mathbb{C}}$ with Lie algebras $\mathfrak{g}^d, \mathfrak{k}^d$.

$\exists!$ algebra homomorphism $D \longmapsto {}^dD$, $\mathbb{D}(G/H) \longrightarrow \mathbb{D}(G^d/K^d)$

$$\begin{tikzcd}
U(\mathfrak{g})^{\mathfrak{h}} \arrow[r, "="] \arrow[d, "r"'] \arrow[dr, phantom, "\circlearrowright"] & U(\mathfrak{g}^d)^{K^d} \arrow[d, "r"] \\
\mathbb{D}(G/H) \arrow[r, "D \mapsto {}^dD"] & \mathbb{D}(G^d/K^d)
\end{tikzcd}$$

The bottom map is an isom. of algebras.

Harish-Chandra isomorphism

$\mathfrak{a}_{\mathfrak{q}}$ extends to Cartan subspace $\mathfrak{b} < \mathfrak{g}$ s.t. $\mathfrak{b}_{\mathfrak{p}} = \mathfrak{a}$

Rem $\mathfrak{b} = \mathfrak{b}_{\mathfrak{k}} \oplus \mathfrak{a}_{\mathfrak{q}}$; $\mathfrak{a}^d = i\mathfrak{b}_{\mathfrak{k}} \oplus \mathfrak{a}$ is max abelian in $\mathfrak{p}^d$.

Recall the iso $\mathbb{D}(G^d/K^d) \xrightarrow[\simeq]{\gamma^d} P(\mathfrak{a}^{d*})^{W(\mathfrak{g}^d, \mathfrak{a}^d)}$

Recall $\gamma^d : \mathbb{D}(G^d/K^d) \longrightarrow P(\mathfrak{a}^{d*})^{W(\mathfrak{a}^d)}$

defined as follows.

- Fix Iwasawa deco $\mathfrak{g}^d = \mathfrak{k}^d \oplus \mathfrak{a}^d \oplus \mathfrak{n}^d$.
- PBW $\Rightarrow$ $\mathcal{U}(\mathfrak{g}^d) \simeq \mathcal{U}(\mathfrak{a}^d) \oplus \mathcal{U}(\mathfrak{g}^d)\,\mathfrak{k}^d + \mathfrak{n}^d\,\mathcal{U}(\mathfrak{g}^d)$.

$$u \longmapsto u_0 + \text{rest}, \quad \mathcal{U}(\mathfrak{g}^d)^{\mathfrak{k}^d} \longrightarrow \mathcal{U}(\mathfrak{a}^d)$$

- $\mathcal{U}(\mathfrak{a}^d) \simeq P(\mathfrak{a}^{d*}) \xrightarrow{T_{\rho^d}} P(\mathfrak{a}^{d*})$
- $\mathcal{U}(\mathfrak{g}^d)^{\mathfrak{k}^d} \ni u \longmapsto T_{\rho^d}(u_0)$ factors through

$$\gamma^d : \mathbb{D}(G/K) \longrightarrow P(\mathfrak{a}^{d*})^{W(\mathfrak{a}^d)}$$