3. Parabolic induction

Def parabolic subgroup of G is a subgroup $P < G$ s.t

$$P = N_G(\mathrm{Lie}(P)).$$

Not^n $\mathfrak{a} \subset \mathfrak{p}$ maximal abelian, $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ pos. system,

$M = Z_K(\mathfrak{a})$, $G = KAN$ Iwasawa decomposition

- $P_0 = MAN$ is a minimal par. subgp.
- $K \cap P_0 = M$, $G = KP_0 \simeq K \times_M P_0$
- $K \subset G$ induces diffeom. $K/M \longrightarrow G/P_0$

Fact Every psg of G is K-conjugate to a psg containing P_0.

Langlands decomposition of a psg $Q < G$.

$M_{1Q} := Q \cap \theta(Q)$

- $Q = M_{1Q} N_Q \simeq M_{1Q} \times N_Q$, where $N_Q = \exp \mathfrak{n}_Q$, $\mathfrak{n}_Q$ the nilpotent radical of $\mathrm{Lie}(Q)$.
- $\mathfrak{a}_Q = \mathrm{Center}(\mathfrak{m}_{1Q}) \cap \mathfrak{p}$, $A_Q = \exp \mathfrak{a}_Q$.
- $M_{1Q} = M_Q A_Q \simeq M_Q \times A_Q = Z_G(\mathfrak{a}_Q)$

$$Q = M_Q A_Q N_Q$$

Notation $\mathcal{P}(A) := \{ Q \text{ psg of } G \mid Q \supset A \}$,

$\mathcal{P}_{st} := \{ Q \in \mathcal{P}(A) \mid Q \supset P_0 \}$. (standard psgs)

Def Given $Q \in \mathcal{P}(A)$, define

$$\mathfrak{a}_Q^+ = \{ X \in \mathfrak{a}_Q \mid \forall(\alpha \in \Sigma(\mathfrak{n}_Q, \mathfrak{a}_Q)) : \alpha(X) > 0 \}$$

Def For $X \in \mathfrak{a}$, define $\Sigma(X) = \{ \alpha \in \Sigma \mid \alpha(X) > 0 \}$.

Rem $X \sim Y :\Longleftrightarrow \Sigma(X) = \Sigma(Y)$

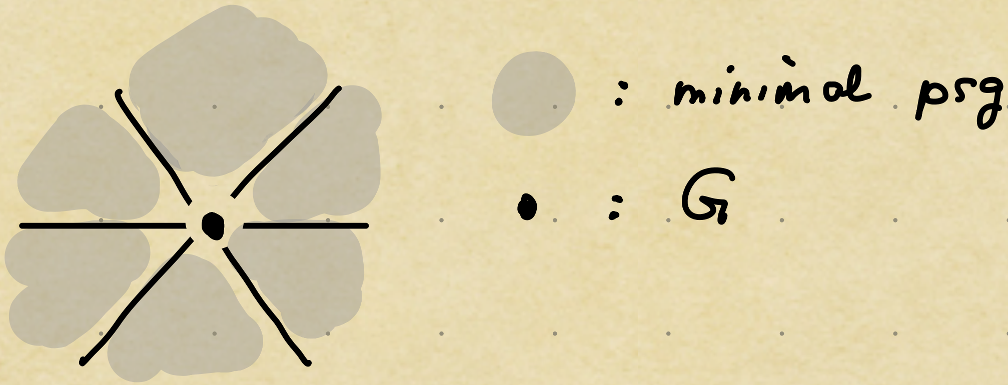
$\sim$ defines equivalence relation on $\mathfrak{a}$.

Lemma $Q \mapsto \mathfrak{a}_Q^+$, $\mathcal{P}(A) \xrightarrow{1-1} \mathfrak{a}/\sim \; = \{\text{classes of } \sim\}$

inverse given by $\Phi \mapsto P_\Phi = M_{1\Phi} N_\Phi$,

$M_{1\Phi} = Z_G(\Phi)$, $\mathfrak{n}_{1\Phi} = \sum_{\alpha \in \Sigma,\, \alpha|\Phi > 0} \mathfrak{g}_\alpha$

classes : facets



Rem

- $P \subseteq Q \iff \overline{\mathfrak{a}_P^+} \supset \mathfrak{a}_Q^+$
- $\mathfrak{a}^+_{wPw^{-1}} = w(\mathfrak{a}_P^+) \quad (w \in W(\Sigma))$

Induction from $P \in \mathcal{P}(A)$ to G.

Data $\xi \in \widehat{M}_F$, $\lambda \in i\mathfrak{a}_P^* \longleftrightarrow \widehat{A}_P$

- $\xi \otimes \lambda$ is unitary rep of $M_{1P} = M_P A_P$ given by

$$(\xi \otimes \lambda)(m, a) = a^\lambda \xi(m) \in GL(\mathcal{H}_\xi), \quad a^\lambda := e^{\lambda(\log a)}$$

- $\mathrm{Ind}_P^G(\xi \otimes \lambda)$ indicates "unitary induction"

Space $L^2(P:\xi:\lambda) := \{ f \in L^2_{loc}(G, \mathcal{H}_\xi) \mid f(man\,x) = a^{\lambda+\rho_P}\xi(m)^{-1} f(x) \}$

$\uparrow$ G/P cpt

equipped with right regular rep $R =: \pi_{P,\xi,\lambda}$.

Here $\rho_P \in \mathfrak{a}_P^* : X \mapsto \frac{1}{2}\mathrm{tr}(\mathrm{ad}(X)|_{\mathfrak{n}_P})$ $\leftrightarrow$ half density on G/P

- $\pi_{P,\xi,\lambda}$ unitary for $\lambda \in i\mathfrak{a}_P^*$.
- Similar defi's for $\lambda \in \mathfrak{a}_{PC}^*$. Then the sesquil. pairing

$$L^2(P:\xi:\lambda) \times L^2(P:\xi:-\bar{\lambda}) \longrightarrow \mathbb{C}$$

$$(f,g) \longmapsto \langle f,g\rangle = \int_K \underbrace{\langle f(k), g(k)\rangle}_{\text{left } K\cap M_P\text{ - invariant}} dk$$

is G-equivariant.

Compact picture

$f \mapsto f|_K$ defines a topol. linear isom.

$$L^2(P:\xi:\lambda) \xrightarrow{\simeq} L^2(K:\xi|_{K\cap M_P}) = \qquad (K_P := K \cap M_P)$$

$$= \text{space } \mathrm{Ind}_{K_P}^{K}(\xi|_{K_P})$$

Transfer: $\pi_{P,\xi,\lambda}$ to cont^s rep on $L^2(K:\xi|_{K_P})$, depending on λ

Thm: $L^2(P:\xi:\lambda)^\infty = C^\infty(P:\xi:\lambda) \quad (\subset C^\infty(G,\mathcal{H}_\xi))$

Dual: $C^{-\infty}(P:\xi:\lambda) = \overline{C^\infty(P;\xi:-\bar\lambda)'}$ ← conjugate space; cont. lin. dual

- $C^\infty(P:\xi:\lambda) \hookrightarrow C^{-\infty}(P:\xi:\lambda)$ naturally via

$$f \longmapsto \langle f, \cdot \rangle \in \overline{C^\infty(P:\xi:\lambda)'}$$

- $\mathcal{H}_\xi^\infty \subset \mathcal{H}_\xi \hookrightarrow \mathcal{H}_\xi^{-\infty} := \overline{(\mathcal{H}_\xi^\infty)'}$ via $\langle \cdot , \cdot \rangle_\xi$.

3.7

Idea: construct $j \in C^{-\infty}(P:\xi:\lambda)^H$, then have matrix coeff

$$m_j : C^\infty(P:\xi:-\bar\lambda) \xrightarrow{G} C^\infty(G/H)$$

given by $m_j(f)(x) = \langle f, \pi^{-\infty}_{P,\xi,\lambda}(x) j \rangle$.

Rem^k $ev_e \circ m_j(f) = \langle f, j \rangle$

Exerc Riemannian case: $\sigma = \theta$, $H = K$, $P = P_\emptyset$, $\xi = 1$

define $j_\lambda(nak) = a^{\lambda+\rho}$. Show that

$j_\lambda \in C^{-\infty}(P:1:\lambda)^K$ and

$$m_{j_\lambda}(f)(x^{-1}) = \mathcal{P}_{-\lambda}(f^\vee)(x).$$

Idea: on open orbit $PvH \subset G$ one must have

$$j|_{PvH} \in C^\infty(PvH, \mathcal{H}_\xi^{-\infty})^H$$

$$\Rightarrow ev_v\, j \in (\mathcal{H}_\xi^{-\infty})^{M_P \cap vHv^{-1},\, A_P \cap vHv^{-1}} \quad \text{(obstructions)}$$

To overcome obstructions, require:

(a) $\sigma\theta P = P$ (σ-parabolic)

(b) $\mathfrak{a}_q \subset \mathfrak{p} \cap \mathfrak{q}$ max abelian, $\mathfrak{a}_q \subset \mathfrak{a} \subset \mathfrak{p}$

(c) $\lambda|_{\mathfrak{a}_q \cap \mathfrak{h}} = 0$ ($\Leftrightarrow \lambda \in \mathfrak{a}^*_{\mathfrak{p}q\mathbb{C}}$)

<u>Fact</u> $\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a}_q)$ is a root system,

- fix positive system Σ^+,
- Δ = simple roots in Σ^+
- $W = W(\Sigma) \simeq N_K(\mathfrak{a}_q) / Z_K(\mathfrak{a}_q)$ $\#(P\backslash G/H) < \infty$ and (if H essentially conn^d)

<u>Notation</u> $\mathcal{P}_\sigma(A_q) = \{ P \text{ psg} \mid P \supset A_q, \ \sigma\theta P = P \}$

- $\mathcal{P}_\sigma(A_q) \subset \mathcal{P}(A)$.

(3.9)

- $P \in \mathcal{P}_\sigma(A_q) \Rightarrow \sigma\ominus M_p = M_p,\ \sigma\ominus N_p = N_p,\ \sigma\ominus A_p = A_p$

$$\Rightarrow \sigma_p = \sigma_{ph} \oplus \sigma_{pq}.$$

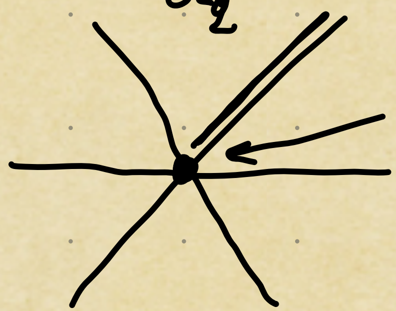
$$\Rightarrow \forall_{\lambda \in \sigma_{pq}^*}: \quad \lambda|_{\sigma_{ph}} = 0 \iff \lambda \in \sigma_{pq}^*.$$

- $P \mapsto \sigma_{pq}^+,\quad \mathcal{P}_\sigma(A_q) \longrightarrow \sigma_q/\sim$

$= \text{faceth for } \Sigma(\sigma_f, \sigma_q)$

- $\sigma_{pq}^+ \subset \overline{\sigma_q^+} \iff \exists!\, F \subset \Delta:\ \sigma_{pq}^+ = \sigma_{Fq}^+$

- $\mathcal{P}_{\sigma,st} = \{ P_F \mid F \subset \Delta \}$



Generalized principal series of reps for G/H

$\mathrm{Ind}_P^G(\sigma \otimes \lambda)$ where $P \in \mathcal{P}_\sigma(A_q)$, $\xi \in \widehat{M}_P$, $\lambda \in \mathfrak{a}_{q\mathbb{C}}^*$

Notation $W = W(\mathfrak{a}_q) \simeq N_K(\mathfrak{a}_q)/Z_K(\mathfrak{a}_q)$, $W_{K\cap H}$, $W_P = Z_W(\mathfrak{a}_{Pq})$.

Fact There exists a set ${}_P\mathcal{W} \subset N_K(\mathfrak{a}) \cap N_K(\mathfrak{a}_q)$ of representatives for $W_P \backslash W / W_{K\cap H}$.

Lemma $\#(P\backslash G/H) < \infty$ and (if H essentially connected)

$v \longmapsto PvH$ defines bijection

$${}_P\mathcal{W} \xrightarrow{\simeq} (P\backslash G/H)_{\text{open}}.$$

Let $P \in \mathcal{P}_\sigma(A_q)$, $\xi \in \widehat{M_P}$, $\lambda \in \mathfrak{a}^*_{q\mathbb{C}}$.

Lemma Let $j \in C^{-\infty}(P:\xi:\lambda)^H$, $v \in {}_P\mathcal{W}$.

Then $j|_{PvH}$ is a smooth function $PvH \to \mathcal{H}_\xi^{-\infty}$,

and $ev_v(j) = j(v) \in (\mathcal{H}_\xi^{-\infty})^{M_P \cap vHv^{-1}}$.

Lemma For generic $\lambda \in \mathfrak{a}^*_{q\mathbb{C}}$ the map

$ev : C^{-\infty}(P:\xi:\lambda)^H \longrightarrow \underbrace{\bigoplus_{v \in {}_P\mathcal{W}} (\mathcal{H}_\xi^{-\infty})^{M_P \cap vHv^{-1}}}_{\text{notation } {}^\circ V_P(\xi)}$ is injective.

Lemma There exists a unique meromorphic family of maps $j(P,\xi,\lambda) : {}^\circ V_P(\xi) \longrightarrow C^{-\infty}(P:\xi:\lambda)^H$, for $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$, s.t. $ev \circ j(P,\xi,\lambda) = id_{{}^\circ V_P(\xi)}$. Meromorphy means that

$j(P,\xi)$ is meromorphic $\mathfrak{a}^*_{Pq\mathbb{C}} \longrightarrow C^{-\infty}(K : \xi|_{K\cap M_P})$. (3.12)

(Compact picture)

Restriction on η to ensure that the matrix coefficients

$\langle f, \pi_{P,\xi,\lambda}(\cdot)\, j(P,\xi,\lambda)\eta \rangle$ are tempered, for $\lambda \in i\mathfrak{a}^*_P$.

i.e. $\in L^{2+\varepsilon}(G/H)$ $(\forall \varepsilon > 0)$

$$\eta \in V_P(\xi) := \bigoplus_{v \in {}_P\mathcal{W}} \left(\mathcal{H}_\xi^{-\infty}\right)^{M \cap vHv^{-1}}_{ds}$$

means: $\langle \cdot\,, \xi^{-\infty}(\cdot)\eta_v \rangle \in \mathrm{Hom}_G(\mathcal{H}_\xi, L^2_d(M_P / M_P \cap vHv^{-1}))$

<u>Notation</u> $X_{P,*,ds} := \{ \xi \in \widehat{M}_P \mid V_P(\xi) \neq 0 \}$.

<u>Recall</u> $\mathrm{Hom}_G(\mathcal{H}_\xi, L^2_d(X_{P,v})) \hookrightarrow \mathrm{Hom}_G(\mathcal{H}_\xi^\infty, L^2_d(X_{P,v})^\infty)$