Lecture 4: The representation theoretic Fourier transform

Def Define unnormalized FT ${}^u\hat{f}$ of $f \in C_c^\infty(G/H)$ by

$$ {}^u\hat{f}(P,\xi,\lambda) := \underbrace{\int_{G/H} f(x)\,\pi_{P,\xi,\lambda}(x)\, j(P,\xi,\lambda)\, dx}_{\in\, V(P,\xi)^* \otimes C^\infty(\xi : K_P\backslash K)} $$

for $P \in \mathcal{P}_\sigma(A_q)$, $\xi \in \widehat{X}_{P,*,ds}$, $\lambda \in \mathfrak{a}^*_{P\mathbb{C}}$

Rem resembles $\pi(f)\, j(P,\xi,\lambda)$

Example $H = K$, $\sigma = \theta$,

$$ \hat{f}(\lambda) = \int_{G/K} f(x)\,\pi_{P_\emptyset,1,\lambda}(x)\,\mathbb{1}_{P,\lambda}\, dx . $$

$$ = \pi_{P_\emptyset,1,\lambda}(f)\,\mathbb{1}_{P,\lambda} . $$

Rem The map $f \mapsto {}^{u}\hat{f}(P,\xi,\lambda)$ intertwines L with $1 \otimes \pi_{P,\xi,\lambda}$.

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Thm (Plancherel identity) For $f \in C_c^\infty(G/H)$,

$$\|f\|^2_{L^2(X)} = \sum_{P \in \mathbb{P}_\sigma} [W : W_P^*] \sum_{\xi \in \hat{X}_{P,*,ds}} \int_{i\mathfrak{a}^*_{Pq}} \|{}^{u}\hat{f}(P,\xi,\lambda)\|^2_{HS} \, d\mu_{P,\xi}(\lambda).$$

Pl measure

Notation

- $P_1, P_2 \in \mathcal{P}_\sigma(A_q)$ are associated if $\mathfrak{a}_{P_1q}$ and $\mathfrak{a}_{P_2,q}$ are W-conjugate, notation $P_1 \sim P_2$
- $\mathbb{P}_\sigma$ is a complete set of representatives for $\mathcal{P}_\sigma(A_q)/\sim$.
- $W_P^* = N_W(\mathfrak{a}_P)$
- Recall $\hat{X}_{P,*,ds} = \bigcup_{v \in {}_P\mathcal{W}} (M_P / M_P \cap vHv^{-1})^\wedge_{ds}$

Plancherel thm, part II

(4.3)

$$f \mapsto \hat{f} \text{ intertwines } L \text{ with } \bigoplus_P \bigoplus_\xi \int_{i\mathfrak{a}_{Pq}^*} 1_{V_P(\xi)} \otimes \pi_{P,\xi,\lambda} \, d\mu_{P,\xi}(\lambda)$$

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Rem $V_P(\xi)$ plays role of multiplicity space.

Standard intertwiners

Suppose $P, Q \in \mathcal{P}$ with $\mathfrak{a}_P = \mathfrak{a}_Q$. Then $M_P = M_Q$. Let $\xi \in \hat{M}_P$ have real inf char. Then $\exists!$ meromorphic family

$$\mathfrak{a}_{P\mathbb{C}}^* \ni \lambda \longmapsto A(Q, P, \xi, \lambda)$$

of intertw ops $\pi_{P,\xi,\lambda} \to \pi_{Q,\xi,\lambda}$ s.t. for $\langle \mathrm{Re}\,\lambda, \alpha \rangle \gg 0$ $(\alpha \in \Sigma(\mathfrak{n}_P) \cap \Sigma(\bar{\mathfrak{n}}_Q))$, $f \in C^\infty(P : \xi : \lambda)$,

$$(A(Q, P, \xi, \lambda) f)(x) = \int_{\bar{N}_P \cap N_Q} f(n x) \, dn \qquad (x \in G).$$

Remark $A(\bar{P}, P, \xi, \lambda)\, A(P, \bar{P}, \xi, \lambda) = \eta(P, \bar{P}, \xi, \lambda)\, I$.

with $\eta(P, \bar{P}, \xi, \cdot) : \mathfrak{a}^*_{PC} \to \mathbb{C}$ meromorphic function

Pf for generic λ, $\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$ irreducible &

$A(\bar{P}, P, \xi, \lambda)\, A(P, \bar{P}, \xi, \lambda)$ intertwines $\pi_{P, \xi, \lambda}$ with itself.

- Apply Schur's lemma. $\boxtimes$

Plancherel part III

$$d\mu_{P,\xi}(\lambda) = \eta(\bar{P}, P, \xi, \lambda)^{-1}\, d\mu_P(\lambda)$$

Lebesgue measure on $i\mathfrak{a}_P^*$

Normalize j

Def. $j^0(P, \xi, \lambda) := A(P, \bar{P}, \xi, \lambda)^{-1}\, j(P, \xi, \lambda)$.

Def $\hat{f}$ as ${}^u\hat{f}$ with j^0 in place of j, so

Def For $f \in C_c^\infty(G)$,

$$\hat{f}(P, \xi, \lambda) = A(P, \bar{P}, \xi, \lambda)^{-1}\, {}^u\hat{f}(P, \xi, \lambda).$$

Cor $\quad \| {}^u\hat{f}(P, \xi, \lambda) \|^2 = \eta(P, \bar{P}, \xi, \lambda)\, \| \hat{f}(P, \xi, \lambda) \| \quad (\lambda \in i\mathfrak{a}_P^*)$

Thm (normalized version of Plancherel): for $f \in C_c^\infty(G/H)$,

$$\|f\|^2_{L^2(G/H)} = \sum_{P \in \mathcal{P}_\sigma} [W : W_P^+] \sum_{\xi \in X_{P,+,ds}} \int_{i\mathfrak{a}_P^*} \| \hat{f}(P, \xi, \lambda) \|^2 \, d\mu_P(\lambda)$$

Lebesgue

Strategy of proof restrict to left K-finite f.

Fix $\delta \in \hat{K}$ and assume $f \in C_c^\infty(G/H)[\delta]$.

$$C_c^\infty(G/H)[\delta] \simeq V_\delta \otimes \mathrm{Hom}_K(V_\delta, C_c^\infty(G/H)) \simeq$$

$$\big([V_\delta \otimes V_\delta^*] \otimes C_c^\infty(G/H)\big)^K \simeq C^\infty(\tau : G/H)$$

$$\tau = 1 \otimes \delta^\vee \qquad L \qquad (V_\tau = V_\delta \otimes V_\delta^*)$$

For (τ, V_τ) f.d. unitary rep of K, define

$$C_c^\infty(\tau : G/H) := \{ f \in C_c^\infty(G/H, V_\tau) \mid f(kx) = \tau(k)^{-1} f(x) \;\; \forall x \in G/H,\ k \in K \}.$$

Accordingly $\hat{f}(P, \xi, \lambda) \in V_P(\xi)^* \otimes C^\infty(\xi : K_P \backslash K : \tau)$ (4.7)

<u>Lemma</u> For $\xi \in \hat{X}_{P, *, ds}$, $v \in {}_P\mathcal{W}$,

$$V_P(\xi, v)^* \otimes C^\infty(\xi : K_P \backslash K : \tau) \simeq L^2_\xi(\tau_P : \overbrace{M_P / M_P \cap vHv^{-1}}^{X_{P,v}})$$ (isometric)

<u>Proof</u> $$\text{LHS} = \Big[\underset{1}{V_P(\xi, v)^*} \otimes \overbrace{\underset{R}{C^\infty \mathrm{Ind}_{K_P}^{K}(\xi_P : K)} \otimes \underset{\tau}{V_\tau}}\Big]^K$$

$$\overset{\text{Frob..}}{=} \Big[\underbrace{V_P(\xi, v)^* \otimes \mathcal{H}_{\xi_P}}_{\underset{12}{\cong}\ L^2_\xi(X_{P,v})} \otimes V_{\tau_P},\Big]^{K_P} = \text{RHS} \quad \text{(isometrically)} \ \boxtimes$$

Here $V_P(\xi, v)^*$ has been equipped with the unique inner product which makes 12 an isometry. Accordingly $V_P(\xi) \simeq \bigoplus_{v \in {}_P\mathcal{W}} V_P(\xi, v)$ is equipped with the direct sum inner product.

Taking the direct sum over the finitely many $\xi \in \hat{X}_{P, *, ds}$ for which $\xi|_{K_P}$ and $\tau|_{K_P}$ have a K-type in common we find

$$\bigoplus_{\xi} \overline{V_P(\xi)} \otimes C^\infty(\xi : K_P\backslash K : \tau) \simeq \bigoplus_{v} L^2_d(\tau : X_{P,v}) \qquad (4.8)$$

Accordingly, for $\varphi \otimes \eta \in C^\infty(\xi : K_P \backslash K : \tau) \otimes \overline{V_P(\xi)}$

$$\langle \hat{f}(P, \xi, \lambda), \varphi \otimes \eta \rangle =$$

$$\int_{G/H} f(x) \langle \pi_{P,\xi,\lambda}(x) j^\circ(P, \xi, \lambda | \eta, \varphi \rangle \, d\dot{x} =$$

$$\int_{G/H} \langle f(x), E^\circ_-(P, \psi_{\varphi \otimes \eta}, \lambda)(x) \rangle \, d\dot{x}$$

<u>Def</u> $\mathcal{A}_{2,P,\tau} := \bigoplus_{v \in {}^P\mathcal{W}} L^2_d(\tau : X_{P,v})$, and for $\psi \in \mathcal{A}_{2,P,\tau}$,

<u>Def</u> $E^\circ(P, \psi, \lambda) \in C^\infty(\tau : G/H)$, linear in $\psi \in \mathcal{A}_{2,P,\tau}$

by $E^\circ(P, \psi_{\varphi \otimes \eta}, \lambda)(x) = \overline{\langle \pi_{P,\xi,\lambda}(x) j^\circ(P, \xi, \lambda | \eta, \varphi \rangle}$

$(\varphi \in C^\infty(\xi_P : K_P \backslash K : \tau),\ \eta \in V_P(\xi, v))$.

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Def $\mathcal{F}_P : C_c^\infty(\tau : G/H) \longrightarrow \mathcal{M}(\mathfrak{a}_{P\mathbb{C}}^*) \otimes \mathcal{A}_{2,P,\tau}$

$$\langle \mathcal{F}_P f(\lambda), \psi \rangle := \int_{G/H} \langle f(x), E^\circ(P, \psi, -\bar\lambda)(x) \rangle \, dx$$

Cor: For $f \in C_c^\infty(\tau : G/H)$:

$$\langle \sum_\xi \hat f(P, \xi, \lambda), T \rangle = \langle \mathcal{F}_P f(\lambda), \psi_T \rangle$$

Since $T \longmapsto \psi_T$, $\bigoplus_\xi V(\xi)_P \otimes C^\infty(\xi_P : K_P \backslash K : \tau) \longrightarrow \mathcal{A}_{2,P,\tau}$ is an isometry, it now follows that

$$\mathcal{F}_P f(\lambda) = \psi\Big(\sum_\xi \hat f(P : \xi : \lambda)\Big)$$

$$\|\mathcal{F}_P f(\lambda)\|^2 = \sum_\xi \|\hat f(P : \xi : \lambda)\|^2$$

Thm (Spherical version of Plancherel).

For $f \in C_c^\infty(\tau : G/H)$:

$$\|f\|^2_{L^2(\tau : G/H)} = \sum_{P \in \mathcal{P}_\sigma} [W_P : W_P^*] \int_{i\mathfrak{a}_P^*} \|\mathcal{F} f(\lambda)\|^2 \underbrace{d\mu_P}_{\text{Lebesgue}}(\lambda).$$

In last lecture: aspects of proof of this theorem.