The Parseval identity for the Whittaker Fourier transform

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Delorme's conference

Representation Theory, Harmonic Analysis and Spherical Varieties

Porquerolles/Marseille

19-23 June, 2023

Whittaker functions

Setting

- G real reductive group
- K maximal compact, $G = KAN_0$ Iwasawa decomposition
- $\chi : N_0 \to \mathrm{U}(1)$ unitary character, regular (!)

$$\text{i.e.: } \forall \alpha \in \Sigma(\mathfrak{n}_0, \mathfrak{a}) \text{ simple: } d\chi(e)|_{\mathfrak{g}_\alpha} \neq 0.$$

Whittaker functions

$$\mathcal{M}(G/N_0 : \chi) \quad := \quad \{f : G \xrightarrow{\text{meas}} \mathbb{C} \mid f(xn) = \chi(n)^{-1} f(x) \quad (x \in G, n \in N_0)\}$$

$$L^2(G/N_0 : \chi) \quad := \quad \{f \in \mathcal{M}(G/N_0 : \chi) \mid |f| \in L^2(G/N_0)\}$$

- Left reg^r rep^n: $L = \mathrm{Ind}_{N_0}^G(\chi)$ is unitary
- $\exists$: abstract Plancherel decomposition: $\mathrm{Ind}_{N_0}^G(\chi) \simeq \int_{\widehat{G}}^{\oplus} m_\pi \pi \, d\mu(\pi)$.
- Realized by a unitary Fourier transform $f \mapsto \hat{f}$.

  abstract Parseval identity: $\|f\|_{L^2}^2 = \int_{\widehat{G}} \|\hat{f}(\pi)\|_\pi^2 \; d\mu(\pi)$.
- Today's goal: describe this as explicitly as possible

Whittaker Plancherel theory

Sources

- Harish-Chandra, Announcement AMS Toronto 1982.

  details in Collected Papers Vol 5 (posthumous), 141- 307, eds. R. Gangolli, V.S. Varadarajan, Springer 2018. Formulation of Parseval, proof incomplete.

- Wallach, RRG II, 1992: discrete part, cusp forms, holomorphic dependence of Jacquet integral, inversion formula, error addressed in [arXiv: 1705.06787]. No proof of Parseval.

- Today: complete proof of Parseval identity, based on [arXiv:2304.11044] and forthcoming paper.

Discrete part of Plancherel decomposition

Discrete part

$\pi \in \widehat{G}$ (unitary dual) is said to appear discretely in $L^2(G/N_0 : \chi)$ if it can be realized as a closed subrepresentation.
The closed span of such π is denoted by $L^2_d(G/N_0 : \chi)$.

Theorem (HC, W)

If $\pi \in \widehat{G}$ appears in $L^2_d(G/N_0 : \chi)$, then it appears in $L^2_d(G)$, i.e., it belongs to the discrete series of G.

Spherical functions

Let (τ, V_τ) be a finite dimensional unitary representation of K.

$$L^2_d(\tau : G/N_0 : \chi) := (L^2_d(G/N_0 : \chi) \otimes V_\tau)^K$$

$$\hookrightarrow \{f \in \mathcal{M}(G, V_\tau) \mid f(kxn) = \chi(n)^{-1}\tau(k)f(x)\}.$$

Whittaker functions of Schwartz type

- Define $\rho \in \mathfrak{a}^*$ by $\rho(X) = \frac{1}{2}\mathrm{tr}(\mathrm{ad}(X)|_{N_0})$.

Definition (Schwartz space, HC, W)

$\mathcal{C}(G/N_0 : \chi)$: the space of $f \in C^\infty(G/N_0 : \chi)$ s.t. $\forall u \in U(\mathfrak{g}), N \in \mathbb{N}$,

$$\exists C_{u,N} > 0 : \quad |L_u f(kan)| \le C_{u,N}\,(1 + |\log(a)|)^{-N} a^{-\rho} \qquad (kan \in KAN_0).$$

Put $\mathfrak{Z} := \mathrm{center}\, U(\mathfrak{g})$. For (τ, V_τ) a finite dim$^\ell$ unitary representation of K,

$$\mathcal{A}_2(\tau : G/N_0 : \chi) := \{f \in \mathcal{C}(\tau : G/N_0 : \chi) \mid \dim \mathfrak{Z}f < \infty\}.$$

Thm (HC, W)

(a) $L^2_d(\tau : G/N_0 : \chi) = \mathcal{A}_2(\tau : G/N_0 : \chi)$.

(b) *The space is finite dimensional.*

(c) $\mathcal{A}_2(\tau : G/N_0 : \chi) = \oplus_{\sigma \in \widehat{G}_{\mathrm{ds}}} \mathcal{A}_2(\tau : G/N_0 : \chi)_\sigma$.

Parabolic induction and Whittaker integral

- $P_0 := Z_K(A)A\,N_0$, minimal psg.
- $\mathcal{P}_{\rm st}$: (finite) set of psg's $P = M_P A_P N_P < G$ with $P \supset P_0$ (standard psg's).

For $P \in \mathcal{P}_{\rm st}$,

- $\chi_P := \chi|_{M_P \cap N_0}$ is regular for $M_P/(M_P \cap N_0)$.
- $K_P := M_P \cap K$, $\tau_P := \tau|_{K_P}$.
- $\mathcal{A}_{2,P} := \mathcal{A}_2(\tau_P : M_P/M_P \cap N_0 : \chi_P)$.

Let $\psi \in \mathcal{A}_{2,P}$. For $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ define $\psi_\nu : G \to V_\tau$ by

$$\psi_\nu(kma\bar{n}) := a^{\nu+\rho_P}\tau(k)\psi(m) \qquad ((k, man) \in K \times \bar{P}).$$

Lemma For $\mathrm{Re}\,\nu >_P 0$, the integral

$$\mathrm{Wh}(P, \psi, \nu, x) := \int_{N_P} \chi(n)\psi_\nu(xn)\, dn \qquad (x \in G)$$

is abs^y conv^t and defines $\mathrm{Wh}(P, \psi, \nu) \in C^\infty(\tau : G/N_0 : \chi)$.

Holomorphic extension

Remark For $\sigma \in \widehat{M}_{P\mathrm{ds}}$, $\psi \in \mathcal{A}_{2,P,\sigma}$,

$\mathrm{Wh}(P, \psi, \nu) \in C^\infty(\tau : G/N_0 : \chi)$ *is essentially a finite sum of generalized matrix coefficients (defined by Jacquet integrals) of* $\mathrm{Ind}_P^G(\sigma \otimes -\nu \otimes 1)$

(analogue of Eisenstein integral for groups and symmetric spaces).

Theorem (W)

$\mathrm{Wh}(P, \psi, \nu)$, *initially defined for* $\mathrm{Re}\nu >_P 0$, *extends to entire holom*^{c} *function of* $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ *with values in* $C^\infty(\tau : G/N_0 : \chi)$.

Remark HC: there exists a merom^c extension, regular on $i\mathfrak{a}^*_P$.

Theorem (Uniformly tempered estimates, $\sim$)

Let $\varepsilon > 0$ *be suff*^{t}*ly small. If* $u \in U(\mathfrak{g})$ *then* $\exists C, N, r > 0$ *s.t.*

$$|\mathrm{Wh}(P, \psi, \nu, u; ka)| \leq C(1 + |\nu|)^N (1 + |\log a|)^N e^{r|\mathrm{Re}\nu||\log a|} a^{-\rho},$$

for all $k \in K$, $a \in A$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ *with* $|\mathrm{Re}\,\nu| < \varepsilon$.

Proof of uniform temperedness

Theorem (Uniformly tempered estimates, $\sim$)

Let $\varepsilon > 0$ be suff^{t}ly small. If $u \in U(\mathfrak{g})$ then $\exists C, N, r > 0$ s.t.

$$|\mathrm{Wh}(P, \psi, \nu, u; ka)| \leq C(1+|\nu|)^N (1+|\log a|)^N e^{r|\mathrm{Re}\nu||\log a|} a^{-\rho},$$

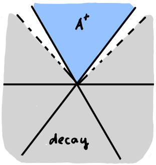
for all $k \in K$, $a \in A$, $\nu \in \mathfrak{a}^*_{P\mathbb{C}}$ with $|\mathrm{Re}\nu| < \varepsilon$.

Steps in proof

- Bernstein-Sato type functional equation for Jacquet integrals (viewed as generalized vectors of principal series reps)
- Uniformly moderate estimates.
- Wallach's method of improving estimates of matrix coefficients along max psg's, with uniformity in parameters.

Strong analogy with the theory of reductive symmetric spaces ($\sim$, Delorme, Carmona)

- $\mathrm{Wh}(P, \psi, \nu)$ is finite under $\mathfrak{Z} := \mathrm{center}(U(\mathfrak{g}))$,
- top order asymptotic behavior of $\exp'$ type along $\mathrm{cl}(A^+)$,
- very rapid decay outside $\mathrm{cl}(A^+)$.



Lemma

Let $P \in \mathcal{P}_{\mathrm{st}}$. For $\psi \in \mathcal{A}_{2,P}$, $\mathrm{Re}\,\nu \in \mathfrak{a}_P^{*+}$, $m \in M_P$, $a \to \infty$ in A_P^+,

$$\mathrm{Wh}(P, \psi, \nu)(ma) \sim a^{\nu - \rho_P}[C_P(\nu)\psi](m),$$

with $C_P(\nu) \in \mathrm{End}(\mathcal{A}_{2,P})$, merom^c in $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$ (reg^r for $\mathrm{Re}\,\nu \in \mathfrak{a}_P^{*+}$).

Definition (HC)

$$\mathrm{Wh}^\circ(P, \psi, \nu) := \mathrm{Wh}(P, C_P(\nu)^{-1}\psi, \lambda) \qquad (\text{merom}^c \text{ in } \nu).$$

- $P \sim Q :\iff \exists w \in W(\mathfrak{a}) : \; w(\mathfrak{a}_P) = \mathfrak{a}_Q$ (associated).
- $W(\mathfrak{a}_Q|\mathfrak{a}_P) := \{s \in \mathrm{Hom}(\mathfrak{a}_P, \mathfrak{a}_Q) \mid \exists w \in W(a) : \; s = w|_{\mathfrak{a}_P}\}$.

Asymptotic behavior Let $P, Q \in \mathcal{P}_{\rm st}$.
There exist unique merom^c functions $C^\circ_{Q|P}(s, \cdot\,) : \mathfrak{a}^*_{P\mathbb{C}} \to \mathrm{Hom}(\mathcal{A}_{2,P}, \mathcal{A}_{2,Q})$, for $s \in W(\mathfrak{a}_Q \mid \mathfrak{a}_P)$, such that for generic $\nu \in i\mathfrak{a}^*_P$ and $a \to \infty$ in A^+_Q.

$$\mathrm{Wh}^\circ(P, \psi, \nu)(ma) \sim \sum_{s \in W(\mathfrak{a}_Q|\mathfrak{a}_P)} a^{s\nu - \rho_Q}[C^\circ_{Q|P}(s, \nu)\psi](m), \quad (m \in M_Q).$$

Maass-Selberg relations (HC)
For $P, Q \in \mathcal{P}_{\rm st}$ and $s \in W(\mathfrak{a}_Q \mid \mathfrak{a}_P)$,

$$C^\circ_{Q|P}(s, -\bar{\nu})^* C^\circ_{Q|P}(s, \nu) = I, \qquad (\nu \in \mathfrak{a}^*_{P\mathbb{C}}).$$

Remark Equivalently: $C^\circ_{Q|P}(s : \nu) \in \mathrm{U}(\mathcal{A}_{2,P}, \mathcal{A}_{2,Q})$ for $s \in i\mathfrak{a}^*_P$.

Corollary $\nu \mapsto \mathrm{Wh}^\circ(P, \psi, \nu) \in C^\infty(\tau : G/N_0 : \chi)$ is meromorphic, regular on $i\mathfrak{a}^*_P$. Satisfies uniform tempered estimates.

Fourier transform, functional equation

For $f \in \mathcal{C}(\tau : G/N_0 : \chi)$, $P \in \mathcal{P}_{st}$, $\nu \in i\mathfrak{a}_P^*$, the Fourier transform $\mathcal{F}_P f(\nu) \in \mathcal{A}_{2,P}$ is defined by

$$\langle \mathcal{F}_P f(\nu), \psi \rangle := \int_{G/N_0} \langle f(x), \mathrm{Wh}^\circ(P, \psi, \nu, x) \rangle_{V_\tau} \, dx, \quad (\psi \in \mathcal{A}_{2,P}).$$

Theorem (∼) $\mathcal{F}_P : \mathcal{C}(\tau : G/N_0 : \chi) \to \mathcal{S}(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}$, *cont^s linearly.*

Proof this follows from the uniformly tempered estimates.

Remark HC proves this for $\mathcal{F}_P$ restricted to $C_c^\infty(\tau : G/N_0 : \chi)$.

Remark $\mathcal{F}_G = L^2$-orth^l proj^n $\mathcal{C}(\tau : G/N_0 : \chi) \to \mathcal{A}_2(\tau : G/N_0 : \chi)$.

Lemma (Functional equations, HC)

Let $P, Q \in \mathcal{P}_{\rm st}$, $P \sim Q$. Then for all $s \in W(\mathfrak{a}_Q|\mathfrak{a}_P)$,

$$\mathrm{Wh}^\circ(Q, C^\circ_{Q|P}(s, \nu)\psi, s\nu) = \mathrm{Wh}^\circ(P, \psi, \nu), \quad (\nu \in \mathfrak{a}^*_{P\mathbb{C}}).$$

Corollary $\mathcal{F}_Q(f)(s\nu) = C^\circ_{Q|P}(s, \nu)\mathcal{F}_P f(\nu)$

Wave packets

Definition For $P \in \mathcal{P}_{\rm st}$, $\psi \in \mathcal{S}(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}$, $x \in G$,

$$\mathcal{W}_P(\psi)(x) := \int_{i\mathfrak{a}_P^*} \mathrm{Wh}^\circ(P, \psi(\nu), \nu, x)\, d\nu.$$

Theorem ($\sim$)

$$\mathcal{W}_P : \mathcal{S}(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P} \to \mathcal{C}(\tau : G/N_0 : \chi)$$

is continuous linear.

Remark HC proves this for $\mathcal{W}_P$ restricted to dense subspace.

Proof requires

- the uniformly tempered estimates
- theory of constant term with parameter
- families of type $\mathrm{II}_{\rm hol}(\Lambda)$ (as in previous joint work with Carmona and Delorme for reductive symmetric space G/H).

Fourier transform of a wave packet

Lemma (adjoints)

$\langle \mathcal{W}_P\psi, f \rangle = \langle \psi, \mathcal{F}_P f \rangle \quad (\psi \in \mathcal{S}(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}, f \in \mathcal{C}(\tau : G/N_0 : \chi)).$

Let $P \in \mathcal{P}_{\rm st}$ and put $c_P = [W(\mathfrak{a}) : N_{W(\mathfrak{a})}(\mathfrak{a}_P)]$.

Lemma (projection)

(a) *If $Q \in \mathcal{P}_{\rm st}$, $Q \not\sim P$ then $\mathcal{F}_Q\mathcal{W}_P = 0$.*

(b) $\Pi_P := c_P\mathcal{F}_P\mathcal{W}_P$ *defines a projection operator in $\mathcal{S}(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}$. Moreover,*

$$\Pi_P \circ \mathcal{F}_P = \mathcal{F}_P.$$

Proof: Put $\mathcal{T} = \mathcal{F}_Q\mathcal{W}_P$ and use that $\mathcal{T} \circ \mu_P(Z) = \mu_Q(Z) \circ \mathcal{T}$ for all $Z \in \mathfrak{Z}$. $\mathcal{T}$ must be of the form

$$\mathcal{T}(\psi)(\nu) = \sum_{s \in W(\mathfrak{a}_Q|\mathfrak{a}_P)} T_s(\nu)\psi(s^{-1}\nu),$$

where $T_s(\nu) \in \mathrm{Hom}(\mathcal{A}_{2,P}, \mathcal{A}_{2,Q})$.
Next, use the asympt ^{cs} of the Whittaker integrals and the MS rel ^{s}.

Lemma *Let $P \in \mathcal{P}_{\mathrm{st}}$. Then $c_P \mathcal{W}_P \mathcal{F}_P \in \mathrm{End}(\mathcal{C}(\tau : G/N_0 : \chi))$ depends on P through its class $[P]$ in $\mathcal{P}_{\mathrm{st}}/\sim$.*

Proof This follows from the MS relations.

Lemma *Let $P, Q \in \mathcal{P}_{\mathrm{st}}$. Then $\mathcal{F}_Q c_P \mathcal{W}_P \mathcal{F}_P = \delta_{[Q],[P]}\, \mathcal{F}_Q$.*

Proof If $[P] \neq [Q]$, use Lemma (projection) (a). If $P \sim Q$, then by (b),

$$\mathcal{F}_Q c_P \mathcal{W}_P \mathcal{F}_P = \mathcal{F}_Q c_Q \mathcal{W}_Q \mathcal{F}_Q = \Pi_Q \circ \mathcal{F}_Q = \mathcal{F}_Q.$$

Define

$$\ker \mathcal{F} := \bigcap_{P \in \mathcal{P}_{\mathrm{st}}} \ker \mathcal{F}_P \quad \subset \quad \mathcal{C}(\tau : G/N_0 : \chi).$$

Thm *For all $f \in \mathcal{C}(\tau : G/N_0 : \chi)$,*

$$f - \sum_{[P] \in \mathcal{P}_{\mathrm{st}}/\sim} c_P \mathcal{W}_P \mathcal{F}_P f \in \ker \mathcal{F}.$$

Proof Fix $Q \in \mathcal{P}_{\mathrm{st}}$. Then

$$\mathcal{F}_Q \sum_{[P] \in \mathcal{P}_{\mathrm{st}}/\sim} c_P \mathcal{W}_P \mathcal{F}_P f = \sum_{[P] \in \mathcal{P}_{\mathrm{st}}/\sim} \delta_{[P],[Q]} \mathcal{F}_Q f = \mathcal{F}_Q f.$$

The Parseval identity

Thm *If $f \perp \ker \mathcal{F}$ then*

$$\|f\|_2^2 = \sum_{P \in \mathcal{P}_{\mathrm{st}}/\sim} c_P \int_{i\mathfrak{a}_P^*} \|\mathcal{F}_P f(\nu)\|^2 d\nu.$$

Proof From $f \perp \ker \mathcal{F}$ it follows that

$$\langle\, f, f \,\rangle = \sum_{P \in \mathcal{P}_{\mathrm{st}}/\sim} \langle\, c_P \mathcal{W}_P \mathcal{F}_P f, f \,\rangle = \sum_{P \in \mathcal{P}_{\mathrm{st}}/\sim} \langle\, c_P \mathcal{F}_P f, \mathcal{F}_P f \,\rangle.$$

In order to establish Parseval it is now sufficient to prove

Thm $\ker \mathcal{F} = 0$.

Proofs There are three proofs.

Three proofs of injectivity of $\mathcal{F}$

HC's philosophy of cusp forms

Let $f \in \ker \mathcal{F}$ and $P \in \mathcal{P}_{\rm st}$.

- From $\mathcal{F}_P f = 0$ it follows that $f^{(\bar{P})} \sim 0$ where $f^{(\bar{P})}$ indicates the descent transformation.
- By a result of HC it now follows that $f = 0$.

Starting with Plancherel for the group

Wallach's Whittaker inversion formula implies $\ker \mathcal{F} = 0$.

Residue method

The residue method for semisimple symmetric spaces ($\sim$ & Schlichtkrull) can be adapted to the present Whittaker setting.

It starts with an inversion formula for $f \in C_c^\infty(\tau : G/N_0 : \chi)$, from $\mathcal{F}_{P_0} f \in \mathcal{M}(\mathfrak{a}_{\mathbb{C}}^*) \otimes \mathcal{A}_{2,P_0}$, where $P_0 = MAN_0$. Note that $\mathcal{A}_{2,P_0} = C^\infty(\tau_0 : M)$.

The residue method

Lemma $\exists!\ \mathrm{Wh}_+(P_0,\nu) \in C^\infty(\tau : G/N_0 : \chi) \otimes \mathcal{A}^*_{2,P_0}$, *meromorphic in* ν, *finite under the action of* $\mathfrak{Z}$, *such that*

$$\mathrm{Wh}_+(P_0,\nu)(ma)\psi = \tau(m)\, a^{\nu-\rho_0} \sum_{\xi\in\mathbb{N}\Delta} a^{-\xi}\Gamma_\xi(\nu)(\psi)(e)$$

with $\Gamma_\xi(\nu) \in \mathrm{End}(\mathcal{A}_{2,P_0})$, $\Gamma_0(\nu) = I_{\mathcal{A}_{2,P_0}}$. *The series is convergent for all* $a \in A$.

Fourier inversion theorem *For all* $f \in C_c^\infty(\tau : G/N_0 : \chi)$ *and for* $\eta \in \mathfrak{a}^*$ *sufficiently* $\bar{P}_0$*-dominant*

$$f(x) = \int_{\eta + i\mathfrak{a}^*} \mathrm{Wh}_+(P_0,\nu)(x)\mathcal{F}_{P_0}f(\nu)\, d\nu, \qquad (x \in G).$$

Proof uses a PW-technique and Holmgren's uniqueness thm for PDE.

Shifting $\eta \to 0$ and organizing the residues, the integral may be rewritten as

$$\sum_{[P]\in\mathcal{P}_{\mathrm{st}}/\sim} c_P \int_{i\mathfrak{a}_P^*} \mathrm{Wh}^\circ(P, \mathcal{F}_P f(\nu), x)\, d\nu.$$

hence $f = \sum_{[P]} c_P \mathcal{W}_P \mathcal{F}_P(f)$, $\quad (f \in C_c^\infty(\tau : G/N_0 : \chi))$.
By continuity and density, formula extends to $f \in \mathcal{C}(\tau : G/N_0 : \chi)$.

Mes félicitations!