A comparison of Paley-Wiener theorems

for groups and symmetric spaces

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Outline

Euclidean space

Setting

- $\mathfrak{a}$: finite dimensional Euclidean space;
- Fourier transform $\mathcal{F} : C_c^\infty(\mathfrak{a}) \to \mathcal{O}(\mathfrak{a}^*_{\mathbb{C}})$,

$$\mathcal{F}f(\lambda) = \int_{\mathfrak{a}} f(X) e^{-\lambda(X)}\, dX;$$

- $C_R^\infty(\mathfrak{a}) := \{f \in C_c^\infty(\mathfrak{a}) \mid \operatorname{supp} f \subset B(0; R)\}$.

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Definition (Paley-Wiener space)

$$\mathrm{PW}_R(\mathfrak{a}) := \{\varphi \in \mathcal{O}(\mathfrak{a}_\mathbb{C}^*) \mid \sup(1+|\lambda|)^N e^{-R|\mathrm{Re}\lambda|}|\varphi(\lambda)| < \infty\}.$$

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Paley-Wiener theorem

$$\mathcal{F} : \quad C_R^\infty(\mathfrak{a}) \xrightarrow{\simeq} \mathrm{PW}_R(\mathfrak{a}) \qquad \text{(topol. lin. isom. onto).}$$

Non-compact Riemannian symmetric space

Setting

- G semisimple, connected, finite center;
- K maximal compact;
- $G = KAN$ Iwasawa decomposition;
- $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}), \quad \Sigma^+$ positive system;
- A^+ positive chamber;
- W : Weylgroup.

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Polar decomposition

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Spherical Fourier transform

Spherical principal series

- $P = MAN$ minimal parabolic subgroup
- $\pi_\lambda := \mathrm{Ind}_P^G(1 \otimes -\lambda \otimes 1)$ in $\Gamma_{\mathrm{L}^2}(G \times_P \mathbb{C}_{-\lambda+\rho})$
- K-fixed element: $1_\lambda(kan) = a^{\lambda-\rho}$
- π_λ is realizable in $L^2(K/M)$ (since $G \simeq K \times_M P$).

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$$\mathcal{F} : C_c^\infty(G/K)_K \to \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*) \otimes L^2(K/M)_K,$$

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Note

$\mathcal{F} : C_c^\infty(K\backslash G/K) \to \mathcal{O}(\mathfrak{a}_\mathbb{C}^*) \otimes \mathbb{C} \simeq \mathcal{O}(\mathfrak{a}_\mathbb{C}^*)$.

Helgason's Paley-Wiener theorems

Theorem (Helgason '66, estimates by Gangolli '71)

$$\mathcal{F}: \; C_R^\infty(K\backslash G/K) \xrightarrow{\simeq} \mathrm{PW}_R(\mathfrak{a})^W.$$

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Definition (Paley-Wiener space)

- $\mathrm{PW}_R(G/K) = \{\varphi \in \mathrm{PW}_R(\mathfrak{a}) \otimes L^2(K/M)_K \mid \varphi(w\lambda) = A_w(\lambda)\varphi(\lambda) \quad \forall w \in W\}$
- where $A_w(\lambda) \in \mathrm{End}(L^2(K/M)_K)$ normalized standard intertwiner (coeffs rational in λ).

The group

Minimal principal series

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  $V(\xi) := \{f \in L^2(K) \otimes \mathcal{H}_\xi \mid f(km) = \xi(m)^{-1} f(k)\}$.

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$$\mathcal{F} : C_R^\infty(G)_{K\times K} \to \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*) \otimes \mathrm{End}(*).$$

where $\mathrm{End}(*) := [\ \oplus_\xi \mathrm{End}\, V(\xi)\ ]_{K\times K}$.

$$\mathcal{F}f(\lambda)_\xi := \pi_{\xi,\lambda}(f) \in \mathrm{End}(V(\xi))_{K\times K}$$

for $f \in C_c^\infty(G)_{K\times K}$, $\xi \in \widehat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$,

Arthur's Paley-Wiener space

Definition (Taylor functionals)

$$\mathcal{O}(\mathfrak{a}^*_{\mathbb{C}})^*_{\mathrm{tayl}} := \{\mathcal{L} : \ \mathfrak{a}^*_{\mathbb{C}} \to S(\mathfrak{a}) \mid \mathrm{supp}\ \mathcal{L} \ \ \text{finite}\}.$$

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$$\mathrm{AC}(G, K) = \{\mathcal{L} \in \mathcal{O}(\mathfrak{a}_{\mathbb{C}}^*)^*_{\mathrm{tayl}} \otimes \mathrm{End}(*)^*_{K \times K} \mid \\ \mathcal{L} = 0 \ \text{ on } \ \lambda \mapsto (\pi_{\xi,\lambda}(x) \mid \xi \in \widehat{M})\}.$$

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PW space

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PW theorems for the group

Theorem (Arthur '82, Campoli for split rank one)

$$\mathcal{F} : C_c^\infty(G, K) \xrightarrow{\simeq} \mathrm{PW}_{\mathrm{Arth}}(G, K).$$

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- Helgason '66, '73: G/K (not a special case);
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- Delorme '82: one conjugacy class of Cartan;
- Arthur '82: general G;
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- vdB & Schlichtkrull '03: generalization to $C_c^\infty(G/H)_K$;
- Delorme '04: general G, different formulation & proof;
- vdB & Schlichtkrull '05: generalization to $\mathcal{E}'(G/H)_K$.

Families of representations

Differentiation of family

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- $\varphi \in \mathrm{PW}(\mathfrak{a}) \otimes \mathrm{End}(*)$ yields $\varphi^{(D)} \in \mathrm{End}(V^{(D)})$.

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Theorem (Delorme '05)

$$\mathcal{F} : C_c^\infty(G)_{K\times K} \xrightarrow{\simeq} \mathrm{PW}_{\mathrm{Del}}(G, K).$$

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Prove that

$$\mathrm{PW}_{\mathrm{Arth}}(G, K) = \mathrm{PW}_{\mathrm{Del}}(G, K)$$

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Involves three key ideas

(1) Use jets instead of repeated differentiations[1];

(2) Reformulate Delorme's intertwining conditions;

(3) Use the Hecke algebra (multipl'n = convolution)

$$\mathbb{H}(G, K) := \mathcal{E}'_K(G)_{K \times K}.$$

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Jet space

- $\mathcal{O}_0$: space of germs at 0 of holomorphic functions on $\mathfrak{a}_{\mathbb{C}}^*$;
- $\mathfrak{m}_0 \lhd \mathcal{O}_0$: unique maximal ideal;
- $I \lhd \mathfrak{m}_0$: ideal such that $\exists k$: $\mathfrak{m}_0^k \subset I \subset \mathfrak{m}_0$.
- Jet space: $\mathcal{O}_0/I$ (finite dimensional $/\mathbb{C}$).

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- $\pi_{\xi,\lambda}, \ V(\xi) \ \rightsquigarrow \ \pi_{\xi,\lambda}^{(I)}, \ \mathcal{O}_0/I \otimes V(\xi)$.

Reformulation of Delorme's intertwining conditions

Direct sums of jets

Datum: $I \lhd \mathcal{O}_0$, $\xi_1, \ldots, \xi_n \in \widehat{M}$, $\mu_1, \ldots, \mu_n \in \mathfrak{a}^*_{\mathbb{C}}$;

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Definition

For V a Harish-Chandra module:

$$\mathrm{End}(V)^{\#} := \{\varphi \in \mathrm{End}(V)_{K\times K} \mid \ \forall n \forall U < V^{\oplus n} : \ \ \varphi^{\oplus n}(U) \subset U\}.$$

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Delorme's condition: $\varphi \in \mathrm{PW}(\mathfrak{a}) \otimes \mathrm{End}(*)$ belongs to $\mathrm{PW}_{\mathrm{Del}}(G,K)$ if and only if for all data (I, ξ, μ) :

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Reformulation of Arthur-Campoli conditions

Definition

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$$\pi(G)^{\perp} = \{\psi \in \mathrm{End}(V)^*_{K\times K} \mid \forall g \in G : \langle \psi, \pi(g) \rangle = 0\}.$$

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Lemma (Arthur-Campoli conditions)

$\varphi \in \mathrm{PW}(\mathfrak{a}) \otimes \mathrm{End}(*)$ belongs to $\mathrm{PW}_{\mathrm{Arth}}(G, K)$ if and only if for all data (I, ξ, μ),

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Category of Harish-Chandra modules $\simeq$ category of finite modules for $\mathbb{H}(G, K)$ $(= \mathcal{E}'_K(G)_{K\times K} \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} \mathcal{E}'(K)_{K\times K})$;

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Theorem (vdB - Souaifi)

Let (π, V) be a Harish-Chandra module. Then

- $\mathrm{End}(V)^{\#} = \pi(\mathbb{H}(G,K))$;
- $\pi(G)^{\perp\perp} = \pi(\mathbb{H}(G,K))$.

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Proof. For all data (I, ξ, μ),

$$\varphi^{(I)}_{\xi,\mu} \in \mathrm{End}(\oplus_j V(\xi_j)^{(I)})^{\#} \iff \varphi^{(I)}_{\xi,\mu} \in \pi^{(I)}_{\xi,\mu}(G)^{\perp\perp}.$$

$\square$

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Lemma

(a) $\mathcal{F}(\mathrm{pr}_1 \mathbb{H}(G,K) * \delta_K) = P(\mathfrak{a}^*)^W$.

(b) $\mathcal{F}(\mathbb{H}(G,K) * \delta_K) = \{\varphi \in P(\mathfrak{a}^*) \otimes L^2(K/M)_K \mid \forall w \in W : \ \varphi(w\lambda) = A_w(\lambda)\varphi(\lambda)\}$.

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- Proof of (b) uses Helgason's thm.
- Arthur's thm + Lemma $\Longrightarrow$ Helgason's thm.

Dear Siggi

Thanks for inspiring us with your beautiful math

Happy Birthday!

Ann. Math. 1973

The surjectivity of invariant differential operators on symmetric spaces I

Dedicated to Leifur Ásgeirsson on the occasion of his 70th birthday.

By SIGURDUR HELGASON*

1. Introduction

The principal result of this paper is that if D is an invariant differential operator on a symmetric space X of the noncompact type then for each function $f \in C^\infty(X)$ the differential equation $Du = f$ has a solution $u \in C^\infty(X)$ (Theorem 8.2). The proof which is given in §§ 3–8 can be outlined as follows.

In § 3 we show (Theorem 3.3) that D is surjective on $C^\infty(X)$ if for each closed ball $V \subset X$ the implication

(1) $$f \in C_c^\infty(X), \qquad \operatorname{supp}(Df) \subset V \Longrightarrow \operatorname{supp}(f) \subset V$$

holds. (Here supp = support and subscript c means compact support.) This is based on the existence of a fundamental solution for D ([10(d)]) and on the elementary Corollary 3.2, which depends on the symmetry of $X = G/K$.

The Paley-Wiener theorem

THEOREM 8.3. *The Fourier transform* $f(x) \to \tilde{f}(\lambda, b)$ *is a bijection of* $\mathcal{D}(X)$ *onto the set of holomorphic functions* $\psi(\lambda, b)$ *of uniform exponential type satisfying the identities*

$$\int_B e^{(is\lambda+\rho)(A(x,b))}\psi(s\lambda, b)db = \int_B e^{(i\lambda+\rho)(A(x,b))}\psi(\lambda, b)db \tag{9}$$

for $s \in W$, $\lambda \in \mathfrak{a}_c^*$, $x \in X$. *Moreover* $\tilde{f}$ *satisfies* (8) *if and only if* f *has support in the closed ball* $V^R(o)$.

Proof. Because of formulas (7) and (9) in § 4 only the "onto" part remains to be verified. In other words we must show that if $\psi \in \mathcal{E}(\mathfrak{a}_c^* \times B)$ is holomorphic in the first variable and satisfies (8) and (9) above, then there exists $f \in \mathcal{D}(X)$ with support in $V^R(o)$ such that $\tilde{f} = \psi$. Because of the Fourier inversion formula (10) § 4 we define f on X by

$$f(x) = w^{-1}\int_{\mathfrak{a}^*\times B} \psi(\lambda, b)e^{(i\lambda+\rho)(A(x,b))}\,|\,\boldsymbol{c}(\lambda)\,|^{-2}d\lambda db\,. \tag{10}$$

Then $f \in \mathcal{E}(X)$. The next thing to prove is that $f(x) = 0$ for $d(o, x) > R$. For this we use the method of Lemma 8.1. For $\delta \in \hat{K}$ we derive from (10)

$$f^\delta(x) = w^{-1}\int_{\mathfrak{a}^*}\left(\int_K e^{(i\lambda+\rho)(A(x,kM))}\delta(k)dk\right)\psi^\delta(\lambda)\,|\,\boldsymbol{c}(\lambda)\,|^{-2}d\lambda\,, \tag{11}$$