# Plancherel formulas for reductive groups, symmetric spaces and Whittaker functions

## II. Spherical functions and Fourier inversion

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# Plancherel identity

**Definition Fourier transform**

For $f \in C_c^\infty(G/H, \chi)$, the Fourier transform $\hat{f}(P, \sigma, \nu)$ is the element of $(\mathcal{V}^\chi_{\sigma, ds})^* \otimes L^2(K/K_P : \sigma_P)$, defined by

$$\hat{f}(P, \sigma, \nu)(\eta) := \int_{G/H} f(x)\, \pi_{P,\sigma,-\nu}(x)\, j^\circ(P, \sigma, -\nu)(\eta)\, dx$$

**Theorem (Plancherel)**

$$\|f\|^2_{L^2} = \sum_{P \in \mathcal{P}_{\mathrm{st}}/\sim}\ \sum_{\sigma \in (M_P)^\wedge_\chi} \int_{i\,\mathfrak{a}_P^*} \|\hat{f}(P, \sigma, \nu)\|^2\, d\lambda_P(\nu)$$

**Strategy**

Prove identity on the dense subspace $C^\infty(G/H, \chi)_K$ of $K$-finite functions. Technical tool: sphericalization. Let $(\tau, V_\tau)$ be an arbitrary finite dimensional unitary representation of $K$. Suff$^t$ to prove the result for functions in $(C_c^\infty(G/H, \chi) \otimes V_\tau)^K$.

# $\tau$-spherical functions

Definition For $X$ a left $K$-manifold:

$$\begin{aligned} C^\infty(\tau : X) : &= \{f : X \to V_\tau \mid f(kx) = \tau(k)f(x)\} \\ &\simeq (C^\infty(X) \otimes V_\tau)^K. \end{aligned}$$

Likewise: $C_c^\infty(\tau : G/H : \chi) \simeq (C_c^\infty(G/H : \chi) \otimes V_\tau)^K$.

By triviality on tensor component $V_\tau$, Fourier transform becomes

$$\begin{array}{ccc} C_c^\infty(\tau : G/H : \chi) & \xrightarrow{\mathrm{ft}_{P,\sigma,\nu}} & \mathcal{V}^*_{\sigma,ds} \otimes L^2(\tau : K/K_P : \sigma_P) \\ & & \downarrow I \otimes \mathrm{ev}_e \\ \mathcal{F}_{P,\sigma,\nu} & \searrow & \mathcal{V}^*_{\sigma,ds} \otimes (\mathcal{H}_\sigma \otimes V_\tau)^{K_P} \\ & & \downarrow \ \simeq \text{(matrix coefficient)} \\ & & \oplus_{v \in {}_P\mathcal{W}} \ L^2_\sigma(\tau_P{:}M_P/M_P \cap vHv^{-1}{:}(v\chi)_P) \\ & & := \mathcal{A}_{2,P,\sigma} \end{array}$$

Notation: $T \mapsto \psi_T$ for composition of vertical maps (isometric).

Assumption: (to simplify exposition) ${}_P\mathcal{W} = \{1\}$ (automatic for group, Riemannian symmetric , complex symmetric, Whittaker case). Then

$$\mathcal{A}_{2,P,\sigma} = L^2_\sigma(\tau_P : M_P/M_P \cap H : \chi_P).$$

# Plancherel identity for spherical functions

**Definition**

$$\begin{aligned}\mathcal{A}_{2,P} &= \oplus_{\sigma \in \widehat{M}^{\chi}_{P,ds}} \mathcal{A}_{2,P,\sigma} \\ &= L^2_{ds}(\tau_P : M_P / M_P \cap H : \chi_P)\end{aligned}$$

**Lemma** $\mathcal{A}_{2,P}$ is finite dimensional
(gp: HC, ss: Oshima-Matsuki, wh: HC, Wallach).

**Definition** $\mathcal{F}_P : C_c^\infty(\tau : G/H : \chi) \to C^\omega(i\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}$ by

$$\mathcal{F}_P(f)(\nu) := \oplus_{\sigma \in \widehat{M}^{\chi}_{P,ds}} \mathcal{F}_{P,\sigma,\nu}(f).$$

**Plancherel identity** is equivalent to

$$\|f\|^2_{L^2} = \sum_{P \in \mathcal{P}_{\mathrm{st}}/\sim} \int_{i\mathfrak{a}_P^*} \|\mathcal{F}_P f(\nu)\|^2 d\lambda_P(\nu), \qquad (f \in C_c^\infty(\tau : G/H : \chi)).$$

# Normalized Eisenstein, Whittaker integrals

**Definition**
$E^\circ(P,\psi,\nu) \in C^\infty(\tau : G/H : \chi)$ is linear in $\psi \in \mathcal{A}_{2,P}$. For $\psi = \psi_T$ with $T = \eta \otimes \varphi \in \mathcal{V}_\sigma^\chi \otimes L^2(\tau_P : K/K_P : \chi_P)$ it is given as matrix coefficient

$$E^\circ(P,\psi_T,\nu,x) = \langle\, \varphi, \pi_{P,\sigma,\bar\nu}(x) j^\circ(P,\sigma,\bar\nu)\eta \,\rangle.$$

**Remark** In the Whittaker case, Harish-Chandra calls this the normalized Whittaker function

**Lemma**

$$\langle\, \mathcal{F}_P f(\nu), \psi \,\rangle = \int_{G/H} f(x)\overline{E^\circ(P,\psi,-\bar\nu,x)}\, dx = \langle\, f, E^\circ(P,\psi,-\bar\nu) \,\rangle$$

**Lemma** $E^\circ(P,\psi,\nu)$ depends meromorphically on $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$. For generic $\nu$ it satisfies the following differential equations

$$R_Z E^\circ(P,\psi,\nu) = E^\circ(P,\underline{\mu}_P(Z,\nu)\psi,\nu), \qquad (Z \in \mathfrak{Z}(\mathfrak{g})).$$

Here $\underline{\mu}_P(Z,\nu) \in \mathrm{End}(\mathcal{A}_{2,P})$ is polynomial in $\nu$, algebra homomorphism in $Z$.

# C-functions, Maass-Selberg relations

**Asymptotic behavior** Let $P, Q \in \mathcal{P}_{\rm st}$. There exist unique meromorphic functions $C^\circ_{Q|P}(s, \cdot\,) : \mathfrak{a}^*_{P\mathbb{C}} \to \mathrm{Hom}(\mathcal{A}_{2,P}, \mathcal{A}_{2,Q})$, for $s \in W({}_{\rm p}\mathfrak{a}_Q \mid {}_{\rm p}\mathfrak{a}_P)$ such that for generic $\nu \in i\,{}_{\rm p}\mathfrak{a}^*_P$ and $a \to \infty$ in ${}_P A^+_Q$.

$$E^\circ(P, \psi, \nu)(kam) \sim \sum_{s \in W({}_{\rm p}\mathfrak{a}_Q | {}_{\rm p}\mathfrak{a}_P)} a^{s\nu - \rho_Q}[C^\circ_{Q|P}(s, \nu)\psi](m), \quad (m \in M_P)$$

**Maass-Selberg relations** $C^\circ_{Q|P}(s, -\bar{\nu})^* C^\circ_{Q|P}(s, \nu)$ indep$^t$ of $Q, s$.

(gp: HC, ss: vdB, Delorme-Carmona, wh: HC)

**Lemma** $C^\circ_{P|P}(1, \nu) = \mathrm{id}_{\mathcal{A}_{2,P}}$.

**Proof** For $\mathrm{Re}\,\nu$ sufficiently dominant in ${}_P\mathfrak{a}^{*+}_P$, Langlands' limit formula for matrix coefficients of $\mathrm{Ind}^G_{\bar{P}}(\sigma \otimes \bar{\nu})$ gives ($\psi = \psi_T$, $T = \eta \otimes \varphi$), for $a \to \infty$ in ${}_{\rm p}A^+_P$ that

$$\begin{aligned} a^{-\nu+\rho_P} E^\circ(P, \psi, \nu, am) &= a^{-\nu+\rho_P}\langle\, [A(\cdots)^{-1}\varphi], \pi_{\bar{P},\sigma,\bar{\nu}}(ma) j(\bar{P}, \sigma, \bar{\nu})\eta\,\rangle \\ &\sim \langle\, A(\cdots)[A(\cdots)^{-1}\varphi](m), \mathrm{ev}_e j(\bar{P}, \sigma, \bar{\nu})\eta\,\rangle \\ &= \langle\, \varphi(m), \eta\,\rangle = \psi(m). \end{aligned}$$

# Regularity

**Corollary** For $P, Q \in \mathcal{P}_{\mathrm{st}}$, $s \in W({}_{\mathrm{P}}\mathfrak{a}_Q \mid {}_{\mathrm{P}}\mathfrak{a}_P)$,

$$C^{\circ}_{Q|P}(s, -\bar{\nu})^* C^{\circ}_{Q|P}(s, \nu) = \mathrm{id}_{\mathcal{A}_{2,P}}.$$

In particular, $C^{\circ}_{Q|P}(s, \nu) \in \mathrm{U}(\mathcal{A}_{2,P}, \mathcal{A}_{2,Q})$ for $\nu$ imaginary.

**Corollary** The meromorphic functions $\nu \mapsto C^{\circ}_{Q|P}(s, \nu)$ are regular on $i\,{}_{P}\mathfrak{a}^*_P$.

**Remark** This implies that $E^{\circ}(P, \psi, \nu)$ is regular for imaginary $\nu$, hence that $j^{\circ}(P, \sigma, \nu)$ is regular for such $\nu$.

# Extension to the Schwartz space

Recall that $\mathcal{C}(G/H:\chi)$ is the space of functions $f \in C^\infty(G/H:\chi)$ such that

$$w^N L_u f \in L^2(G/H:\chi) \qquad (u \in U(\mathfrak{g}), w \in \mathbb{N}).$$

Here $w(kah) = (1 + |\log a|)$, for $a \in {}_{\mathrm{p}}A$.

Let $\mathcal{S}(i\,{}_{\mathrm{p}}\mathfrak{a}_P^*)$ denote the usual space of Schwartz functions on the finite dimensional real linear space $i\,{}_{\mathrm{p}}\mathfrak{a}_P^*$.

**Theorem** For each $P \in \mathcal{P}_{\mathrm{st}}$ the map $\mathcal{F}_P$ is continuous linear

$$\mathcal{C}(\tau : G/H:\chi) \to \mathcal{S}(i\,{}_{\mathrm{p}}\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}.$$

**Proof** for gp: HC, for ss: vdB, Carmona–Delorme, for wh: vdB. The following strategy works in all cases.

(a) the generalized vector map $j(\bar{P}:\sigma:\nu)$ is defined for $\mathrm{Re}\,\nu$ sufficiently $P$-dominant.

(b) derive a Bernstein-Sato type functional equation for $j(\bar{P}:\sigma:\nu)$

# Extension to the Schwartz space, II

**Theorem** $\mathcal{F}_P\colon \mathcal{C}(\tau : G/H : \chi) \to \mathcal{S}(i\,{}_{\mathrm{p}}\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P}$ is cont$^s$ linear.

**Strategy of Proof**

(a) the generalized vector map $j(\bar{P} : \sigma : \nu)$ is defined for $\mathrm{Re}\nu$ sufficiently $P$-dominant.

(b) derive a Bernstein-Sato type functional equation for $j(\bar{P} : \sigma : \nu)$

(c) use (b) to extend $j(\bar{P} : \sigma : \nu)$ meromorphically. Singular set is a locally finite union of real translates of root hyperplanes. Gives estimates for $j(\bar{P} : \sigma : \nu)$ with uniformity for $\mathrm{Re}\nu$ in translates of the cone of $P$-dominant elements.

(d) get moderate estimates for $E^\circ(P : \sigma : \nu)$ on $G/H$ which are of the type of uniformity mentioned in (c).

(e) use estimate improvements by repeated application of the differential equations coming from $\mathfrak{Z}(\mathfrak{g})$.

(f) estimates lead to uniformly tempered estimates in the range $\nu \in i\,{}_{\mathrm{p}}\mathfrak{a}_P^*$, hence to estimates for $\langle \mathcal{F}_P f, \psi \rangle = \langle f, E^\circ(P, \psi, \nu) \rangle$.

# Wave packets, Spherical Fourier inversion

**Definition** For $P \in \mathcal{P}_{\rm st}$ define $\mathcal{W}_P : \mathcal{S}(i\,_{\rm p}\mathfrak{a}_P^*) \otimes \mathcal{A}_{2,P} \to C^\infty(\tau : G/H : \chi)$ by

$$\mathcal{W}_P(\psi)(x) = \int_{i\,_{\rm p}\mathfrak{a}_P^*} E^\circ(P, \psi(\nu), \nu, x)\, d\lambda_P(\nu).$$

**Theorem** $\mathcal{W}_P$ maps continuously to $\mathcal{C}(\tau : G/H : \chi)$.

(gp: HC, ss: vdB–C–D, wh: vdB).

**Proof** In all cases: a theory of the constant term with parameters: holomorphic version of HC's functions of type $\mathrm{II}(\lambda)$.

**Lemma** The composition $\mathcal{W}_P\mathcal{F}_P$ depends on $P$ through $[P] \in \mathcal{P}_{\rm st}/\sim$

(consequence of Maass-Selberg relations).

**Lemma** $\mathcal{F}_P$ and $\mathcal{W}_P$ are adjoint.

Since $\|\mathcal{F}_P f\|^2 = \langle\, f, \mathcal{W}_P\mathcal{F}_P f\,\rangle$ the spherical Plancherel identity follows from:

**Theorem: spherical fourier inversion**

$$I = \sum_{P \in \mathcal{P}_{\rm st}/\sim} \mathcal{W}_P\mathcal{F}_P \qquad \text{on } \mathcal{C}(\tau : G/H : \chi) \qquad \text{(SFI)}.$$

**Final part of the talk:** sketch of proof for both ss (vdB–S) and wh (vdB).

# Cone supported functions

There exists an open polyhedral cone ${}_{\mathrm{p}}\mathfrak{a}^+$ such that (${}_{\mathrm{p}}A^+ = \exp({}_{\mathrm{p}}\mathfrak{a}^+)$)

$$G_+ := K{}_{\mathrm{p}}A^+H = K\exp({}_{\mathrm{p}}\mathfrak{a}^+)H \qquad \text{open dense in } G.$$

Cases:

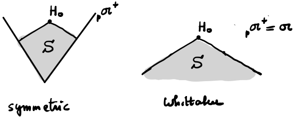
(a) Symmetric space: ${}_{\mathrm{p}}A^+$ is positive chamber for $\Sigma^+({}_{\mathrm{p}}\mathfrak{a})$.

(b) Group: ${}_{\mathrm{p}}\mathfrak{a}^+ = \underline{\mathfrak{a}}^+ \times -\underline{\mathfrak{a}}^+$.

(c) Whittaker: ${}_{\mathrm{p}}A^+ = A$.

Notation

- $\mathcal{C} \subset {}_{\mathrm{p}}\mathfrak{a}$ is the cone dual to ${}_{\mathrm{p}}\mathfrak{a}^+(P_0)$.
- $C^\infty_{cs}(G/H : \chi)$ is the collection of $f \in C^\infty(G/H : \chi)$ such that there exists a subset of ${}_{\mathrm{p}}\mathfrak{a}$ of the form $S_X := \mathrm{cl}((X - \mathcal{C}) \cap {}_{\mathrm{p}}\mathfrak{a}^+)$ such that $\mathrm{supp} f \subset K\exp(S_X)H$.




Remark For ss: $C^\infty_{cs}(G/H : \chi) = C^\infty_c(G/H)$. For wh: not the case.

# Series expansions

Let $P_0 = M_0 A_0 N_0$ be the minimal element in $\mathcal{P}_{\rm st}$. Then $M_0/M_0 \cap H$ is compact, so $\sigma \in \widehat{M}^{\chi}_{0,ds} \implies \dim(\sigma) < \infty$.

First step towards proof of (SIF): investigation of $\mathcal{W}_0\mathcal{F}_0 = \mathcal{W}_{P_0}\mathcal{F}_{P_0}$.

Recall:

$$G_+ = K_{\rm p} A^+ H \qquad \text{open dense in } G.$$

Theorem: There exists unique functions $E_+(\nu) \in \mathcal{A}^*_{2,0} \otimes C^\infty(\tau : G_+/H : \chi)$ depending meromorphically on $\nu \in {}_{\rm p}\mathfrak{a}^*_{\mathbb{C}}$ such that, for $\psi \in \mathcal{A}_{2,0} = \mathcal{A}_{2,P_0}$,

$$E(P_0, \psi, \nu)(x) = \sum_{s \in W({}_{\rm p}\mathfrak{a})} E_+(s\nu, x) C^\circ(s : \nu)(\psi), \qquad (x \in G_+/H).$$

$$E_+(\nu, a)(\psi) = a^{\nu - \rho} \sum_{m \in \mathbb{N}\Sigma^+({}_{\rm p}\mathfrak{a})} a^{-m} \Gamma_m(\nu)(\psi), \qquad (a \in {}_{\rm p}A^+).$$

Here $C^\circ(s, \nu) := C^\circ_{P_0|P_0}(s, \nu)$, $\Gamma_m(\nu) \in \mathcal{A}^*_{2,0} \otimes V_\tau$, and $\Gamma_0(\nu)(\psi) = \psi(e)$.

# Contour shift à la Helgason (G/K)

For $f \in C_c^\infty(\tau : G/H : \chi)$, $x \in G_+$,

$$\begin{aligned}
\mathcal{W}_0\mathcal{F}_0 f(x) &= \int_{i\,_{\mathrm{p}}\mathfrak{a}^*} \sum_{s \in W} E_+(s\nu, x) C^\circ(s : \nu)\mathcal{F}_0 f(\nu)\, d\lambda(\nu) \\
&= \sum_{s \in W} \int_{i\,_{\mathrm{p}}\mathfrak{a}^*} E_+(\nu, x) C^\circ(s : s^{-1}\nu)\mathcal{F}_0 f(s^{-1}\nu)\, d\lambda(\nu) \\
&= |W| \int_{i\,_{\mathrm{p}}\mathfrak{a}^*} E_+(\nu, x)\mathcal{F}_0(f)(\nu)\, d\lambda(\nu) \\
&= |W| \int_{i\,_{\mathrm{p}}\mathfrak{a}^* - \eta} E_+(\nu, x)\mathcal{F}_0(f)(\nu)\, d\lambda(\nu) + \text{residual integrals} \\
&= \mathcal{T}_\eta f(x) + \mathrm{ResInt}(f),
\end{aligned}$$

with $\eta \in {}_{\mathrm{p}}\mathfrak{a}^*$ sufficiently $P_0$-dominant. These residues are picked up along finitely many real translates of root hyperplanes. $R_Z$ acts by $\underline{\mu}(Z : \nu)$ in the integrals on the right. For suitable $Z_0 \in \mathfrak{Z}(\mathfrak{g})$ the residues are cancelled so that

$$R_{Z_0}\mathcal{W}_0\mathcal{F}_0 f(x) = R_{Z_0}\mathcal{T}_\eta f(x)$$

By sending $\eta \to \infty$ and applying a Paley-Wiener type estimation one concludes, for $f \in C_c^\infty(\tau : G_+/H : \chi)$,

$$\mathrm{supp}(f) \subset K\exp(S_X)H \implies \mathrm{supp} R_{Z_0}\mathcal{W}_0\mathcal{F}_0 f \subset K\exp(S_X)H.$$

# To be named

**Lemma** The operator $R_{Z_0}\mathcal{W}_0\mathcal{F}_0 \in \mathrm{End}(C_c^\infty(\tau : G_+/H : \chi))$ is support preserving.

**Proof:** By combining above with symmetry of the operator.

**Lemma** $R_{Z_0}\mathcal{W}_0\mathcal{F}_0 = R_{Z_0}$.

**Proof:**

- The radial part of the operator on the left is essentially a differential operator $D$ on ${}_{\mathrm{p}}A^+$.
- $D$ commutes with the radial parts of all $Z \in \mathfrak{Z}(\mathfrak{g})$.
- coefficients of $D$ satisfy cofinite system of differential equations, which makes that $D$ is determined by behavior at infinity.
- asymptotically, $D \sim \mathrm{rad}(R_{Z_0})$, hence $D = \mathrm{rad}(R_{Z_0})$.

**Theorem** For all $f \in C_c^\infty(\tau : G/H : \chi)$ and $\eta$ sufficiently $P_0$-dominant, one has

$$f = \mathcal{T}_\eta(f) \qquad \text{on} \quad G_+.$$

**Proof:**

- Induction $\rightsquigarrow \mathrm{ResInt}(f) \in C^\infty(\tau : G/H : \chi)$, hence $\mathcal{T}_\eta f \in C^\infty(\tau : G/H : \chi)$.
- By Paley-Wiener type estimation, $\mathcal{T}_\eta f \in C_{cs}^\infty(\tau : G/H : \chi)$.
- $\rightsquigarrow f - \mathcal{T}_\eta f \in C_{cs}^\infty(\tau : G/H : \chi)$.
- $\rightsquigarrow f - \mathcal{T}_\eta f$ is annihilated by the analytic linear partial differential operator $R_{Z_0}$.
- By Holmgren uniqueness, $f - \mathcal{T}_\eta f = 0$.

# Identification of Residual integrals

Have found:

$$\mathcal{W}_{P_0}\mathcal{F}_{P_0}f = \mathcal{T}_\eta f - \mathrm{ResInt}(f), \qquad \mathcal{T}_\eta f = f.$$

Corollary

$$f = \mathcal{W}_0\mathcal{F}_0 f + \mathrm{ResInt}(f).$$

One can organize the residue scheme so that it allows induction over *M*-components of parabolic subgroups. By comparison of asymptotic behavior along *A*-components, one obtains:

$$\mathrm{ResInt}(f) = \sum_{P \in \mathcal{P}_{\mathrm{st}}/\sim, P \neq P_0} \mathcal{W}_P\mathcal{F}_P f$$

This completes the proof of (SFI), hence of the Plancherel identity.