

Maass–Selberg relations for Whittaker integrals

Erik van den Ban

Utrecht University

62. Seminar Sophus Lie
FAU Erlangen–Nürnberg

Organizers:

Karl-Hermann Neeb,
Thomas Creutzig

Whittaker functions

Setting

- ▶ G real reductive group (HC class).
- ▶ K maximal compact, $G = KAN_0$ Iwasawa decomposition.
- ▶ $\chi : N_0 \rightarrow \mathbb{U}(1)$ unitary character, **regular (!)**

i.e.: $\forall \alpha \in \Sigma(\mathfrak{n}_0, \mathfrak{a})$ simple: $d\chi(\mathbf{e})|_{\mathfrak{g}_\alpha} \neq 0$.

Whittaker functions

$$\mathcal{M}(G/N_0 : \chi) := \{f : G \xrightarrow{\text{meas}} \mathbb{C} \mid f(xn) = \chi(n)^{-1}f(x) \quad (x \in G, n \in N_0)\}.$$

$$L^2(G/N_0 : \chi) := \{f \in \mathcal{M}(G/N_0 : \chi) \mid |f| \in L^2(G/N_0)\}$$

- ▶ Left reg^r repⁿ: $L = \text{Ind}_{N_0}^G(\chi)$ is unitary.
- ▶ $\exists$: Plancherel decomposition: $\text{Ind}_{N_0}^G(\chi) \simeq \int_{\widehat{G}}^{\oplus} m_\pi \pi d\mu(\pi)$.
- ▶ Realized by a **unitary** Fourier transform $f \mapsto \hat{f}$.
Plancherel identity: $\|f\|_{L^2}^2 = \int_{\widehat{G}} \|\hat{f}(\pi)\|_\pi^2 d\mu(\pi)$.
- ▶ **Today:** Maass-Selberg relations and their role in this.

Sources

- ▶ Harish-Chandra, Announcement AMS Toronto 1982.
details in Collected Papers Vol 5 (posthumous), 141- 307,
eds. R. Gangolli, V.S. Varadarajan, Springer 2018.
Formulation of Parseval, proof incomplete.
MS relations crucial.
- ▶ Wallach, RRG II, 1992: discrete part, cusp forms, holomorphic
dependence of Jacquet integral, inversion formula, error addressed in
Jpn. J. Math 19 (2024). Uses HC's Plancherel formula for the group, No
use of MS relations.
- ▶ **Today: sketch proof of MS relations and their role in Plancherel.** See
arXiv:2511.19224.

Discrete part of Plancherel decomposition

Distribution vectors

For $(\pi, H_\pi) \in \widehat{G}$ (unitary dual), we put $H_\pi^{-\infty} := \overline{H_\pi^\infty}'$. Then $H_\pi \hookrightarrow H_\pi^{-\infty}$. The space of **Whittaker vectors** of π is defined by

$$H_{\pi, \chi}^{-\infty} := \{\eta \in H_\pi^{-\infty} \mid (\forall n \in N_0) : \pi^{-\infty}(n)\eta = \chi(n)\eta\}.$$

- ▶ **Matrix coefficient** $m_\eta : v \mapsto \langle \pi(\cdot)^{-1}v, \eta \rangle$, $H_\pi^\infty \rightarrow C^\infty(G/N_0 : \chi)$.
- ▶ $\dim H_{\pi, \chi}^{-\infty} < \infty$.

Discrete part

$\pi \in \widehat{G}$ is said belong to the discrete series of $L^2(G/N_0 : \chi)$ if

$$\mathrm{Hom}_G(H_\pi, L^2(G/N_0 : \chi)) \neq 0, \quad (*)$$

Theorem (HC, W) If $(*)$ then

- (a) π belongs to the discrete series $\mathrm{DS}(G)$ of **G** .
- (b) $\overline{H_{\pi, \chi}^{-\infty}} \simeq \mathrm{Hom}_G(H_\pi, L^2(G/N_0 : \chi))$, induced by $\eta \mapsto m_\eta$.

Discrete part of Plancherel $L_d^2(G/N_0 : \chi) \simeq \widehat{\bigoplus_{\pi \in \mathrm{DS}(G)} (H_\pi \otimes \overline{H_{\pi, \chi}^{-\infty}})}$.

Parabolic induction

- ▶ $\mathcal{P}$: (finite) set of psg's $P = M_P A_P N_P < G$ with $P \supset A$.
- ▶ $\mathcal{P}_{\text{st}}$: set of $P \in \mathcal{P}$ such that $P \supset P_0 := Z_K(A) A N_0$
- ▶ $\bar{\mathcal{P}}_{\text{st}} = \{P \in \mathcal{P} \mid \bar{P} \in \mathcal{P}_{\text{st}}\}$ (opposite standard psgs).
- ▶ For $P \in \mathcal{P}_{\text{st}}$, $\chi_P := \chi|_{M_P \cap N_0}$ is regular for $M_P/(M_P \cap N_0)$.

Define

For $\sigma \in \text{DS}(M_P)$, $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$,

$$C^\infty(G/P : \sigma : \nu) := \{\varphi \in C^\infty(G, H_\sigma) \mid \varphi(xman) = a^{-\nu - \rho_P} \sigma(m)^{-1} \varphi(x)\},$$

Equipped with $\pi_{P, \sigma, \nu}$, the restriction of the left regular rep, this is the space of smooth vectors for $\text{Ind}_P^G(\sigma \otimes \nu \otimes 1)$.

Sesquilinear pairing

$$L^2(G/P : \sigma : \nu) \times L^2(G/P : \sigma : -\bar{\nu}) \rightarrow \mathbb{C},$$

$$(f, g) \mapsto \int_K \langle f(k), g(k) \rangle_\sigma dk$$

is G -equivariant.

Whittaker vectors

The pairing induces a G -equivariant linear injection $f \mapsto \langle f, \cdot \rangle$,

$$L^2(G/P : \sigma : \nu) \hookrightarrow C^{-\infty}(G/P : \sigma : \nu) := \overline{C^{\infty}(G/P : \sigma : -\bar{\nu})}'.$$

► If $P \in \bar{\mathcal{P}}_{\text{st}}$ then $N_0 P$ is open dense in G , contains e .

For $\eta \in H_{\sigma, \chi_P}^{-\infty}$ and $\text{Re } \nu$ P -dominant, define

$$j(P, \sigma, \nu, \eta)(n_0 man) := \chi(n_0) a^{-\nu - \rho_P} \sigma(m)^{-1} \eta.$$

Lemma ($W, \sim$)

$j(P, \sigma, \nu, \eta)$ defines a Whittaker vector in $C^{-\infty}(G/P : \sigma : \nu)_{\chi}$ and extends holomorphically in $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$.

Thm ($W, \sim$)

For each $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$ the map $\eta \mapsto j(P, \sigma, \nu, \eta)$ is a linear bijection

$$H_{\sigma, \chi_P}^{-\infty} \longrightarrow C^{-\infty}(G/P : \sigma : \nu)_{\chi}.$$

Its inverse is given by evaluation at e .

Arbitrary parabolic subgroups

HC uses $j(P, \sigma, \nu)$ only for $P \in \bar{\mathcal{P}}_{\text{st}}$. This is too limited if one wishes to understand the action of KS-intertwining operators on $j(P, \sigma, \nu)$. The following definition allows more flexibility.

Def

- ▶ for $P \in \mathcal{P}$ chose $v_P \in N_K(\mathfrak{a})$ s.t. $v_P P v_P^{-1} \in \bar{\mathcal{P}}_{\text{st}} \implies N_0 v_P P$ is open in G
- ▶ Put $\chi^P := \chi|_{N_0 \cap v_P M_P v_P^{-1}}$: regular for $(v_P M_P v_P^{-1}, N_0 \cap v_P M_P v_P^{-1})$.

For $\eta \in H_{\sigma, \chi^P}^{-\infty}$ and $\text{Re } \nu$ $\bar{P}$ -dominant, define

$$j(P, \sigma, \nu, \eta)(n_0 v_P m a n) := \chi(n_0) a^{-\nu - \rho_P} \sigma(m)^{-1} \eta.$$

Lemma (W, $\sim$)

$j(P, \sigma, \nu, \eta)$ defines a Whittaker vector in $C^{-\infty}(G/P : \sigma : \nu)_{\chi}$ and extends holomorphically in $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$.

Thm (W, $\sim$)

For each $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$ the map $\eta \mapsto j(P, \sigma, \nu, \eta)$ is a linear bijection

$$H_{\sigma, \chi^P}^{-\infty} \longrightarrow C^{-\infty}(G/P : \sigma : \nu)_{\chi}.$$

Its inverse is given by evaluation at v_P (possible by smoothness of j on open orbit $N_0 v_P P$).

Spherical functions, Whittaker integral

Spherical functions

Let (τ, V_τ) be a finite dimensional unitary representation of K .

$$\begin{aligned}
C^\infty(\tau : G/N_0 : \chi) &:= (V_\tau \otimes C^\infty(G/N_0 : \chi))^K \\
&\simeq \{f \in C^\infty(G, V_\tau) \mid f(kxn) = \chi(n)^{-1} \tau(k)f(x)\}.
\end{aligned}$$

$$\blacktriangleright K_P := M_P \cap K, \sigma_P := \sigma|_{K_P}$$

$$\begin{aligned}
\text{Def } C^\infty(\tau : G/\bar{P} : \sigma : \nu) &:= [V_\tau \otimes C^\infty(G/\bar{P} : \sigma : \nu)]^K \\
&\simeq [V_\tau \otimes C^\infty(K/K_P : \sigma_P)]^K \\
&=: C^\infty(\tau : K/K_P : \sigma_P).
\end{aligned}$$

$$\varphi_{\bar{P}, \sigma, \nu} \leftrightarrow \varphi$$

Def Whittaker integral

For $\varphi \in C^\infty(\tau : K/K_P : \sigma_P)$, $\eta \in H_{\sigma, \chi_P}^{-\infty}$, $\psi := \varphi \otimes \eta$,

$$\text{Wh}(\bar{P}, \psi, \nu)(x) := \langle \varphi_{\bar{P}, \sigma, -\nu}, \pi_{\bar{P}, \sigma, \bar{\nu}}(x)j(\bar{P}, \sigma, \bar{\nu})\eta \rangle$$

Properties

- ▶ $\text{Wh}(P, \psi, \nu) \in C^\infty(\tau : G/N_0 : \chi)$,
- ▶ linear in $\psi \in \mathcal{A}_{2,P,\sigma} := C^\infty(\tau : K/K_P : \sigma_P) \otimes \overline{H_{\sigma,\chi_P}^{-\infty}}$.
- ▶ holomorphic in $\nu \in \mathfrak{a}_{\mathbb{C}}^*$
- ▶ 3-finite ($\mathfrak{z}$ = center $U(\mathfrak{g})$).

The linearity mentioned in the second item allows us to extend the definition of $\psi \mapsto \text{Wh}(P, \psi, \nu)$ linearly to the finite direct sum

$$\mathcal{A}_{2,P} := \bigoplus_{\sigma \in \text{DS}(M_P)} \mathcal{A}_{2,P,\sigma}$$

Theorem (Uniformly tempered estimates, $\sim$)

Let $\varepsilon > 0$ be suff^{tly} small. If $u \in U(\mathfrak{g})$ then $\exists C, N, r > 0$ s.t.

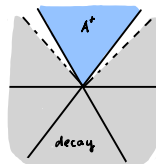
$$|\text{Wh}(P, \psi, \nu, u; kan)| \leq C(1 + |\nu|)^N (1 + |\log a|)^N e^{r|\text{Re } \nu| |\log a|} a^{-\rho},$$

for all $k \in K$, $a \in A$, $n \in N_0$, $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$ with $|\text{Re } \nu| < \varepsilon$.

Note: for $\text{Re } \nu = 0$, the estimate simplifies to the usual tempered estimates.

C-functions

- ▶ $\text{Wh}(P, \psi, \nu)$ is finite under $\mathfrak{J} := \text{center}(U(\mathfrak{g}))$,
- ▶ top order asymptotic behavior of $\exp'$ type along $\text{cl}(A^+)$,
- ▶ **very** rapid decay outside $\text{cl}(A^+)$.



- ▶ $P \sim Q : \iff \exists w \in W(\mathfrak{a}) : w(\mathfrak{a}_P) = \mathfrak{a}_Q \quad (\text{associated}).$
- ▶ $W(\mathfrak{a}_Q | \mathfrak{a}_P) := \{s \in \text{Hom}(\mathfrak{a}_P, \mathfrak{a}_Q) \mid \exists w \in W(\mathfrak{a}) : s = w|_{\mathfrak{a}_P}\}.$

Asymptotic behavior Let $P \in \mathcal{P}$, $Q \in \mathcal{P}_{\text{st}}$ and suppose $P \sim Q$.

There exist unique merom^c functions $C_{Q|P}(s, \cdot) : \mathfrak{a}_{P\mathbb{C}}^* \rightarrow \text{Hom}(\mathcal{A}_{2,P}, \mathcal{A}_{2,Q})$, for $s \in W(\mathfrak{a}_Q | \mathfrak{a}_P)$, such that for $\psi \in \mathcal{A}_{2,P}$, generic $\nu \in i\mathfrak{a}_P^*$ and $m \in M_Q$,

$$\text{Wh}(P, \psi, \nu)(ma) \sim \sum_{s \in W(\mathfrak{a}_Q | \mathfrak{a}_P)} a^{s\nu - \rho_Q} [C_{Q|P}(s, \nu)\psi](m),$$

for $a \rightarrow \infty$ in A_Q^+ .

C-functions, Maass-Selberg relations

Asymptotic behavior Let $P \in \mathcal{P}$, $Q \in \mathcal{P}_{\text{st}}$ and suppose $P \sim Q$.

There exist unique merom^c functions $C_{Q|P}(s, \cdot) : \mathfrak{a}_{P\mathbb{C}}^* \rightarrow \text{Hom}(\mathcal{A}_{2,P}, \mathcal{A}_{2,Q})$, for $s \in W(\mathfrak{a}_Q | \mathfrak{a}_P)$, such that for $\psi \in \mathcal{A}_{2,P}$, generic $\nu \in i\mathfrak{a}_P^*$ and $m \in M_Q$,

$$\text{Wh}(P, \psi, \nu)(ma) \sim \sum_{s \in W(\mathfrak{a}_Q | \mathfrak{a}_P)} a^{s\nu - \rho_Q} [C_{Q|P}(s, \nu)\psi](m), \quad (a \rightarrow \infty \text{ in } A_Q^+).$$

Maass-Selberg relations (HC)

Let $P \in \mathcal{P}$ and $\sigma \in \text{DS}(M_P)$. There exists a meromorphic function $\mathfrak{a}_{P\mathbb{C}}^* \ni \nu \mapsto \eta_\sigma(P, \nu) \in \mathbb{C}$ such that for all $Q \in \mathcal{P}_{\text{st}}$, $Q \sim P$ and $s \in W(\mathfrak{a}_Q | \mathfrak{a}_P)$,

$$C_{Q|P}(s, -\bar{\nu})^* C_{Q|P}(s, \nu) = \eta_\sigma(P, \nu) I$$

on $\mathcal{A}_{2,P,\sigma}$, for $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$.

Remark Equivalently: for $\sigma \in \text{DS}(M_P)$ and $\psi \in \mathcal{A}_{2,P,\sigma}$, the scalar function

$$i\mathfrak{a}_P^* \ni \nu \mapsto \|C_{Q|P}(s : \nu)\psi\|$$

is independent of $Q \in \mathcal{P}_{\text{st}}$, ($Q \sim P$), $s \in W(\mathfrak{a}_Q | \mathfrak{a}_P)$.

Application of MS rels

Normalized Whittaker integral For $P \in \mathcal{P}_{\text{st}}$, $\psi \in \mathcal{A}_{2,P}$, $\nu \in \mathfrak{a}_P^* \mathbb{C}$,

$$\text{Wh}^\circ(P, \nu, \psi) := \text{Wh}(P, \nu, C_{P|P}(1, \nu)^{-1} \psi).$$

gives rise to **normalized** asymptotic coeffs

$$C_{Q|P}^\circ(s : \nu) = C_{Q|P}(s : \nu) C_{P|P}(1, \nu)^{-1}$$

MS rels become $C_{Q|P}^\circ(s : \nu) \in U(\mathcal{A}_{2,P}, \mathcal{A}_{2,Q})$ ($\nu \in i\mathfrak{a}_P^*$)

Hence $\nu \mapsto \text{Wh}^\circ(P, \nu, \psi)$ regular on $i\mathfrak{a}_P^*$.

Def For $f \in C_c^\infty(\tau : G/N_0 : \chi)$ define $\mathcal{F}_P(f) : i\mathfrak{a}_P^* \rightarrow \mathcal{A}_{2,P}$ by

$$\langle \mathcal{F}_P(f)(\nu), \psi \rangle = \int_{G/N_0} \langle f(x), \text{Wh}^\circ(P, \psi, \nu) \rangle_\tau d\dot{x} \quad (\psi \in \mathcal{A}_{2,P}).$$

Thm ($\sim$) $\mathcal{F}_P : C_c^\infty(\tau : G/N_0 : \chi) \rightarrow \mathcal{S}(i\mathfrak{a}_P^*, \mathcal{A}_{2,P})$ is continuous linear and extends continuously to appropriate Schwartz space.

Lemma If $P, Q \in \mathcal{P}_{\text{st}}$, $P \sim Q$, then $\mathcal{F}_Q(f)(s\nu) = C_{Q|P}(s, \nu) \mathcal{F}_P(\nu)$.

Plancherel: $\|f\|_2^2 = \sum_{P \in \mathcal{P}_{\text{st}}/\sim} \|\mathcal{F}_P f\|_2^2$.

Intertwining operators

For $P, Q \in \mathcal{P}$ with $\mathfrak{a}_P = \mathfrak{a}_Q$, consider the Knapp-Stein intertwining operator

$$A(Q, P, \sigma, \nu) : C^\infty(G/P : \sigma : \nu) \rightarrow C^\infty(G/Q : \sigma : \nu).$$

Using double transposition, we extend it to a continuous linear operator on generalized vectors. By G -equivariance it gives a commutative diagram

$$\begin{array}{ccc} C^{-\infty}(G/P : \sigma : \nu)_\chi & \xrightarrow{A(Q, P, \sigma, \nu)} & C^{-\infty}(G/Q : \sigma : \nu)_\chi \\ \simeq \uparrow j(P, \sigma, \nu) & & j(Q, \sigma, \nu) \uparrow \simeq \\ H_{\sigma, \chi_P}^{-\infty} & \xrightarrow{B(Q, P, \sigma, \nu)} & H_{\sigma, \chi_Q}^{-\infty} \end{array}$$

Lemma

The map $B(Q, P, \sigma, \nu) \in \text{Hom}(H_{\sigma, \chi_P}^{-\infty}, H_{\sigma, \chi_Q}^{-\infty})$ depends meromorphically on $\nu \in \mathfrak{a}_P^*_{\mathbb{C}}$.

Fact

$$A(Q, P, \sigma, -\bar{\nu})^* A(Q, P, \sigma, \nu) = \eta(Q, P, \sigma, \nu) I$$

with $\nu \mapsto \eta(Q, P, \sigma, \nu) \in \mathbb{C}$ meromorphic in $\nu \in \mathfrak{a}_P^*_{\mathbb{C}}$.

Theorem (MS for B) Let $P, Q \in \mathcal{P}$ and suppose $\mathfrak{a}_P = \mathfrak{a}_Q$ (then also $M_P = M_Q$). If $\sigma \in \text{DS}(M_P)$ then

$$B(Q, P, \sigma, -\bar{\nu})^* B(Q, P, \sigma, \nu) = \eta(Q, P, \sigma, \nu) I$$

as meromorphic $\text{End}(H_{\sigma, \chi_P}^{-\infty})$ -valued meromorphic functions of $\nu \in \mathfrak{a}_{P\mathbb{C}}^*$.

Theorem (MS B $\implies$ C)

The assertion of Thm (MS for B) implies the MS for the C-functions $C_{Q|P}(s, \nu)$, for $P \in \mathcal{P}$, $Q \in \mathcal{P}_{\text{st}}$, $\mathfrak{a}_P = \mathfrak{a}_Q$.

Proof: This follows from

$$C_{Q|P}(s, \nu)\psi = [A(Q, s\bar{P}s^{-1}, s\sigma, -s\nu) \otimes B(\bar{Q}, s\bar{P}s^{-1}, s\sigma, -s\nu) R_s]\psi.$$



The product decomposition of the KS-intertwiner in terms of intertwiners for adjacent parabolic subgroups leads to a reduction of the proof of Thm (MS for B) to the case that P, Q are adjacent.

Conjugation by the Weyl group leads to the case that P, Q are adjacent and P standard. Assume this now.

Reduction of MS for B to the basic case

Let $\alpha \in \Delta$ be such that $\ker \alpha$ separates $\mathfrak{a}_P^+$ and $\mathfrak{a}_Q^+$ while $\text{cl}(\mathfrak{a}_P^+) \cap \text{cl}(\mathfrak{a}_Q^+)$ has an interior point $X \in \ker \alpha$.

Let P_X be the standard parabolic subgroup with Lie algebra equal to the vector sum of the centralizer of X and $\mathfrak{n}_0$. Let $P_X = M_X A_X N_X$ be its Langlands decomposition.

Then $G^{(\alpha)} := M_X$ has compact center, $P^{(\alpha)} := P \cap G^{(\alpha)}$ is a standard maximal parabolic in $G^{(\alpha)}$, and $Q^{(\alpha)} := Q \cap G^{(\alpha)}$ is a maximal parabolic subgroup of $G^{(\alpha)}$. Furthermore, $P^{(\alpha)}$ and $Q^{(\alpha)}$ are opposite hence adjacent.

Theorem ($\sim$)

For all $\sigma \in \text{DS}(M_P)$, we have

$$B(\bar{Q}, \bar{P}, \sigma, \nu) = B^{(\alpha)}(P^{(\alpha)}, \bar{P}^{(\alpha)}, \sigma, \nu)$$

as $\text{End}(H_{\sigma, \chi_P}^{-\infty})$ -valued meromorphic functions of $\nu \in \mathfrak{a}_P^* \mathbb{C}$.

This is a familiar strategy, but the proof is hard since distributions on manifolds of different dimensions need to be compared.

HC's proof for the basic case, 1

Basic case The above result leads to reduction of the proof of Theorem "MS for B" to the basic case:

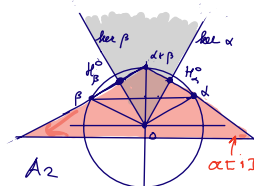
- ▶ G has compact center,
- ▶ P, Q are maximal parabolic subgroups of G with $a_P = a_Q$,
- ▶ P, Q are adjacent, hence opposite.

Thm *In the basic case the implication of Thm $[MS\ B \implies C]$ is an equivalence.*

Conclusion It suffices to prove the MS rels for the C -functions in the basic case.

HC's proof in the basic case, 1

- ▶ For $\alpha \in \Delta$ define H_α^0 by $\beta(H_\alpha^0) = 0$ for $\beta \in \Delta \setminus \{\alpha\}$ and by $\langle H_\alpha^0, H_\alpha \rangle = 1$.
- ▶ $\mathfrak{a}[1]$ is the subset of $\mathfrak{a}$ consisting of $H \in \mathfrak{a}$ with $\langle H, H_\alpha^0 \rangle \leq 1$ for $\alpha \in \Delta$ and $\sum_{\alpha \in \Delta} \langle H_\alpha^0, H \rangle \geq -1$.
- ▶ $\mathfrak{a}[t] := t\mathfrak{a}[1]$ ($t > 0$), and $G[t] = K\mathfrak{a}[t]N_0$.



We consider

$$I(t) = \int_{G[t]/N_0} [f, g](x) dx, \quad (t \rightarrow \infty)$$

where $f, g \in C^\infty(\tau : G/N_0 : \chi)$, and (Ω Casimir):

$$[f, g] := \langle \Omega f, g \rangle_\tau - \langle f, \Omega g \rangle_\tau \in C^\infty(K \backslash G/N_0).$$

By radial component analysis, and application of Gauss' divergence theorem:

$$I(t) = \int_{A[t]} \sum_{\alpha \in \Delta_e} H_\alpha(f, g)_{H_\alpha^0} da = \sum_{\alpha \in \Delta_e} \int_{\partial A[t]} \langle \nu, H_\alpha \rangle (f, g)_{H_\alpha^0} ds.$$

Here $\Delta_e = \Delta \cup \{ \langle \sum_{\alpha \in \Delta} H_\alpha^0, \cdot \rangle \}$.

HC's proof for the basic case, 2

Assume $f, g \in \mathcal{A}(\tau : G/N_0 : \chi)$ have constant term zero along parabolics P_F with $|\Delta \setminus F| \geq 2$. Then by an asymptotic analysis it follows that

$$I(t) = \sum_{\alpha \in \Delta} J_{\alpha}(f, g, t) + \mathcal{O}(e^{-\delta t}),$$

where

$$J_{\alpha}(f, g, t) = \int_{M_{F_{\alpha}}/M_{F_{\alpha}} \cap N_0} \{f, g\}_{\alpha}(m \exp tH_{\alpha}^0) \, d\dot{m}.$$

and

$$\{f, g\}_{\alpha}(m) = \langle R(H_{\alpha}^0)f_{F_{\alpha}}, g_{F_{\alpha}} \rangle - \langle f_{F_{\alpha}}, R(H_{\alpha}^0)g_{F_{\alpha}} \rangle.$$

Final argument: suppose f is an eigen function of Ω with a real eigen value, then

$$\sum_{\alpha \in \Delta} J_{\alpha}(f, f, t) \rightarrow 0, \quad (t \rightarrow \infty)$$

HC's proof for the basic case, final part

Setting Basic case G has compact center, $P \in \mathcal{P}$ is maximal, $\bar{P}$ is adjacent. Let $\sigma \in \text{DS}(M_P)$, $\nu \in i\mathfrak{a}_P^*$. Then $W_\nu = \text{Wh}(P, \psi, \nu)$ is eigen for Ω with real eigenvalue.

Since $\dim \mathfrak{a}_P = 1$, there are two possibilities: $|W(\mathfrak{a}_P)| = 1$ or $|W(\mathfrak{a}_P)| = 2$. Assume e.g. the second.

Then $\exists! \alpha \in \Delta$ s.t. $P_{F_\alpha} \sim P$. It follows that for $\beta \in \Delta \setminus \{\alpha\}$, the constant term of W_ν along P_{F_β} is zero. Along P_{F_α} the constant term of W_ν takes the form

$$W_{\nu, F_\alpha}(m \exp tH_\alpha^0) = e^{it\lambda} \varphi_1(m) + e^{-it\lambda} \varphi_2(m) \quad (m \in M_{F_\alpha}, t > 0),$$

with $i\lambda = \nu(H_\alpha^0)$, $\varphi_1, \varphi_2 \in \mathcal{C}(\tau : M_{F_\alpha} : \chi|_{M_{F_\alpha} \cap N_0})$. Now

$$\begin{aligned} J_\alpha(f, f, t) &= i\lambda \langle e^{it\lambda} \varphi_1 - e^{-i\lambda t} \varphi_2, e^{i\lambda t} \varphi_1 + e^{-i\lambda t} \varphi_2 \rangle \\ &\quad - i\lambda \langle e^{i\lambda t} \varphi_1 + e^{-i\lambda t} \varphi_2, e^{i\lambda t} \varphi_1 - e^{-i\lambda t} \varphi_2 \rangle \\ &= 2i\lambda (\|\varphi_1\|^2 - \|\varphi_2\|^2). \end{aligned}$$

Since $J_\alpha(f, f, t) \rightarrow 0$ for $t \rightarrow \infty$, we conclude $\|\varphi_1\| = \|\varphi_2\|$.