Analysis on real semisimple Lie groups, II

1. We will consider representation theory in a more systematic way.

In the following we assume that V is a Fréchet space (more generally we may assume that V is locally convex, Hausdorff, quasi-complete and barreled; the latter to ensure validity of the principal of uniform boundedness).

A representation π of G in V is <u>continuous</u> if the map $G \times V \longrightarrow V$, $(x, v) \longmapsto \pi(x)v$ is continuous (equivalently, $\pi : G \longrightarrow \mathrm{End}(V)$ should be strongly continuous; this involves uniform bddness).
We assume G $\mathbb{R}$ semisimple, connected, $\# Z(G) < \infty$, $\theta : G \to G$ Cartan involution, so $K = G^\theta$ is maximal compact.

2. <u>Def</u>: $V_K = \{ v \in V \mid \mathrm{span}(\pi(K)v) \text{ finite dimensional} \}$.

If $v \in V_K$, then $W := \mathrm{span}(\pi(K)v)$ decomposes canonically as $W \simeq \bigoplus_{\delta \in \hat{K}} \mathrm{Hom}_K(V_\delta, W) \otimes V_\delta$ (finitely many terms non-trivial). Here $\hat{K}$ denotes the collection of (equivalence classes of) irreducible finit dim cont reps of K.

The inclusion $\mathrm{Hom}_K(V_\delta, W) \otimes V_\delta \longrightarrow W$ is given by $T \otimes u \longmapsto T(u)$, and is K–

equivariant for $1 \otimes \delta$.

It follows that also (K-equivariantly)

$$V_K \simeq \bigoplus_{\delta \in \hat{K}} \mathrm{Hom}_K(V_\delta, V) \otimes V_\delta$$

but now $\mathrm{Hom}(V_\delta, V)$ may be infinite dimensional. We agree to write

$$V[\delta] = \mathrm{image}\left(\mathrm{Hom}_K(V_\delta, V) \otimes V_\delta\right)$$

$$= \{ v \in V_K \mid \exists_N \ \mathrm{span}(\pi(K)v) \simeq \delta^N \}.$$

This is called the isotypical component of V of type δ.

3. <u>Def</u> V is said to be <u>admissible</u> if $\forall_{\delta \in \hat{K}} \ \dim_{\mathbb{C}} V[\delta] < \infty$.

<u>Lemma</u> Let $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Then, as a K-module,

$$C^\infty(G : P : \lambda)_K \simeq C^\infty(K/M)_K$$

$$\simeq \bigoplus_{\delta \in K} (V_\delta^*)^M \otimes V_\delta .$$

In particular, $\mathrm{Ind}_P^G(\lambda)$ is <u>admissible</u>.

Proof. The first isomorphism has been established. For the second one, we note that

$$C^\infty(K/M)_M \simeq \bigoplus_{\delta\in\hat{K}} \mathrm{Hom}_k(V_\delta, C^\infty(K/M)) \otimes V_\delta.$$

Now $C^\infty(K/M)$ is the space of $\mathrm{ind}_M^K(1)^\infty$ and by Frobenius reciprocity for compact groups, the map $T \mapsto ev_{eM}\circ T$ (with $ev_{eM}: C^\infty(K/M) \to \mathbb{C}$, $\varphi \mapsto \varphi(eM)$) is a linear isomorphism

$$\mathrm{Hom}_K(\delta, \mathrm{ind}_M^K(1)) \simeq \mathrm{Hom}_M(\delta|_M, \mathbb{C})$$
$$\simeq (V_\delta)^{*M}.$$

Remark more generally, if $\tau\in\hat{M}$ is a finite dimensional unitary representation of M in $\mathcal{H}_\tau$, and $\lambda\in\mathfrak{a}_\mathbb{C}^*$, we denote by $\tau\otimes\lambda\otimes 1$ the representation of P in $\mathcal{H}_\tau$ given by

$$(\tau\otimes\lambda\otimes 1)(man) = a^\lambda\,\tau(m).$$

Then

$$\mathrm{Ind}_P^G(\tau\otimes\lambda\otimes 1)^\infty \simeq \mathrm{ind}_P^G(\tau\otimes(\lambda+\rho)\otimes 1)^\infty$$

is admissible as well, by similar reasoning.

The representations

$$\mathrm{Ind}_P^G(\tau \otimes \lambda \otimes 1)^\infty \quad \text{of } G,$$

for $\tau \in \hat{M}$ (fin. dim^l. irr. unitary) and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ are said to form the (minimal) principal series of representations

4. We assume (π, V) continuous Fréchet rep^n of G.

<u>Def</u> A vector $v \in V$ is said to be <u>smooth</u> if the map $x \longmapsto \pi(x)v$ is C^∞ from G to V. The subspace of such vectors is denoted by V^∞.

<u>Lemma</u> V^∞ is dense in V.

<u>Proof</u> This follows essentially by a convolution argument. Let dx be left- (hence also right) invariant positive density on G.

For $f \in C_c(G)$ define

$$\pi(f) : V \longrightarrow V$$

by

$$\pi(f)v = \int_G f(x)\pi(x)v \, dx$$

Then $\pi(L_y f) = \pi(y) \circ \pi(f)$, $\forall y \in G$ (use left invariance of the measure). If $v \in V$ and $f \in C_c^\infty(G)$ then $\pi(f) v \in V^\infty$. Now take $(f_j) \subset C_c^\infty(G)$ s.t.:

(1) $\operatorname{supp} f_j \to \{e\}$

(2) $f_j \geq 0$, $\int_G f_j(x)\,dx = 1$.

Then $f_j dx \to \delta_e$ in $C(G)'$. It is rather straight forward to show that

$$\pi(f_j) v \to v. \qquad \boxtimes$$

5 <u>Lemma</u> $V^\infty \cap V_K$ is dense in V.

<u>Proof.</u> One may take the sequence (f_j) of the previous proof to consist of K-finite (f_j) (by applying representation theory of K). $\boxtimes$

<u>Cor</u> If (π, V) is admissible, then

1) $V_K \subset V^\infty$

2) V_K is dense in V

<u>Proof</u> $V^\infty \cap V[\delta]$ is dense in $V[\delta]$ and $V[\delta]$ is finite dimensional $\forall_{\delta \in \hat{K}}$.

Therefore, $V^\infty \cap V[\delta] = V[\delta]$. It follows that $V^\infty \cap V_K = V_K$. 1) & 2) follow. $\boxtimes$.

We note that V^∞ is G-invariant. By smoothness, the representation of G in V^∞ induces a representation of $\mathfrak{g}$ in V^∞ given by

$$\pi_*(X)v = \frac{d}{dt}\Big|_{t=0} [\pi(\exp tX)v].$$

By the universal property, we see that V^∞ is a $U(\mathfrak{g})$-module.

6 <u>Lemma</u> Let $x \in G$ and $Y \in U(\mathfrak{g})$. Then on V^∞ we have

$$\pi(x) \circ \pi_*(Y) = \pi_*(Ad(x)Y) \circ \pi(x).$$

<u>Proof</u> It suffices to prove this for $Y \in \mathfrak{g}$. Then

$$\begin{aligned}\pi(x)\pi_*(Y)v &= \pi(x) \frac{d}{dt}\Big|_{t=0} \pi(\exp tY)v \\ &= \frac{d}{dt}\Big|_{t=0} \pi(x)\pi(\exp tY)v \\ &= \frac{d}{dt}\Big|_{t=0} \pi(\exp t\,Ad(x)Y)\pi(x)v \\ &= \pi_*(Ad(x)Y)\pi(x)v. \quad \boxtimes\end{aligned}$$

<u>Prop^n</u>: Assume (π, V) is admissible. Then

(1) $V_K \subset V^\infty$;

(2) $\forall_{X \in \mathfrak{g}}$ $\pi(X) V_K \subset V_K$.

<u>Proof</u> (1) has been established already.

We now note that for every $\delta \in \hat{K}$,

$$\mathfrak{g} \otimes V[\delta] \xrightarrow{t} V$$

is a K-module map; here the K-action on $\mathfrak{g}$ is given by $(k, X) \mapsto Ad(k) X$. The space $\mathfrak{g} \otimes V[\delta]$ is a finite dimensional K-module. Hence $t(\mathfrak{g} \otimes V[\delta]) \subset V_K$. The result follows. $\boxtimes$.

7 <u>Def</u> A $(\mathfrak{g}, K)$-module is a complex linear space V, equipped with representations of $\mathfrak{g}$ and K such that.

(1) $\forall_{v \in V}$ $\dim_{\mathbb{C}} \operatorname{span}\{Kv\} < \infty$ and the rep of K on $\operatorname{span}(Kv)$ is continuous

(2) $\forall_{k \in K, X \in \mathfrak{g}}$ $(k\cdot) \circ (X\cdot) = (Ad(k)X \cdot) \circ k\cdot$

(3) $\forall_{X \in \mathfrak{k}}$ $X\cdot = \frac{d}{dt}\Big|_{t=0} \exp tX \cdot$

8 <u>Def</u> Let V be $(\mathfrak{g}, K)$-module. For $\delta \in \hat{K}$ we denote by $V[\delta]$ the subspace of $v \in V$ s.t. the K-module span(Kv) is a sum of copies of δ.

Note that $V[\delta] \simeq \mathrm{Hom}_K(V_\delta, V) \otimes V_\delta$ naturally. Also,

$$V = \bigoplus_{\delta \in \hat{K}} V[\delta]$$

9 <u>Def</u> V is said to be <u>admissible</u> if and only if $\forall \delta \in \hat{K}$: $\dim V[\delta] < \infty$.

<u>Lemma</u> Let (π, V) be a continuous Fréchet representation of G. Then $V_K \cap V^\infty$ is a $(\mathfrak{g}, K)$-module.

Furthermore, V is admissible $\iff$ $V_K \cap V^\infty$ is admissible. In this case $V_K = V_K \cap V^\infty$.

<u>Proof</u> Easy. ☒

10 <u>Def</u> Let (π, V) be a continuous Fréchet representation of G. Then π is called <u>irreducible</u> if the only <u>closed</u> invariant subspaces of V are $\{0\}$ and V.

<u>Lemma</u> Let (π, V) be an admissible Fréchet representation. Then the map $W \longmapsto W_K$ establishes a bijection between closed G-invariant subspaces of V and $(\mathfrak{g}, K)$-submodules of V_K. Its inverse is given by $\mathcal{U} \longmapsto \overline{\mathcal{U}}$.

<u>proof</u> easy ☒.

<u>Theorem</u> (Harish-Chandra). Let $(\pi, \mathcal{H})$ be an irreducible unitary representation. Then π is admissible.

11 <u>Def</u> An admissible $(\mathfrak{g}, K)$-module V is said to be <u>unitarizable</u> if and only if there exists a Hermitian inner product $\langle \cdot, \cdot \rangle$ on V such that

$$(1) \quad \langle kv, kw \rangle = \langle v, w \rangle \quad \left(\forall_{v, w \in V} \ \forall_{k \in K} \right)$$

$$(2) \quad \langle Xv, w \rangle = - \langle v, Xw \rangle \quad \left(\forall_{v, w \in V} \ \forall_{X \in \mathfrak{g}} \right)$$

Clearly, if $(\pi, \mathcal{H})$ is irreducible unitary, then $\mathcal{H}_K$ is irreducible unitarizable $(\mathfrak{g}, K)$-module.

<u>Theorem</u> (Harish-Chandra) Let V be an irreducible unitarizable $(\mathfrak{g}, K)$-module. Then there exists an

irreducible unitary representation $(\pi, \mathcal{H})$ of G s.t. $\mathcal{H}_K \simeq V$.

12 <u>Theorem</u> (Harish-Chandra & Casselman subrepresentation thm)
Let V be an irreducible $(\mathfrak{g}, K)$-module. Then there exist $\tau \in \hat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and an embedding $j : V \hookrightarrow C^\infty(P : \tau : \lambda)_K$ of $(\mathfrak{g}, K)$-modules.

13 <u>Distribution vectors</u>

Let (π, V) be a continuous Fréchet representation. We denote by $V^{-\infty}$ the continuous anti-linear dual of V^∞, equipped with the strong dual topology.

If V is a Hilbert space and π unitary then the map $v \mapsto \langle v, \cdot \rangle$ defines a G-equivariant embedding $V \hookrightarrow V^{-\infty}$. Thus we have G-equivariant embeddings

$$V^\infty \hookrightarrow V \hookrightarrow V^{-\infty}$$

Let σ be an involution on G and

H an open (hence closed) subgroup of G^σ. Then G/H is a so called semisimple symmetric space. It is known that there exists a Cartan involution θ of G s.t. $\sigma \circ \theta = \theta \circ \sigma$. Let σ also denote the involution $d\sigma(e)$ of $\mathfrak{g}$, and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ the decomposition in the associated $+1$ and -1 eigenspaces.

The homogeneous space G/H has a G-invariant density dx. The corresponding space $L^2(G/H) = L^2(G/H, dx)$ therefore comes equipped with the unitary left regular representation. From general principles of functional analysis and group theory it can be shown that there exists a Plancherel decomposition

$$L^2(G/H) \simeq \int_{\widehat{G}_H}^{\oplus} m_\pi \, \pi \, d\mu(\pi),$$

i.e. $L^2(G/H)$ allows a direct integral decomposition into irreducible unitary representations π, with multiplicities m_π. The integral ranges over $\widehat{G}_H$ the set of (equivalence classes

of) irreducible unitary representations $(\pi, \mathcal{H}_\pi)$ with $(\mathcal{H}_\pi^{-\infty})^H \neq 0$.
This set carries a natural topology for which $d\mu$ is a Borel measure, the so-called Plancherel measure.

The condition $(\mathcal{H}_\pi^{-\infty})^H \neq 0$ is equivalent to the requirement that

$$\mathrm{Hom}_G(\mathcal{H}_\pi^\infty, C^\infty(G/H)) \neq 0.$$

(one needs to have intertwiners $\mathcal{H}_\pi^\infty \hookrightarrow C^\infty(G/H)$ for π to appear in $L^2(G/H)$).

To find an explicit Plancherel formula, one of the first steps is to find principal series of representations possessing an H-fixed distribution vector.

Let $\mathfrak{a}_q \subset \mathfrak{p} \cap \mathfrak{q}$ be maximal abelian and let $\mathfrak{a} \subset \mathfrak{p}$ be maximal abelian st $\mathfrak{a} \supset \mathfrak{a}_q$. Then both $\Sigma(\mathfrak{g}, \mathfrak{a})$ and $\Sigma(\mathfrak{g}, \mathfrak{a}_q)$ are (possibly non-reduced) root systems.

Let $P = MAN_P$ be a minimal parabolic

subgroup of G. I.e.,

$$\Sigma(P) := \{\alpha \in \Sigma(\mathfrak{g}, \mathfrak{a}) \mid \mathfrak{g}_\alpha \subset \mathfrak{n}_P\}$$

is a positive system for $\Sigma(\mathfrak{g}, \mathfrak{a})$.

13 We consider a finite dimensional irreducible unitary representation (ξ, V_ξ) of M with $V_\xi^{M\cap H} \neq 0$, and are interested in embeddings

$$\mathrm{Ind}_P^G(\xi \otimes (-\mu) \otimes 1)^\infty \hookrightarrow C^\infty(G/H),$$

for $\mu \in \mathfrak{a}_{\mathbb{C}}^*$. These correspond to continuous linear functionals

$$u \in C^\infty(P : \xi : -\mu)'^{H}.$$

We attempt to construct these as follows. Let $\eta \in (V_\xi^*)^{H\cap M} \simeq \overline{V_\xi}^{H\cap M}$. Let ω be a positive density of $\mathfrak{h}/\mathfrak{h}\cap \underline{P} \simeq T_{e(H\cap P)}((H\cap P)\backslash H)$. We would like to define $u = u_\eta$ by

$$u_\eta(f) = \int_{H\cap P\backslash H} \langle f(h), \eta\rangle \, dr_h(e)^{-1*}\omega \qquad (13.1)$$

14 For this we need the integrand to be a genuine density on $(H\cap P)\backslash H$, or, equivalently

$$\langle f(ph),\eta\rangle\, dr_{ph}(e)^{-1*}\omega = \langle f(h),\eta\rangle\, dr_h(e)^{-1*}\omega$$

for all $h\in H$ and $p\in P\cap H$. It can be shown that $P\cap H = (N_P\cap H)(A\cap H)(M\cap H)$. Since

$$f(namh) = a^{\mu-\rho_P}\,\xi(m)^{-1} f(h)$$

and

$$dr_{ph}(e)^{-1*}\omega = \left|\det \mathrm{Ad}(p^{-1})|_{\mathfrak h/\mathfrak h\cap \mathfrak p}\right|\cdot dr_h(e)^{-1*}\omega$$

we see that we need

$$a^{\mu-\rho_P} = \left|\det \mathrm{Ad}(a)|_{\mathfrak h/\mathfrak h\cap\mathfrak p}\right|$$

for all $a\in A\cap H$. The map $X\mapsto X+\sigma X$,

$$\mathfrak n_P/\mathfrak n_P\cap\mathfrak h \longrightarrow \mathfrak h/\mathfrak h\cap \mathfrak p$$

is an isomorphism of $A\cap H$-modules. Thus we need that

$$(\mu-\rho_P)|_{\mathfrak a_{\mathfrak h}} = 2\rho_P|_{\mathfrak a_{\mathfrak h}} - \mathrm{tr}\,\mathrm{ad}(\cdot)|_{\mathfrak n_P\cap\mathfrak h}\big|_{\mathfrak a_{\mathfrak h}}$$

or

$$\mu|_{\mathfrak a_{\mathfrak h}} = \rho_P|_{\mathfrak a_{\mathfrak h}} - 2\rho_{Ph}|_{\mathfrak a_{\mathfrak h}} - \rho_{P\sigma}|_{\mathfrak a_{\mathfrak h}} \qquad (\#)$$

where $\rho_{P\mathfrak{h}} = \frac{1}{2} \sum_{\alpha \in \Sigma(P) \cap \mathfrak{a}_{\mathfrak{h}}^*} m_\alpha \alpha$

$$\rho_{P\sigma} = \frac{1}{2} \sum_{\substack{\alpha \in \Sigma(P) \setminus \mathfrak{a}_{\mathfrak{h}}^* \\ \sigma\alpha \in \Sigma(P)}} m_\alpha \alpha .$$

Condition (#) is equivalent to

$$\mu|_{\mathfrak{a}_{\mathfrak{h}}} = -\rho_{P\mathfrak{h}}|_{\mathfrak{a}_{\mathfrak{h}}} .$$

Thus

$$\mu = \lambda - \rho_{P\mathfrak{h}} , \quad \lambda \in \mathfrak{a}_{\mathfrak{q}\mathbb{C}}^* .$$

We now recall the convexity theorem of Balibanu & vdB which tells us that

$$\mathrm{pr}_{\mathfrak{q}} \circ \mathcal{H}_P(H) = \sum_{\alpha \in \Sigma(P)_-} \mathbb{R}_{\geq 0} \, \mathrm{pr}_{\mathfrak{q}} H_\alpha$$

where $\Sigma(P)_- = \left\{ \alpha \in \Sigma(P) \cap \sigma\theta\Sigma(P) \;\middle|\; \begin{matrix} \alpha \in \mathfrak{a}_{\mathfrak{q}}^* \Rightarrow \\ \sigma\theta|_{\mathfrak{g}_\alpha} \neq I_{\mathfrak{g}_\alpha} \end{matrix} \right\}$

Using this convexity theorem one can show

<u>Theorem</u> There exist constants $c_\alpha > 0$, $\alpha \in \Sigma(P)_-$ such that the integral (13.1) converges for $\mu = \lambda - \rho_{P\mathfrak{h}}$, $\lambda \in \mathfrak{a}_{\mathfrak{q}\mathbb{C}}^*$,

$$\langle \mathrm{Re}\, \lambda, \alpha \rangle > c_\alpha \quad \forall_{\alpha \in \Sigma(P)_-} .$$

15. Via the sesquilinear pairing

$$C^\infty(P:\xi:-\mu) \times C^\infty(P:\xi:\bar{\mu}) \to \mathbb{C}$$

we identify $\overline{C^\infty(P:\xi:-\mu)}'$ with

$$C^{-\infty}(P:\xi:\bar{\mu}) \simeq C^{-\infty}(K:\xi)$$

(generalized sections of $G \times_P (V_\xi \otimes \mathbb{C}_{\bar{\lambda}})$).

- <u>Theorem</u> For every λ as in Theorem 14, the map

$$u_\eta(\lambda) : C^\infty(P:\xi:-\lambda+\rho_{Ph}) \to \mathbb{C}$$

defines an element of

$$C^{-\infty}(P:\xi:\bar{\lambda}-\rho_{Ph})^H!$$

- Moreover, $\lambda \mapsto u_\eta(\lambda)$ is holomorphic on $\Omega(P) = \{\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbb{C}} \mid \langle \mathrm{Re}\,\lambda, \alpha\rangle > c_\alpha \ \forall_{\alpha \in \Sigma(P)_-}\}$ as a map with values in (the compact picture)

$$C^{-\infty}(K:\xi).$$

<u>Proof</u>. Also involves convexity theorem of Balibanu – vd Ban.