Cusp forms for semisimple symmetric spaces

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Schwartz functions for the group

Setting

- G real semisimple Lie group, (connected, finite center)
- K maximal compact, $\quad G = K \exp \mathfrak{p}$
- $\mathfrak{a} \subset \mathfrak{p}$ maximal abelian, $\quad A := \exp \mathfrak{a}$,
- $G = KAK, \quad \tau(k_1 \exp X \, k_2) = \|X\|$.

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Harish-Chandra Schwartz space

$\mathcal{C}(G)$ consists of all $f \in C^\infty(G)$ such that

$$\forall u, v \in U(\mathfrak{g}) \;\; \forall N \in \mathbb{N} : \quad (1 + \tau)^N L_u R_v f \in L^2(G).$$

The representation $L \times R$ of $G \times G$ is continuous.

Cusp forms for the group

Theorem (Harish-Chandra)

If $P = M_P A_P N_P$ is a parabolic subgroup of G then for all $f \in \mathcal{C}(G)$

$$\int_{N_P} |f(n)| \, dn \; < \infty.$$

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Cusp form on G

- A function $f \in \mathcal{C}(G)$ such that

$$\forall x, y \in G \ \forall P < G : \quad \int_{N_P} f(xny) \, dn = 0.$$

- $\mathcal{C}(G)_{\mathrm{cusp}} \subset \mathcal{C}(G)$: space of cusp forms

Cusp forms on G, II

- $\mathcal{P}(A)$: the (finite) collection of parabolics $P < G$ with $P \supset A$
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Theorem (HC, 60's)

$$\mathcal{C}(G)_{\rm d} = \mathcal{C}(G)_{\rm cusp}.$$

Remark

In theory of automorphic forms, in general

$$\mathcal{C}(G/\Gamma)_{\rm cusp} \subsetneq \mathcal{C}(G/\Gamma)_{\rm d}$$

Semisimple symmetric spaces

Setting

- σ involution of G such that $\sigma(K) = K$
- $(G^\sigma)_e < H < G^\sigma$,
- $X = G/H$ semisimple symmetric space

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- $G = {}^\backprime G \times {}^\backprime G$, $H = \mathrm{diag}({}^\backprime G)$. Then $X \simeq {}^\backprime G$ (‘group case’)

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- Hyperbolic spaes $X_{p,q} = \mathrm{SO}(p,q)/\mathrm{SO}(p-1,q) \simeq$

$$\{x \in \mathbb{R}^{p+q} \mid x_1^2 + \cdots x_p^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2) = 1\}.$$

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- $\mathrm{SL}(n,\mathbb{R})/\mathrm{GL}(n-1,\mathbb{R})$.

Semisimple symmetric spaces

Structure

- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}$ (± 1 eigenspaces for θ, σ)
- $\mathfrak{a}_\mathrm{q} \subset \mathfrak{p} \cap \mathfrak{q}$ maximal abelian, $A_\mathrm{q} = \exp(\mathfrak{a}_\mathrm{q})$
- $G = KA_\mathrm{q}H, \quad \tau_X : X \to \mathbb{R},\ k \exp X\, H \mapsto \|X\|.$

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Schwartz space

The Schwartz space $\mathcal{C}(X)$ consists of $f \in C^\infty(X)$ such that

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σ-parabolic subgroups

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Plancherel decomposition of $L^2(X)$:

Building blocks are the discrete series for X and the induced reps

$$\mathrm{Ind}_P^G(\xi \otimes \lambda \otimes 1)$$

with

- $P \in \mathcal{P}_\sigma(A_\mathrm{q})$,
- ξ a discrete series rep of $M_P/M_P \cap H$,
- $\lambda \in i\mathfrak{a}_{P\mathrm{q}}^*$.

Cusp forms, first attempt

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Cusp form: $f \in \mathcal{C}(G/H)$ such that

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Problems

- Integral need not converge (e.g. hyperbolic spaces, Andersen's talk; $\mathrm{SL}(n,\mathbb{R})/\mathrm{GL}(n-1,\mathbb{R})$).
- In the groups case:

$$\mathrm{New}: \int_{N_P \times \bar{N}_P} f(xn\bar{n}y)\, dn\, d\bar{n} = 0, \quad \mathrm{Old}: \int_{N_P} f(xny)\, dn = 0.$$

Idea of Flensted-Jensen

Minimal σ-parabolics

- $\mathfrak{a}_\mathrm{q} \subset \mathfrak{p} \cap \mathfrak{q}$ maximal abelian;
- $\Sigma(\mathfrak{a}_\mathrm{q})$ roots of $\mathfrak{a}_\mathrm{q}$ in $\mathfrak{g}$, fix $\Sigma^+(\mathfrak{a}_\mathrm{q}) \quad \rightsquigarrow \mathfrak{n}_0,\ N_0$
- $P_0 = Z_G(\mathfrak{a}_\mathrm{q})N_0$ is a minimal σ-parabolic.

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Minimal parabolics

- Extend $\mathfrak{a}_\mathrm{q}$ to $\mathfrak{a}$: max abelian in $\mathfrak{p}$; $\quad A = \exp \mathfrak{a}$.
- Fix $P \in \mathcal{P}_{\mathrm{min}}(A)$ with $P \subset P_0$,

Then P has $N_P \cap H$ of minimal dimension and $N_0 \simeq N_P / N_P \cap H$.

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Then P has $N_P \cap H$ of minimal dimension and $N_0 \simeq N_P/N_P \cap H$.

Flensted-Jensen's idea

- Use $Q \in \mathcal{P}_{\min}(A)$ with $N_Q \cap H$ of maximal dimension.
- There exists $N^* < N_Q$ such that $N_Q \simeq N^* \times (N_Q \cap H)$.

Cusp forms, II

Group case

Let $G = {}^\backprime G \times {}^\backprime G$, and ${}^\backprime P < {}^\backprime G$ a minimal parabolic then

- $P_0 = P = {}^\backprime P \times {}^\backprime \overline{P}, \qquad Q = {}^\backprime P \times {}^\backprime P.$
- $N_P / N_P \cap H \simeq N_{{}^\backprime P} \times \overline{N}_{{}^\backprime P}, \qquad N_Q / N_Q \cap H \simeq N_{{}^\backprime P}.$

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Theorem: Andersen, Flensted-Jensen, Schlichtkrull

Let $X = X_{p,q}$ be hyperbolic (over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$). Let Q be a minimal parabolic subgroup with $N_Q \cap H$ of maximal possible dimension. Then

$$\forall f \in \mathcal{C}(X): \qquad \int_{N_Q/(N_Q \cap H)} |f(n)| \, dn < \infty$$

This allows [A, F-J, S] to define a cusp form by the requirement

$$\forall x \in G: \qquad \int_{N_Q/(N_Q \cap H)} f(xn) \, dn = 0.$$

σ-split rank one

From now on $\dim \mathfrak{a}_{\mathrm{q}} = 1$

Define $\rho_{Qh} \in \mathfrak{a}^* : \; X \mapsto \frac{1}{2}\mathrm{tr}(\mathrm{ad}(\,\cdot\,)|_{\mathfrak{n}_Q \cap Z(\mathfrak{a}_{\mathrm{q}})})$

Theorem (vdB - K)

(a) There exist $Q \in \mathcal{P}_{\min}(A)$ which are *H*-compatible, i.e.

  (1) $\dim(N_Q \cap H)$ is max

  (2) $\langle \rho_{Qh}, \alpha \rangle \geq 0 \quad \forall \alpha \in \Sigma(\mathfrak{a}, \mathfrak{n}_Q)$.

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(b) If Q as in (a) then for all $f \in \mathcal{C}(X)$,

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Remark (vdB - K -S)

Condition (2) is really needed for $X = \mathrm{SL}(n, \mathbb{R})/\mathrm{GL}(n-1, \mathbb{R})$.
If Q is in (a.1), then (a.2) is restrictive, and

$$(a.2) \iff (b)$$

Outline of proof

Step 1: reduction to K-fixed positive f:

Let $f \in \mathcal{C}(X)$. Then $\quad \exists \varphi \in \mathcal{C}(X)^K : \quad |f| \leq \varphi \text{ on } X.$

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Step 2: Theorem

The operator $\mathcal{H}_Q : C_c(X)^K \to C^\infty(A_{\mathrm{q}})$ defined by

$$\mathcal{H}_Q f(a) = a^{\rho_Q} \int_{N_Q / N_Q \cap H} f(an)\, dn$$

extends to a continuous linear operator $\widetilde{\mathcal{H}}_Q : \mathcal{C}(X)^K \to C^\infty(A_{\mathrm{q}})$.

Proof: uses Plancherel formula for $\mathcal{C}(X)^K$.

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Final step

Let $f \in \mathcal{C}(X)^K, \quad f \geq 0. \quad \exists (f_n) \subset C_c^\infty(X)^K$ s.t. $f_n \nearrow f$ in $\mathcal{C}(X)$.

$\implies \quad \mathcal{H}_Q(f_n)(e) \nearrow \quad \& \quad \mathcal{H}_Q(f)(e) \to \widetilde{\mathcal{H}}_Q(f)(e)$

$\implies \quad \int_{N_Q/(N_Q \cap H)} f(n)\, dn < \infty.$

Eisenstein integrals

For simplicity assume G/P_0 has one open H-orbit.

Let $Q \in \mathcal{P}_{\min}(A)$, $\lambda \in \mathfrak{a}^*_{\mathfrak{q}\mathbb{C}}$.
Define

$$\psi_{Q,\lambda} : G \to \mathbb{C}, \quad kan_Q \mapsto a^{\lambda + \rho_{Qh} - \rho_Q}$$

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Definition: Eisenstein integral

$$E(Q,\lambda)(x) := \int_{H/(H\cap N_Q)} \psi_{Q,\lambda}(xh)\, dl_h(e)^{-1*}|\omega|$$

where

- $\omega \in \wedge^{\mathrm{top}} T_e(H/H\cap Q) \setminus \{0\}$
- $\mathrm{Re}\lambda$ sufficiently Q-dominant.

Extend $E(Q,\lambda) \in C^\infty(G/H)^K$ meromorphically in $\lambda \in \mathfrak{a}_{\mathrm{q}\mathbb{C}}^*$.

Fourier transform and Harish-Chandra transform

Fourier transform

Define $\mathcal{F}_Q^{\mathrm{un}} : C_c^\infty(G/H)^K \to \mathcal{M}(\mathfrak{a}_{\mathrm{q}\mathbb{C}}^*)$ by

$$\mathcal{F}_Q^{\mathrm{un}} f(\lambda) := \int_X f(x)\, E(Q : -\lambda)(x)\, dx.$$

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Relation to HC transform

$$\mathcal{F}_Q^{\mathrm{un}}(f)(\lambda) = \mathcal{F}_{\mathrm{eucl}}(\mathcal{H}_Q f)(\lambda)$$

for $\mathrm{Re}\lambda$ sufficiently dominant.

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$$\Longrightarrow \qquad \mathcal{H}_Q f(a) = \int_{\eta + i\mathfrak{a}_{\mathfrak{q}}^*} a^\lambda \mathcal{F}_Q^{\mathrm{un}}(f)(\lambda)\, d\lambda$$

for $\eta \in \mathfrak{a}_{\mathfrak{q}}^*$ sufficiently dominant.

Residual operators

Residual formula

Let $f \in C_c^\infty(X)^K$. Then

$$\mathcal{H}_Q f = T_Q f + R_Q f$$

where

$$T_Q f(a) := \lim_{\varepsilon \downarrow 0} \int_{i\mathfrak{a}_{\mathrm{q}}^* + \varepsilon\eta} a^\lambda \mathcal{F}_Q^{\mathrm{un}} f(\lambda)\, d\lambda$$

and

$$R_Q(f)(a) := 2\pi i \sum_j \operatorname*{Res}_{\lambda=\mu_j} \left(a^\lambda \mathcal{F}_Q^{\mathrm{un}} f(\lambda)\right)$$

Extension of T

Residual formula

Let $f \in C_c^\infty(X)^K$. Then

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Extension of T

There exists a tempered distribution v_Q on A_q such that

$$T_Q f = v_Q * \mathcal{F}_{\mathrm{eucl}}^{-1} \mathcal{F}_{\bar{P}_0} f$$

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Corollary

The operator T_Q extends to a continuous linear operator

$$\mathcal{C}(X)^K \to C^\infty(A_{\mathrm{q}})_{\mathrm{temp}}$$

Extension of R

Kernel for R

Let $f \in C_c^\infty(X)^K$. Then, for $a \in A_{\mathrm{q}}$,

$$R_Q f(a) = 2\pi i \sum_j \int_X f(x)\, R_j(a,x)\; dx$$

where $R_j(a,x) = \mathrm{Res}_{\lambda=\mu_j}\, a^\lambda E_Q(-\lambda, x)$

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Theorem

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Theorem

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Corollary

Let Q be H-compatible. Then R_Q extends to a continuous linear map

$$\mathcal{C}(X)^K \to \mathcal{E}(A_{\mathrm{q}}).$$

Final conclusions

Assumption: $\dim \mathfrak{a}_\mathfrak{q} = 1$.

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$$\mathcal{C}(X)_{\mathrm{res}}^{K} = \mathrm{span}\{\underset{\lambda=\mu_j}{\mathrm{Res}}(a^{\lambda} E(Q, -\lambda)) \mid a \in A_{\mathrm{q}}\}$$

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Theorem

$$\mathcal{C}(X)_{\mathrm{res}}^{K} = 0 \quad \Longrightarrow \quad \mathcal{C}(X)_{\mathrm{res}} = 0.$$

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Hartelijk gefeliciteerd!