

Bicoloured torus loop groups

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Printed by: Ipskamp Printing, Enschede
ISBN: 978-90-393-6743-8

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Bicoloured torus loop groups

Bigekleurde toruslusgroepen
(met een samenvatting in het Nederlands)

Proefschrift

ter verkrijging van
de graad van doctor aan de Universiteit Utrecht
op gezag van de rector magnificus, prof.dr. G.J. van der Zwaan,
ingevolge het besluit van het college voor promoties
in het openbaar te verdedigen op woensdag 15 maart 2017
des ochtends te 10.30 uur

door

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geboren op 6 mei 1989
te Lelystad

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Funding:

The research that resulted in this thesis was supported by a Graduate Funding Grant for the project ‘The classification of abelian Chern–Simons theories’, obtained from the ‘Geometry and Quantum Theory’ (GQT) mathematics cluster of the Netherlands Organisation for Scientific Research (NWO).

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Chapter 1

Introduction

The material collected and developed in this thesis is motivated by the following question:

Given that certain centrally extended loop groups and their positive energy representations give rise to conformal nets, does there exist a generalisation of the theory of loop groups which allows one to analogously construct defects between conformal nets?

We focus exclusively on a special class of loop groups, namely the *torus loop groups* which give rise to *lattice conformal nets*, and manage to make progress towards answering the above question. Our results can be summarised by the claim

Torus loop groups generalise to so called *bicoloured torus loop groups*. The latter enjoy many properties one expects them to have for constructing defects between lattice conformal nets.

Our first aim in this Introduction will be to explain the various terms used in the above question. We begin with a short review of von Neumann algebras in Section 1.1 because these feature prominently in the definition of conformal nets that follows next in Section 1.2. In that section we list some (mathematical) motivations for the theory of conformal nets and basic examples of these objects. We notably give an overview of the construction of certain conformal nets from central extensions of loop groups.

We then turn to introducing the notion of a defect between two conformal nets in Section 1.3 of which we present elementary examples. The scarcity of these examples will immediately spark the question of finding other methods of constructing defects. To answer this question we formulate in Section 1.4 a

loose, hypothetical notion of a bicoloured loop group as a method of producing conformal net defects. Our proposal starts properly in Section 1.5, where we make precise the special case of a definition of a bicoloured torus loop group and summarise the results we obtain in this thesis on this new notion.

We refer to Section 1.7 and Appendix A for various notations, conventions and definitions used in this Introduction.

Remark 1.0.1 (Advice to the reader). The question posed above and the material on von Neumann algebras, conformal nets and defects treated in Sections 1.1 to 1.3 has been included only to explain the context of our studies, and it will not make a relevant reappearance until Section 4.1. The bulk of our investigations in Chapters 2 and 3 does not strictly require knowledge of these matters. The reader may therefore safely skip Sections 1.1 to 1.3 without missing technical background needed for the rest of the thesis.

1.1 Von Neumann algebras

Von Neumann algebras were introduced by F. Murray and J. von Neumann in a series of papers in the 1930s and 40s with applications to representation theory and physics in mind. Their relevance to this Introduction is their appearance in the definitions of conformal nets and defects we give in Sections 1.2 and 1.3, respectively, and they will make a brief reappearance in Chapter 4. We assume a passing familiarity with the weak, the strong operator and ultraweak topology on the algebra of bounded operators on a Hilbert space from the reader, as treated in for example [Con90, Chapters IV and IX].

Let \mathcal{H} be a Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ its algebra of bounded operators. For a subset $S \subseteq \mathcal{B}(\mathcal{H})$, its *commutant* S' is defined as the set of bounded operators on \mathcal{H} which commute with all operators of S . We list some algebraic properties of taking commutants. It reverses inclusions of subsets, and we have $S \subseteq S''$ and $S''' = S'$. This means that the operation of taking the commutant does not continue indefinitely. It is always true that S' is a unital subalgebra of $\mathcal{B}(\mathcal{H})$ and if S is *self-adjoint*, meaning that $S^* = S$, then S' is a unital $*$ -subalgebra.

A topological property of forming the commutant is that S' , and hence also S'' is weakly closed in $\mathcal{B}(\mathcal{H})$. The following important result strengthens this fact dramatically when S is a unital $*$ -algebra.

Theorem 1.1.1 (Von Neumann's Bicommutant Theorem). *The double commutant A'' of a unital $*$ -subalgebra A of $\mathcal{B}(\mathcal{H})$ is equal to both the closure of A in the weak, and in the strong operator topology.*

That is, the algebraic operation of taking the double commutant of a unital $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$ can be expressed in two, equivalent topological terms. It implies that the following definition is unambiguous.

Definition 1.1.2 ((Concrete) von Neumann algebras). A (concrete) von Neumann algebra is a unital $*$ -subalgebra A of the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space \mathcal{H} which equivalently

- is closed in the weak operator topology on $\mathcal{B}(\mathcal{H})$,
- is closed in the strong operator topology on $\mathcal{B}(\mathcal{H})$, or
- satisfies $A'' = A$.

One can furthermore use the Bicommutant Theorem to prove

Corollary 1.1.3. *The smallest von Neumann algebra $\text{vN}(S)$ containing a subset $S \subseteq \mathcal{B}(\mathcal{H})$ equals both $(S \cup S^*)''$ and the closure of the unital $*$ -subalgebra generated by S in either the weak or strong operator topology.*

So elements of $\text{vN}(S)$ can be thought of as limits (in the weak or strong operator topology) of polynomials with (non-commuting) variables in $S \cup S^*$.

We give some examples of von Neumann algebras. The most obvious one is of course $\mathcal{B}(\mathcal{H})$ itself. Slightly more interesting are

Example 1.1.4 (Algebras of essentially bounded functions). Let (X, Σ, μ) be a σ -finite measure space and $L^2(X) := L^2(X, \Sigma, \mu)$ the Hilbert space of all measurable functions $f : X \rightarrow \mathbb{C}$ which are square integrable, modulo functions that are zero almost everywhere. Define next $L^\infty(X) := L^\infty(X, \Sigma, \mu)$ to be the set of measurable functions $f : X \rightarrow \mathbb{C}$ which are essentially bounded, divided out by the same equivalence relation as for $L^2(X)$. With the pointwise multiplication, the $*$ -operation $f^* := \bar{f}$ and the essential supremum norm this is a C^* -algebra. There is a unital, isometric $*$ -homomorphism

$$L^\infty(X) \hookrightarrow \mathcal{B}(L^2(X)), \quad f \mapsto m_f,$$

where m_f is left multiplication by f . It can then be proved that $L^\infty(X)' = L^\infty(X)$ (see [Con90, Theorem 6.6]), which implies that $L^\infty(X)$ is a von Neumann subalgebra of $\mathcal{B}(L^2(X))$.

The above are examples of *abelian* von Neumann algebras. It can in fact be shown that *all* abelian von Neumann algebras on a separable Hilbert space are of this form (see [Con90, Theorem 7.8]). This is the reason why the general

theory of von Neumann algebras is sometimes referred to as ‘non-commutative measure theory’.

The von Neumann algebras that are relevant to us in this Introduction are far from abelian, though. Their construction will instead be more similar in spirit to that of

Example 1.1.5 (Group von Neumann algebras). Let $Q: G \rightarrow U(\mathcal{H})$ be a unitary representation of a topological group G . Then $Q(G)'$ and $Q(G)''$ are both von Neumann algebras. When G is locally compact and Q_L and Q_R are the left, respectively, right regular representation with respect to Haar measures on G we call $Q_L(G)''$ and $Q_R(G)''$ the left and right *group von Neumann algebras* of G . They have the special feature that they are each others commutant [Tak03, Proposition VII.3.1]. An elementary example showing that a group von Neumann algebra reflects properties of G is the fact that if G is countable then it is amenable if and only if the algebra $Q_L(G)''$ is the weak closure of an ascending sequence of finite-dimensional $*$ -subalgebras.

In order to define homomorphisms of von Neumann algebras it is useful to introduce a fourth, ‘coordinate-free’ definition of the latter, which can be shown to be equivalent to the ones listed in Definition 1.1.2:

Definition 1.1.6 ((Abstract) von Neumann algebras). (Taken from [BDH14a, Definition A.1].) An (*abstract*) *von Neumann algebra* is a topological, unital $*$ -algebra A (we do not require the multiplication to be continuous) for which there exists an injective unital $*$ -homomorphism $A \hookrightarrow \mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} that is a homeomorphism onto the image of A and such that the image of A is closed with respect to the ultraweak topology on $\mathcal{B}(\mathcal{H})$.

A *homomorphism* between two (abstract) von Neumann algebras is a unital, continuous $*$ -homomorphism.

We close this section with a brief discussion of the *standard form* of a von Neumann algebra. Recall that, given a C^* -algebra A and a positive state f on it, the so called GNS-construction allows one to build a Hilbert space $L^2(A, f)$ and a left action Q of A on $L^2(A, f)$ such that Q is cyclic for some cyclic vector Ω and $f(a) = \langle Q(a)\Omega, \Omega \rangle$ for all $a \in A$. This construction is unique in the sense that it only depends on f .

However, when A is a von Neumann algebra and f a faithful, continuous state, remarkably, the GNS-construction is endowed with extra structures sufficient to characterise $L^2(A, f)$ even independently from f . We call it the *standard form* of A and denote it by L^2A . We refer to [BDH14a, p. 46] and [Haa76] for a listing of these characterising structures and we only point out one of them

here, namely that L^2A is an A - A -bimodule. In general, A might not have any faithful, continuous states, so that the GNS-construction can not be used as a model for the standard form. A different, more widely applicable model can be found in [Tak03, Section IX.1].

1.2 Conformal nets

The notion of a conformal net has its origins in the field of *algebraic quantum field theory* (AQFT)—a topic initiated in the 1960s by R. Haag and D. Kastler which seeks to describe and study quantum field theories (QFT's) in a mathematically rigorous way through operator algebraic methods. More specifically, conformal nets are one possible mathematical model, based on general axioms formulated by Haag and Kastler, of a class of QFT's called *chiral conformal field theories* (chiral CFT's). We refer to [Ara99] for general background material on AQFT and to [KR09] for a physical motivation of the study of conformal nets, and we continue by giving their definition.

Consider the topological group

$$\mathrm{SU}(1, 1) := \left\{ \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} \mid \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 - |\beta|^2 = 1 \right\}.$$

It acts by orientation-preserving diffeomorphisms on S^1 by setting for $g \in \mathrm{SU}(1, 1)$ and¹ $z \in S^1$,

$$g \cdot z := \frac{\alpha z + \beta}{\bar{\beta} z + \bar{\alpha}}.$$

Clearly, the central subgroup $\{\pm I\} \subseteq \mathrm{SU}(1, 1)$ then acts trivially and we therefore get a well-defined action of the quotient group $\mathrm{PSU}(1, 1) := \mathrm{SU}(1, 1)/\{\pm I\}$. We call $\mathrm{PSU}(1, 1)$ the *Möbius group* and abbreviate its notation to $\mathrm{Möb}$. The group $\mathrm{Rot}(S^1)$ of counterclockwise rotations of S^1 embeds into $\mathrm{Möb}$ via

$$\varphi_\theta \mapsto \begin{bmatrix} e^{\pi i \theta} & 0 \\ 0 & e^{-\pi i \theta} \end{bmatrix},$$

where φ_θ is the rotation along an angle $\theta \in [0, 1]$.

All representations considered here are strongly continuous and unitary, and some elementary notions regarding them are presented in Appendix A.3. In this Introduction we will also have use for projective representations.

¹This is the sole exception we make in this thesis on our convention of denoting points on S^1 as θ , p or q .

Definition 1.2.1. A *projective representation* of a topological group G on a Hilbert space \mathcal{H} is a continuous homomorphism of G into the quotient group $\text{PU}(\mathcal{H}) := \text{U}(\mathcal{H})/\text{U}(1)$. Here, we give $\text{PU}(\mathcal{H})$ the quotient topology inherited from the strong operator topology on $\text{U}(\mathcal{H})$.

Denote by INT_{S^1} the poset of subintervals of S^1 and for a Hilbert space \mathcal{H} by $\text{VN}_{\mathcal{H}}$ the poset of von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$, both ordered by inclusion.

Definition 1.2.2. A (*concrete, positive energy*) *conformal net* consists of a Hilbert space \mathcal{H} called the *vacuum sector*, a unit vector $\Omega \in \mathcal{H}$ called the *vacuum vector*, a map of posets

$$\mathcal{A}: \text{INT}_{S^1} \rightarrow \text{VN}_{\mathcal{H}}, \quad I \mapsto \mathcal{A}(I) \quad (1.1)$$

and a representation U of Möb on \mathcal{H} satisfying the following properties:

- (i) (*Locality*) if $I, J \in \text{INT}_{S^1}$ are intervals with disjoint interiors, then the algebras $\mathcal{A}(I)$ and $\mathcal{A}(J)$ commute,
- (ii) (*Diffeomorphism covariance*) U extends to a projective representation (which we will still denote by U) of $\text{Diff}_+(S^1)$ on \mathcal{H} such that²

$$U(\varphi)\mathcal{A}(I)U(\varphi)^* = \mathcal{A}(\varphi(I))$$

for all $\varphi \in \text{Diff}_+(S^1)$ and $I \in \text{INT}_{S^1}$, and, moreover, if φ has support in I then $U(\varphi)$ commutes with $\mathcal{A}(I')$,

- (iii) (*Positivity of energy*) the restriction of U to the subgroup $\text{Rot}(S^1)$ of Möb is of positive energy in the sense of Definition A.3.13,
- (iv) (*Vacuum axiom*) the vector Ω is invariant under the action U of Möb and cyclic for the von Neumann algebra $\text{vN}(\cup_{I \subseteq S^1} \mathcal{A}(I))$.

When $I \subseteq J$ is an inclusion of intervals the corresponding inclusion $\mathcal{A}(I) \hookrightarrow \mathcal{A}(J)$ of von Neumann algebras is called an *isotony homomorphism*.

An *isomorphism* from \mathcal{A} to another conformal net \mathcal{A}' is an isomorphism of Hilbert spaces $\mathcal{H} \rightarrow \mathcal{H}'$ which sends Ω to Ω' and intertwines the respective poset maps.

When referring to a conformal net we will often omit the data \mathcal{H} , Ω and U and simply write \mathcal{A} , as we already did in the definition of an isomorphism of nets.

²Even though $U(\varphi)$ for $\varphi \in \text{Diff}_+(S^1)$ might not be a well-defined operator on \mathcal{H} , the two demands that we ask of $U(\varphi)$ in this axiom remain unambiguous.

Remark 1.2.3. We do not include the extension of U to $\text{Diff}_+(S^1)$ as a datum because it is shown in [CW05, Theorem 5.5] that, under an additional, mild assumption, the restriction to Möb and the listed other properties already determine this extension uniquely. Those authors even speculate that that assumption is not necessary.

Remark 1.2.4. For the definition of an isomorphism of conformal nets we do not require the isomorphism of Hilbert spaces to also intertwine the representations of Möb because it is a (highly non-trivial) fact that this holds automatically (see [KL04, p. 499]).

Remark 1.2.5. The usage of the word ‘net’ is slightly strange because a conformal net is not a net in the point-set topological sense. The poset INT_{S^1} namely does not form a directed set because two intervals that cover S^1 do not have an upper bound in INT_{S^1} . The alternative term *conformal pre-cosheaf* that some authors use might be more suitable.

The definition of a conformal net involves by definition a Hilbert space on which the von Neumann algebras are represented. When the algebras can be made to act on a different Hilbert space as well, this is given its own name.

Definition 1.2.6. Let \mathcal{A} be a conformal net. A *sector*³ Q of conformal net \mathcal{A} consists of a Hilbert space \mathcal{H} and a family of von Neumann algebra homomorphisms $Q_I: \mathcal{A}(I) \rightarrow \mathcal{B}(\mathcal{H})$ for all $I \in \text{INT}_{S^1}$ which is compatible with the isotony homomorphisms of \mathcal{A} , meaning that $Q_I|_{\mathcal{A}(J)} = Q_J$ if $J \subseteq I$. A *morphism* from Q to another sector Q' with underlying Hilbert space \mathcal{H}' is a bounded linear map $\mathcal{H} \rightarrow \mathcal{H}'$ which intertwines the homomorphisms Q_I and Q'_I for all $I \in \text{INT}_{S^1}$.

Remark 1.2.7. One may ask why the definition of a sector does not include an action of Möb which, just like for the vacuum sector, intertwines covariantly with the algebra homomorphisms Q_I and restricts to a positive energy representation of $\text{Rot}(S^1)$. The answer is that this has been shown to hold automatically by [DFK04, Theorem 5] and [Wei06, Theorem 3.8], with the subtlety added that on a general sector usually only the universal covering group of Möb acts.

1.2.1 Examples of conformal nets

Let us present some basic examples of families of conformal nets. Our discussions will be deliberately cursory. We start with a family that does not have

³The literature often uses this term for an isomorphism class of what we call sectors. We follow the terminology of [BDH14a] instead.

the focus of this thesis, but whose relatively simple construction will serve to illustrate features common to the constructions of the other, more complicated families.

Example 1.2.8 (Heisenberg nets). For every finite-dimensional real vector space F with a non-degenerate, symmetric, positive definite bilinear form $\langle \cdot, \cdot \rangle: F \times F \rightarrow \mathbb{R}$ on it there exists an associated *Heisenberg conformal net* \mathcal{A}_F . We outline its construction and we refer to [DX06] and [Bis12] for details.

One first forms the *loop group* $LF := C^\infty(S^1, F)$ of all smooth maps from S^1 to F , equipped with the point-wise multiplication. It admits a canonical decomposition $LF \xrightarrow{\sim} F \oplus VF$, where VF is the real vector space of all loops in LF whose average over S^1 is zero. We will explain in Section 2.5.1 how the form $\langle \cdot, \cdot \rangle$ together with the holomorphic structure on the unit disc can be used to turn VF into a complex pre-Hilbert space—hence making LF a topological group.

Next, the form $\langle \cdot, \cdot \rangle$ is used to construct a central extension $\tilde{L}F$ of LF by the group $U(1)$ through an explicit continuous 2-cocycle on LF . The extension satisfies the so called *disjoint-commutativity* property: if $\tilde{L}_I F$ denotes for an interval $I \subseteq S^1$ the pre-image in $\tilde{L}F$ of those loops in LF with support in I , then $\tilde{L}_I F$ and $\tilde{L}_J F$ are commuting subgroups of $\tilde{L}F$ whenever I and J have disjoint interiors. The left action of $\text{Diff}_+(S^1)$ on LF given by precomposition with inverses of circle diffeomorphisms lifts to $\tilde{L}F$ in such a way that if $\varphi \in \text{Diff}_+(S^1)$ then $\varphi \cdot \tilde{L}_I F = \tilde{L}_{\varphi(I)} F$ and if φ has support in I then it acts trivially on $\tilde{L}_I F$.

In particular, $\text{Rot}(S^1)$ acts on $\tilde{L}F$ and it therefore makes sense to discuss its *positive energy* representations. For every $\alpha \in F$ there exists a particular irreducible such representation W_α with underlying Hilbert space \mathcal{S}_α and it turns out that every irreducible, positive energy representation of $\tilde{L}F$ is isomorphic to one of this form up to the character by which the central subgroup $U(1) \subseteq \tilde{L}F$ acts. A general positive energy representation is hence a direct sum of these W_α 's.

The representation \mathcal{S}_0 , which we call the *vacuum representation* of $\tilde{L}F$ in anticipation, carries even more structure than the action R of $\text{Rot}(S^1)$ that is part of the positive energy property. There exists a projective representation U of $\text{Diff}_+(S^1)$ on \mathcal{S}_0 whose interaction with W_0 is described by the intertwining relation

$$U(\varphi)W_0(\tilde{L}_I F)U(\varphi)^* = W_0(\tilde{L}_{\varphi(I)} F)$$

for all $I \in \text{INT}_{S^1}$. It restricts to an honest representation of Möb fixing a certain unit vector $\Omega \in \mathcal{S}_0$ and extends R . (See [Seg81, Section 5] and [Vro13, Subsection 5.3.2] for these facts.)

We now create for every interval $I \in \text{INT}_{S^1}$ the von Neumann algebra

$$\mathcal{A}_F(I) := \text{vN}\left(W_0(\tilde{L}_I F)\right)$$

acting on the Hilbert space \mathcal{S}_0 . That is, we take the $*$ -closed set $W_0(\tilde{L}_I F)$ of unitary operators and then form its von Neumann-algebraic completion inside $\mathcal{B}(\mathcal{S}_0)$. This obviously defines a map of posets \mathcal{A}_F as in (1.1). We claim that this is a conformal net. The locality axiom is deduced from the disjoint-commutative property of $\tilde{L}F$. Given the cited results, diffeomorphism covariance, positivity of energy and invariance of Ω of course also follow. Finally, the cyclicity of Ω holds because Ω is cyclic for $\tilde{L}F$ and this group is generated by its subgroups $\tilde{L}_I F$. This construction of \mathcal{A}_F only makes use of the representation W_0 . The role of the other positive energy representations of $\tilde{L}F$ is that each of them can be equipped with the structure of a sector of the net \mathcal{A}_F .

Heisenberg nets derive their name from the fact that the restriction of $\tilde{L}F$ to VF is a *Heisenberg group* in the sense of Definition A.4.1.

Example 1.2.9 (Lattice nets). The contents of this thesis are exclusively motivated by the class of *lattice nets*. The rough idea behind creating one of them is to replace the real vector space F in Example 1.2.8 by a different real Lie group, namely a torus T . Furthermore, the role of the \mathbb{R} -valued form $\langle \cdot, \cdot \rangle$ on F is taken over by the structure of an even, positive definite lattice on the free, finite rank \mathbb{Z} -module $\ker(\exp)$, where $\exp: \mathfrak{t} \rightarrow T$ is the exponential map on the Lie algebra \mathfrak{t} of T . (See Appendix A.1 for an introduction to lattices.) The construction of the lattice net then proceeds very similar to that of a Heisenberg net, except that the fact that a torus is not simply-connected introduces many new complications. (Essentially, the construction of a lattice net breaks up into two steps: first forging the Heisenberg net $\mathcal{A}_\mathfrak{t}$ and, next, enlarging $\mathcal{A}_\mathfrak{t}$ appropriately to form the lattice net. We will not emphasise this viewpoint, though.) Note that we might as well take an even lattice Λ as the primary datum and then define $T := \Lambda \otimes_{\mathbb{Z}} \text{U}(1)$.

We begin again by forming the loop group $LT := C^\infty(S^1, T)$, which can be made into a topological group. It is not connected: each connected component is labelled by an element of Λ which signifies how (many times, when T is 1-dimensional) a loop winds around T . The bi-additive form on Λ is then, together with some minor extra data, used to construct a $\text{U}(1)$ -central extension $\tilde{L}T$ through a 2-cocycle on LT that is continuous on its identity component and similar to, but more complex than the one on LF . Up to non-canonical isomorphism, the extension is independent of the aforementioned extra data. It is disjoint-commutative and admits an action of $\text{Diff}_+(S^1)$ with identical

properties as for $\tilde{L}F$. A major difference of $\tilde{L}T$ with $\tilde{L}F$ is that the former possesses only finitely many isomorphism classes of irreducible, positive energy representations after fixing the character by which $U(1) \subseteq \tilde{L}T$ acts. These classes are namely labelled by the *discriminant group* $D_\Lambda := \Lambda^\vee / \Lambda$ of Λ .

We single out a particular representation denoted by $\text{Ind } W_0$ of $\tilde{L}T$ of the type just mentioned, representing the class corresponding to $0 \in D_\Lambda$, and name it the *vacuum representation*. It carries a projective representation of $\text{Diff}_+(S^1)$, intertwining with $\text{Ind } W_0$ and satisfying the same properties as mentioned in Example 1.2.8. One then constructs a conformal net \mathcal{A}_Λ via the same method as for a Heisenberg net. (We conjecture that this procedure works for an odd lattice as well, but that this will lead to a $\mathbb{Z}/2\mathbb{Z}$ -graded net in the sense of [DH, Definition 3.7].) The other positive energy representations of $\tilde{L}T$ carry the structure of a sector of \mathcal{A}_Λ (see [DX06, Proposition 3.15]).

Chapter 2 of this thesis is devoted to explaining the details on $\tilde{L}T$ and its representation theory. For example, the crucial formula for the cocycle defining it is stated in (2.7). We refer to [Seg81], [PS86], [BMT88] [Sta95], [DX06] and [Bis12] for further material. (We warn that the statement of [PS86, Proposition (9.5.14)] is likely wrong for representations that are not irreducible because the direct sum of two projective representations is in general not again a projective representation.) Some of these listed authors do not explicitly state that their constructions are valid for an arbitrary even, positive definite lattice, instead of only for one of *ADE*-type.

Example 1.2.10 (Affine Kac–Moody nets). An *affine Kac–Moody net* $\mathcal{A}_{G,k}$ is associated to a pair of a compact, connected, simple, simply-connected Lie group G and a choice of a multiple of a certain normalisation of the Killing form on the Lie algebra of G by a positive integer k called a *level*. The literature sometimes refers to this as a *loop group net*, but we believe this to be slightly confusing since a Heisenberg or lattice net deserves that name as well.

The construction of this net starts again with the loop group LG , that is, $C^\infty(S^1, G)$. The level k then determines a $U(1)$ -central extension $\tilde{L}G$. Unlike in the situations of Examples 1.2.8 and 1.2.9, $\tilde{L}G$ is not topologically trivialisable as a $U(1)$ -bundle over LG . It is therefore not possible to build $\tilde{L}G$ by prescribing a globally continuous 2-cocycle on $(LG)^{\times 2}$ —continuity in a neighbourhood around $(1, 1) \in (LG)^{\times 2}$ is the best one can hope for. Instead, [PS86, Sections 4.4 and 4.5], [GF93, Section III.3] and [Wal16, Example 2.6] present means to build $\tilde{L}G$ in a more geometric manner. Again, $\tilde{L}G$ satisfies disjoint-commutativity and it carries a $\text{Diff}_+(S^1)$ -action with desirable properties regarding supports of loops.

Depending on the group G , there exist various methods for constructing the irreducible, positive energy representations of $\tilde{L}G$ and their intertwining projective $\text{Diff}_+(S^1)$ -actions. A uniform procedure is explained in [BDH14a, Section 4.c] which involves building representations of a dense subalgebra of the complexified Lie algebra of $\tilde{L}G$ first and integrating these next using the techniques in [Tol99]. Up to isomorphism, $\tilde{L}G$ has only finitely many irreducible, positive energy representations after fixing the character by which $U(1) \subseteq \tilde{L}G$ acts (see [PS86, Theorem (9.3.5)]). (We again warn about the absence of the irreducibility assumption in [PS86, Theorems (9.3.1)(v) and (13.4.2)].)

The further construction of the affine Kac–Moody net $\mathcal{A}_{G,k}$ using a vacuum representation of $\tilde{L}G$ is formally similar to that of Heisenberg and lattice nets and we refer for its details to [BDH14a, Section 4.c]. It is worth mentioning that there exists a precise relationship between affine Kac–Moody and lattice nets: when G is *simply-laced* $\mathcal{A}_{G,1}$ is isomorphic to a net associated to a lattice of *ADE*-type (see [Bis12, Proposition 3.19]).

We named affine Kac–Moody nets so because the dense subalgebra of the complexified Lie algebra of $\tilde{L}G$ mentioned earlier is a *Kac–Moody algebra of affine type*.

Remark 1.2.11. The positive (definite) assumptions in Examples 1.2.8 to 1.2.10 are not needed for the constructions of the central extensions of the respective loop groups. They are, however, necessary and sufficient for the natural bilinear forms on their representation spaces to be positive definite, so that the latter can be completed into Hilbert spaces.

Loop groups are not the only source of conformal nets:

Example 1.2.12 (Virasoro nets). For certain real numbers c one can define a so called *Virasoro net* $\mathcal{A}_{\text{Vir},c}$. The range of admissible values of c is the union of the closed half-line $[1, \infty)$ with a particular infinite discrete set in the interval $[0, 1)$, namely

$$\left\{ 1 - \frac{6}{m(m+1)} \mid m = 2, 3, 4, \dots \right\}.$$

The general idea of constructing Virasoro nets is to replace the loop groups in Examples 1.2.8 to 1.2.10 with $\text{Diff}_+(S^1)$. There exists a particular non-trivial $U(1)$ -central extension, named the *Virasoro–Bott group*, of $\text{Diff}_+(S^1)$ which is characterised by the fact that its restriction over the subgroup $\text{PSU}(1, 1)$ of $\text{Diff}_+(S^1)$ is trivial. Because it is topologically trivialisable as a $U(1)$ -bundle over $\text{Diff}_+(S^1)$ it can be defined by a continuous 2-cocycle on $\text{Diff}_+(S^1)$ called the *Thurston–Bott cocycle* (see [KW09, Section II.2]).

One way to build irreducible, positive energy representations of the Virasoro–Bott group is similar to the method mentioned in Example 1.2.10: the integration of certain representations of a dense subalgebra of its complexified Lie algebra using the results of, for example, [Tol99]. The peculiar restrictions on the values of c are the result of investigating when these Lie algebra representations are unitary. We refer to [Car04] and [Wei05, Section 3.3] for further information on the construction of Virasoro nets. A proof that $\text{Diff}_+(S^1)$ is generated by its subgroups of diffeomorphisms supported in an interval can be found in [Lok94, Proposition 1.2]. The irreducible representations of $\mathcal{A}_{\text{Vir},c}$ have been classified in [Wei06, Corollary 3.9] and [Wei16].

Virasoro nets play a special role in the general theory of conformal nets because it can be shown that every net, thanks to the axiom of diffeomorphism covariance and the *Haag duality* theorem, contains some Virasoro net. In turn, Virasoro nets cannot contain subnets strictly themselves [Car98]. This is why (at least for $c < 1$) they are sometimes named *minimal models* in the physics literature.

The examples we listed above by no means exhaust all conformal nets. They have in common that they are constructed in terms of ‘external’ data, such as a loop group or the group $\text{Diff}_+(S^1)$ and a choice of level. There exist, however, also plenty of constructions one can perform ‘internally’ to the category of conformal nets, meaning that they take one or more nets as input to produce another. Examples of such constructions are direct sums, tensor products, extensions, orbifolds, mirror extensions and coset constructions. Using these one can construct many more examples of nets. We will not discuss these techniques further.

A final method of producing conformal nets we mention is the recent work [Car+15] in which those authors show how to construct nets from different mathematical models of chiral CFT’s: the unitary *vertex algebras*.

The technique of orbifolding shows that a naive classification of all conformal nets is as infeasible as that of all finite groups. Nevertheless, successes have been booked on the ‘relative’ problem of classifying nets that contain a fixed one when the inclusion satisfies a certain finiteness condition. See for example [KL04] for a tabulation of all nets for which the embedded Virasoro net $\mathcal{A}_{\text{Vir},c}$ satisfies $c < 1$.

1.2.2 Mathematical applications of conformal nets

While the early literature on conformal nets has a large focus on answering questions motivated by physical considerations, their theory found relevance

to areas closer to pure mathematics also. We mention three of these.

Sporadic groups and Moonshine Even though the original definition of a vertex algebra in [Bor86] heavily used notions inspired by the physics literature, R. Borcherds was at the time not motivated by the question of giving a mathematical formalisation of a chiral CFT [Bor11]. He instead attempted to axiomatise, and exhibit further structure on certain constructions made by I. Frenkel, V. Kac, J. Lepowsky and A. Meurman in order to define and study the so called *Monster vertex algebra*. It is a vertex algebra of which the automorphism group is precisely the *Monster group*—the largest of the 26 sporadic finite simple groups—and it played an important role in Borcherds’ eventual proof in [Bor92] of J.H. Conway and S.P. Norton’s main *Monstrous Moonshine conjecture*.

Because (unitary) vertex algebras and conformal nets are both roughly mathematical models of the same physical notion, it is reasonable to expect that the Monster vertex algebra has a conformal net counterpart. Such a net has indeed been constructed in [KL06] by taking an orbifold subnet of the conformal net associated to the Leech lattice followed by a net extension. A different construction has been given in [Car+15], in which its authors also construct a conformal net analogue of G. Höhn’s *Baby Monster vertex algebra* of which the automorphism group is the *Baby Monster*—the second largest of the sporadic finite simple groups.

Modelling topological field theories The category $\text{Sect } \mathcal{A}$ of sectors of a conformal net \mathcal{A} carries far more structure than we indicated in this Introduction. Most significantly, there is a way of taking the tensor product of two sectors which makes $\text{Sect } \mathcal{A}$ into a *tensor category*. That this tensor product is very different from the naive one already follows from the facts that it is in general not symmetric, but braided, and that the vacuum sector of \mathcal{A} is a monoidal unit object.

When \mathcal{A} satisfies a certain finiteness condition $\text{Sect } \mathcal{A}$ is of an even more special type: it is a *modular tensor category* (MTC) (see [KLM01, Corollary 37] and [BDH14b, Theorem 3.9]). Modular tensor categories are ‘rare’ mathematical objects (in the words of [MNS12]) in the sense that few methods are known for producing them. One motivation for studying them is that each MTC gives rise to a *3-dimensional topological quantum field theory* (3d TQFT) and hence to an invariant of knots and smooth 3-manifolds. The objects of the MTC then correspond to the bulk line operators of the TQFT and the morphisms can be interpreted as local operators sitting at the junction between two bulk line operators.

Lattice nets are known to satisfy this finiteness criterion [DX06, Corollary 3.19] and the same is conjectured to hold for affine Kac–Moody nets, even though this has only been shown in particular cases (see for example [BDH14a, Theorem 4.18] and the references cited therein). The TQFTs arising from lattice and affine Kac–Moody nets are expected to equal the *abelian Chern–Simons*, and (*ordinary*) *Chern–Simons theories*, respectively. (Because these theories are not defined mathematically, this claim contains ample room for interpretation.)

This relation between finite conformal nets and 3d TQFTs is in general many-to-one, though: there might exist non-isomorphic conformal nets of which their categories of sectors are equivalent as MTCs and hence produce the same TQFTs. For example, we conjecture following the work of [Höh03] that this occurs for nets associated to lattices that lie in the same *genus*.

Defining elliptic cohomology geometrically In the last quarter of the 20th century a new family of generalised cohomology theories was discovered in the field of algebraic topology—the so called *elliptic cohomology theories*. There is in general no reason for an arbitrary generalised cohomology theory to have an interpretation in terms of the geometry of the spaces one evaluates it on in the same way that, for example, topological *K*-theory can be defined using vector bundles over the spaces. However, a web of conjectures has gradually been formed suggesting that the elliptic cohomology theories could have a quantum field-theoretic, geometric definition. Based on joint work with A. Bartels, C. Douglas and A. Henriques have started a project investigating whether a certain ‘universal’ elliptic cohomology theory, named TMF, can be modelled via conformal nets (see [DH11] for a survey of some of their results). An early success they and B. Janssens achieved in unpublished work is a definition in terms of conformal nets of the *String groups*. These groups are related to TMF in a way that is similar to the relation between the Spin groups and real topological *K*-theory.

1.3 Defects between conformal nets

Up to this point we have mostly been discussing the theory of conformal nets as it is described in the more traditional literature. We now turn our focus to recent work of A. Bartels, C. Douglas and A. Henriques which revolves around their new notion of a *defect* between conformal nets. Before giving its

definition, we first introduce an alternative definition of a conformal net devised by those authors which simultaneously is a ‘coordinate-free’ formulation of Definition 1.2.2 and mildly generalises it.

Denote by INT the category of which the objects are intervals⁴ and the morphisms are smooth embeddings and write VN for the category of which the objects are (abstract) von Neumann algebras and the morphisms are either von Neumann algebra homomorphisms or antihomomorphisms. If $I \in \text{INT}$ is an interval, $\text{Diff}_+(I)$ stands for the group of its orientation-preserving diffeomorphisms.

Definition 1.3.1. (Taken from [BDH14a, Definition 1.1].) An (*abstract conformal net*) is a continuous covariant functor $\mathcal{A}: \text{INT} \rightarrow \text{VN}$ sending orientation-preserving and reversing embeddings to injective algebra homomorphisms and antihomomorphisms, respectively, which satisfies the following properties:

- (i) (*Locality*) if $I \hookrightarrow K$ and $J \hookrightarrow K$ are embeddings of intervals of which the images have disjoint interiors, then the images of the corresponding homomorphisms or antihomomorphisms $\mathcal{A}(I) \hookrightarrow \mathcal{A}(K)$ and $\mathcal{A}(J) \hookrightarrow \mathcal{A}(K)$ commute inside $\mathcal{A}(K)$,
- (ii) (*Inner covariance*) if $\varphi \in \text{Diff}_+(I)$ is the identity in a neighbourhood of the endpoints of I , then there exists a unitary operator $U \in \mathcal{A}(I)$ such that $\text{Ad}(U) = \mathcal{A}(\varphi)$,
- (iii) (*Existence of a vacuum sector*) assume that $I \subsetneq K$ is a pair of an interval and a properly included subinterval such that I contains a boundary point p of K and write \bar{I} for the same manifold as I , but equipped with the reversed orientation. Then $\mathcal{A}(I)$ and $\mathcal{A}(\bar{I})$ both act on the left of the standard form $L^2\mathcal{A}(K)$ —the former via the homomorphism $\mathcal{A}(I) \hookrightarrow \mathcal{A}(K)$ and the latter through the homomorphism $\mathcal{A}(\bar{I}) \xrightarrow{\sim} \mathcal{A}(I)^{\text{op}} \hookrightarrow \mathcal{A}(K)^{\text{op}}$.

We then demand that the left action of the algebraic tensor product $\mathcal{A}(I) \otimes \mathcal{A}(\bar{I})$ on $L^2\mathcal{A}(K)$ extends to an action of the algebra $\mathcal{A}(I \cup_p \bar{I})$ along the homomorphism $\mathcal{A}(I) \otimes \mathcal{A}(\bar{I}) \rightarrow \mathcal{A}(I \cup_p \bar{I})$. Here, we mean by $I \cup_p \bar{I}$ any interval obtained by gluing I and \bar{I} along p with a smooth structure that extends those of I and \bar{I} and such that the involution which swaps I and \bar{I} is smooth.

A *morphism* between two abstract conformal nets is defined to be a natural transformation.

⁴See Section 1.7 for our definition of an interval.

The meaning of the term *continuous* in the above definition can be found in [BDH14a].

Remark 1.3.2. The authors of [BDH14a] include two additional axioms in their definition of an abstract conformal net, namely those of *strong additivity* and *splitness*. We did not do so both to simplify the definition for the uninitiated reader and because one would otherwise be excluding some natural, non-pathological examples of conformal nets. For example, we did not include strong additivity since this property is not satisfied by Virasoro nets $\mathcal{A}_{\text{Vir},c}$ when $c > 1$ [BS90]. However, we warn that many structural results in [BDH14a] do rely on these two axioms.

It is at first sight far from clear in what way the Definition 1.3.1 of an abstract conformal net is a generalisation of that of a concrete, positive energy net in Definition 1.2.2. Three features in which they differ immediately stand out:

- for an abstract net one no longer posits the existence of an ambient Hilbert space on which the von Neumann algebras associated to intervals act. The reason for this is that the *Reeh–Schlieder* and *Bisognano–Wichmann theorem* imply that the axioms of a concrete, positive energy net \mathcal{A} are sufficiently strong to reconstruct the vacuum sector as the standard form $L^2\mathcal{A}(K)$ for *any* interval $K \subseteq S^1$. Conversely, the purpose of the vacuum sector axiom for an abstract net \mathcal{A} is to ensure that, given an interval $K \subseteq S^1$, the standard form $L^2\mathcal{A}(K)$ not only carries actions of $\mathcal{A}(K)$ and $\mathcal{A}(K')$ via its $\mathcal{A}(K)$ – $\mathcal{A}(K)$ -bimodule structure, but also of all algebras $\mathcal{A}(I)$ with $I \subseteq S^1$,
- an abstract net assigns algebras to all intervals, instead of only to those that are embedded in S^1 . This is not a stronger demand, though, since, given a concrete, positive energy net, one can assign algebras to any interval I by choosing embeddings of I in S^1 . Using the diffeomorphism covariance, these algebras can then be glued together into a single algebra which is independent of the choice of embedding,
- there is no assumption of positivity of energy in the definition of an abstract net, and it is in this direction that such a net is a genuine generalisation of a concrete, positive energy net. The motivation given in [BDH14a] for dropping this requirement is to assemble conformal nets into a symmetric monoidal category with duals, and the natural dual of a positive energy net is of negative energy.

Even though some axioms were left out in Definitions 1.2.2 and 1.3.1 for the sake of exposition, we approximately cite

Theorem 1.3.3. (See [BDH14a, Proposition 4.9] for the precise statement.) *Every concrete, positive energy conformal net can be extended to an abstract conformal net.*

From this point onwards, whenever we will speak of a *conformal net*, we will mean an abstract conformal net in the sense of Definition 1.3.1.

Physically, a defect between two field theories or condensed-matter systems can be thought of as a ‘2-sided’ boundary condition, designating a submanifold at the junction between the domains of the theories where the fields of one theory experience a discontinuity and transition to those of the other theory. Defects have also been named *domain walls* or *surface operators* in the literature and studies of them (for theories we do not treat in this thesis) can be found in for example [Frö+07], [KS11b] and [Guk16].

The formulation of a defect between two conformal nets introduced by [BDH15] is a straightforward translation of their Definition 1.3.1 when the category INT of intervals is enhanced to the category $\text{INT}_{\bullet, \circ}$ of *bicoloured intervals*. An object of this category is an interval equipped with a covering of two subintervals, one of which is seen as being coloured white, the other one as being coloured black. The subintervals are required to overlap in one point, at which the colour changes. We allow that a bicoloured interval is coloured entirely white or entirely black, meaning that the other subinterval is empty, but we rule out that one of the two subintervals is a singleton. *Morphisms* between bicoloured intervals are smooth embeddings that preserve bicolourings. The full subcategories of entirely white and entirely black intervals are denoted by INT_{\circ} and INT_{\bullet} , respectively, and we call a bicoloured interval which is not contained in either of them *genuinely bicoloured*.

After these preliminaries⁵ we are ready for

Definition 1.3.4. (Taken from [BDH15, Definition 1.7].) If \mathcal{A} and \mathcal{B} are two conformal nets, then an \mathcal{A} – \mathcal{B} -defect D is a covariant functor $D: \text{INT}_{\bullet, \circ} \rightarrow \text{VN}$ sending orientation-preserving and reversing morphisms to algebra homomorphisms and antihomomorphisms, respectively, whose restrictions to INT_{\circ} and INT_{\bullet} are equal to \mathcal{A} and \mathcal{B} , respectively, and which satisfies the following properties:

⁵The authors of [BDH15] require one more piece of structure of bicoloured intervals which we left out for brevity, namely a local coordinate around the colour-changing point, and morphisms are required to respect this.

- (*Isotony*) if $I \hookrightarrow K$ is a morphism of genuinely bicoloured intervals, then the corresponding homomorphism or antihomomorphism $D(I) \rightarrow D(K)$ is injective,
- (*Locality*) if $I \hookrightarrow K$ and $J \hookrightarrow K$ are morphisms of bicoloured intervals of which the images have disjoint interiors, then the images of $D(I)$ and $D(J)$ in $D(K)$ commute,
- (*Existence of a vacuum sector*) let K be a genuinely bicoloured interval and $I \in \text{INT}_\circ \cup \text{INT}_\bullet$ a subinterval of K containing a boundary point p of K . Write \bar{I} for the same manifold as I , but equipped with the reversed orientation. Then $D(I)$ and $D(\bar{I})$ both act on the left of the standard form $L^2D(K)$ —the former via the homomorphism $D(I) \rightarrow D(K)$ and the latter through the homomorphism $D(\bar{I}) \xrightarrow{\sim} D(I)^{\text{op}} \rightarrow D(K)^{\text{op}}$.

We then demand that the left action of the algebraic tensor product $D(I) \otimes D(\bar{I})$ on $L^2D(K)$ extends to an action of the algebra $D(I \cup_p \bar{I})$ along the homomorphism $D(I) \otimes D(\bar{I}) \rightarrow D(I \cup_p \bar{I})$.⁶

When $I \hookrightarrow K$ is a morphism of bicoloured intervals the corresponding algebra homomorphism $D(I) \rightarrow D(K)$ is called an *isotony homomorphism*.

A *morphism* between two \mathcal{A} - \mathcal{B} -defects is defined to be a natural transformation.

Remark 1.3.5. The reason the above Definition 1.3.4 does not include requirements of continuity and inner covariance analogous to those in Definition 1.3.1 is that these can be shown to hold automatically (see [BDH15, Proposition 1.21] and [BDH15, Proposition 1.10], respectively) when a *strong additivity* axiom of the nets and the defect is assumed. We removed the latter axiom for brevity.

We stress that the non-trivial information contained in an \mathcal{A} - \mathcal{B} defect consists of the algebras assigned to genuinely bicoloured intervals and the isotony homomorphisms associated to inclusions of white, black or genuinely bicoloured into genuinely bicoloured ones. The rest of the data is already determined by the nets \mathcal{A} and \mathcal{B} .

The notion of a defect between two conformal nets bears a formal similarity to that of a bimodule between two rings. This is not meant to suggest that, just like for a bimodule one can forget the action of one ring and obtain an ordinary module for the other ring, also a net defect possesses an underlying sector structure for both nets. Instead, it is more fruitful to think of defects

⁶See Definition 1.3.1 for the definition of the unicoloured interval $I \cup_p \bar{I}$.

as a generalisation of morphisms of conformal nets. Indeed, we will see in Example 1.3.8 that suitable morphisms give rise to defects.

The analogy between defects and ring bimodules runs deeper still in that both admit a form of composition. Recall that when M is a Q – R -bimodule and N is an R – S -bimodule for rings Q , R and S the tensor product $M \otimes_R N$ is a Q – S -bimodule. This operation is weakly associative and weakly unital. Taking bimodules as 1-morphisms and equivariant bimodule maps as 2-morphisms, rings become in this way a symmetric monoidal bicategory.

Similarly, it is shown in [BDH15; BDH16] that defects (under certain conditions) can be composed and that a restricted class of conformal nets becomes a particular kind of symmetric monoidal 3-category when defects are taken as the 1-morphisms. The 2-morphisms used in this 3-category are not the ordinary morphisms between defects as stated in Definition 1.3.4, though. They are rather a natural generalisation of the notion of sectors as defined in Definition 1.2.6, which explains the increase in the categorical level compared to the situation of rings.

The study of this 3-category has the potential to reveal a richer structure on the family of conformal nets—one that is not easily visible at the level of objects and ordinary morphisms. Unfortunately, there is very little known about it so far. For example, it is very simple to point out for every two nets \mathcal{A} and \mathcal{B} some \mathcal{A} – \mathcal{B} -defect D . As explained in [BDH14a, Proposition 1.23], one namely always has an \mathcal{A} – $\underline{\mathbb{C}}$ - and a $\underline{\mathbb{C}}$ – \mathcal{B} -defect and D can then be defined as their composition over the net $\underline{\mathbb{C}}$. However, this defect is ‘large’ in the sense that it is not *dualisable* (in the higher categorical sense), and no necessary conditions are known to impose on a pair of nets for a dualisable defect to exist between them.

Remark 1.3.6 (Alternative definitions of defects). Definition 1.3.4 is not the only notion of a conformal net defect put forward in the literature: in independent work the authors of [Bis+16] offer an alternative, but more restrictive, formulation. (Their definition applies to 1-dimensional defects between 2d *full* CFT’s as well.) At the moment of this writing the precise relation between their theory and that of [BDH15] is not entirely clear, but we refer to [BDH15, Remark 1.28] and [Bis+16, Remark 3.4] for interesting discussions on this matter.

1.3.1 Examples of defects

A few elementary examples of defects between conformal nets were obtained in [DH] and [BDH15, Section 1.c]. We summarise a selection of those here.

Example 1.3.7 ((Twisted) identity defects). For every conformal net \mathcal{A} there is an *identity* \mathcal{A} - \mathcal{A} -defect denoted by $1_{\mathcal{A}}$. It is defined by simply setting $1_{\mathcal{A}}(I) := \mathcal{A}(I)$ for every bicoloured interval I , where we make sense of the evaluation $\mathcal{A}(I)$ by ignoring the bicolouring of I . Furthermore, the isotony homomorphisms induced by inclusions of white, black and genuinely bicoloured intervals into genuinely bicoloured ones are also defined to be those we get from \mathcal{A} by forgetting bicolourings. This defect serves as the identity 1-morphism in the 3-category of conformal nets.

This example can be tweaked as follows. Let g be an automorphism of \mathcal{A} . Then define an \mathcal{A} - \mathcal{A} -defect D_g again by $D_g(I) := \mathcal{A}(I)$ for every bicoloured interval I , except that we ‘twist’ the isotony homomorphisms on, say, the black side by g . More precisely: if I is a black interval, J is genuinely bicoloured and $f : I \hookrightarrow J$ is an embedding then the homomorphism $D_g(f) : D_g(I) \rightarrow D_g(J)$ is set to be $\mathcal{A}(f) \circ g_I$, where g_I is the automorphism of $\mathcal{A}(I)$ given by conjugation by g . All other types of isotony homomorphisms are inherited from \mathcal{A} , unaffected by g .

The construction of such a twisted identity defect can be compared to the situation when we consider a ring R as an R - R -bimodule. Twisting the right action by an automorphism of R then namely also produces a new R - R -bimodule.

Example 1.3.8 (Defects from conformal embeddings). Continuing the analogy of defects with ring bimodules, we note that if R is a subring of a ring S , then S is an R - S -, S - R - and an R - R -bimodule. The counterpart of this observation for defects is as follows.

A morphism $\tau : \mathcal{A} \rightarrow \mathcal{B}$ between two conformal nets is called a *conformal embedding* in [BDH14a, Definition 1.45] if it respects unitary operators that implement algebra automorphisms induced by interval diffeomorphisms. That is, we demand of τ that $\text{Ad}(U) = \mathcal{A}(\varphi)$ implies that $\text{Ad}(\tau(U)) = \mathcal{B}(\varphi)$ for every interval I , diffeomorphism φ of I which is the identity near ∂I and unitary operator $U \in \mathcal{A}(I)$. It is then shown in [BDH15, Proposition 1.24] that there exists an \mathcal{A} - \mathcal{B} -defect⁷ which evaluates genuinely bicoloured intervals like \mathcal{B} does after forgetting bicolourings and of which the isotony homomorphisms are the obvious ones induced by τ . Similarly, τ gives rise to a \mathcal{B} - \mathcal{A} - and an \mathcal{A} - \mathcal{A} -defect.

Example 1.3.9 (Defects from Q -systems). Let us return to the construction in Example 1.3.8 of an \mathcal{A} - \mathcal{A} -defect from a conformal embedding $\tau : \mathcal{A} \rightarrow \mathcal{B}$ of

⁷Those authors actually only need the morphism τ to be a conformal embedding to prove the *strong additivity* property of the resulting defect.

nets. A closer inspection of the locality axiom of a defect teaches us that it is not necessary for the net \mathcal{B} to be local as well—it is sufficient for \mathcal{B} to be local *relative* to \mathcal{A} , meaning that if $I \hookrightarrow K$ and $J \hookrightarrow K$ are embeddings of intervals of which the images have disjoint interiors, then the images of $\tau(\mathcal{A}(I))$ and $\mathcal{B}(J)$ in $\mathcal{B}(K)$ commute. It is known that, upon imposing a certain finiteness condition on τ , such relatively local extensions \mathcal{B} of \mathcal{A} can be classified in terms of certain algebra objects called *Q-systems* in the category of sectors of \mathcal{A} . In [BDH15, pp. 15–17] those authors demonstrate how to produce an \mathcal{A} – \mathcal{A} -defect from a Q-system directly, without first needing to build the corresponding net extension.

Example 1.3.10 (Defects realising invertibility of nets). The authors of [DH] call a conformal net \mathcal{A} *invertible* if there exist another net \mathcal{A}^{-1} and an invertible defect D from $\mathcal{A} \otimes \mathcal{A}^{-1}$ to $\underline{\mathbb{C}}$, where $\underline{\mathbb{C}}$ is the trivial conformal net that is constantly \mathbb{C} and of which all isotony homomorphisms are $\text{id}_{\mathbb{C}}$. (Here, the invertibility of a defect should be interpreted in the higher categorical sense.) Together with the result [BDH14a, Corollary 3.26] they proved that \mathcal{A} is invertible if and only if all its algebras have trivial centre and its representation theory is trivial. This was done by exhibiting for the ‘if’ direction an explicit defect implementing the invertibility.

Lastly, just like conformal nets, there is also a direct sum operation for defects (see [BDH14a, Lemma 1.29]), which can be used to produce more defects from a set of given ones.

1.4 Bicoloured loop groups

The examples of defects presented in the previous Section are fairly formal in nature. They are built ‘internally’ to the relevant categories, as opposed to the examples of conformal nets presented in Section 1.2.1 that were constructed in terms of ‘external’ data. In order to gain a greater understanding of defects it is natural to ask for a larger source of richer examples of them.

We now outline a proposal suggesting that the construction of conformal nets from loop groups in Examples 1.2.8 to 1.2.10 might admit a generalisation that allows one to build defects between these loop group nets. Let us repeat the essential features of those loop groups:

If G is a real vector space, a real torus or a compact, connected, simple, simply-connected Lie group, then its associated loop group $LG := C^\infty(S^1, G)$ admits certain disjoint-commutative $U(1)$ -central

extensions $\tilde{L}G$. Such an extension carries an action of $\text{Diff}_+(S^1)$ which lifts the natural one on LG and is compatible with respect to supports of loops and circle diffeomorphisms. One is able to classify and explicitly construct the irreducible, positive energy representations of $\tilde{L}G$.

We use the notations ζ , η , p and q as they are explained in Section 1.7. That is, ζ and η are the closed left and right halves of S^1 , respectively, and p and q denote the two points i and $-i$ on S^1 . We shall think of ζ as being coloured white and η as having the colour black. An interval on S^1 which either does not contain the points p and q , or contains exactly one of them, not on its boundary, is then a bicoloured interval.

For every two real Lie groups G_\circ and G_\bullet , we propose that there should exist a list of *matching conditions*. (In this generality, when G_\circ and G_\bullet are not necessarily tori, we will remain deliberately vague on their precise nature.) To every such matching condition M there should be associated a *bicoloured loop group* $L(G_\circ, M, G_\bullet)$. The data of an element of this group consists (at least) of two smooth maps $\gamma_\circ: \zeta \rightarrow G_\circ$ and $\gamma_\bullet: \eta \rightarrow G_\bullet$. The endpoints of these two paths are not placed arbitrarily, though, and this is where the matching condition comes in: the two points $\gamma_\circ(p)$ and $\gamma_\bullet(p)$ are constrained with respect to each other as dictated by M , and the same holds for $\gamma_\circ(q)$ and $\gamma_\bullet(q)$. This constraint need not necessarily be a property satisfied by the pair $(\gamma_\circ, \gamma_\bullet)$ —it might also be expressed in terms of a *matching datum* γ_m , which is an additional piece of information attached to $(\gamma_\circ, \gamma_\bullet)$. In that case we denote an element of $L(G_\circ, M, G_\bullet)$ as a triple $(\gamma_\circ, \gamma_m, \gamma_\bullet)$.

We call elements of $L(G_\circ, M, G_\bullet)$ *bicoloured loops* and we refer to the elements of an ordinary loop group LG as being *unicoloured* to make the distinction with our new notion.

With an eye towards the axioms in Definition 1.3.4 for conformal net defects, we postulate that bicoloured loop groups should satisfy the following properties:

- for every real Lie group G there should exist a particular matching condition M_G from G to itself such that $L(G, M_G, G)$ is isomorphic to the loop group LG . This is the first property allowing one to consider bicoloured loop groups as generalisations of unicoloured ones. It is inspired by the demand that, for every loop group from which one can construct a Heisenberg, lattice or loop group conformal net, there should be a bicoloured loop group which reproduces the identity defect from this net to itself,

- if we write $L_{\subset}G_{\circ}$ for those loops in LG_{\circ} which have support in \subset and define $L_{\supset}G_{\bullet}$ similarly, then there should exist two injective group homomorphisms

$$L_{\subset}G_{\circ} \hookrightarrow L(G_{\circ}, M, G_{\bullet}) \hookleftarrow L_{\supset}G_{\bullet}.$$

This corresponds to the axiom that a defect between two conformal nets restricts to each of the nets on the full subcategories of white and black intervals respectively,

- elements of $L(G_{\circ}, M, G_{\bullet})$ should have a notion of *support* in a bicoloured interval that is embedded in S^1 (see Definition 3.2.9 for the meaning we will assign to the latter term). The support of a bicoloured loop in the image of the homomorphisms $LG \xrightarrow{\sim} L(G, M_G, G)$, $L_{\subset}G_{\circ} \hookrightarrow L(G_{\circ}, M, G_{\bullet})$ or $L_{\supset}G_{\bullet} \hookrightarrow L(G_{\circ}, M, G_{\bullet})$ should coincide with that of its unicoloured pre-image, and
- the group $L(G_{\circ}, M, G_{\bullet})$ should carry an action of the group of orientation preserving circle diffeomorphisms which fix the points p and q . Moreover, there should exist an action of $\text{Rot}(S^1)$ or a cover thereof as well.

Adding to these properties, we require that a bicoloured loop group admits a certain privileged list of $U(1)$ -central extensions $\tilde{L}(G_{\circ}, M, G_{\bullet})$ such that

- when G is a real vector space, a real torus or a compact, connected, simple, simply-connected Lie group and $\tilde{L}G$ is a central extension of LG of the type discussed in Examples 1.2.8 to 1.2.10, then the central extension $\tilde{L}(G, M_G, G)$ that is pulled back from $\tilde{L}G$ under the isomorphism $L(G, M_G, G) \cong LG$ is present on the aforementioned list,
- let $\tilde{L}G_{\circ}$ and $\tilde{L}G_{\bullet}$ be centrally extended unicoloured loop groups of the types in Examples 1.2.8 to 1.2.10. Then there should exist a central extension $\tilde{L}(G_{\circ}, M, G_{\bullet})$ which admits lifted injective group homomorphisms into it:

$$\tilde{L}_{\subset}G_{\circ} \hookrightarrow \tilde{L}(G_{\circ}, M, G_{\bullet}) \hookleftarrow \tilde{L}_{\supset}G_{\bullet},$$

- suppose that two elements of the central extension $\tilde{L}(G_{\circ}, M, G_{\bullet})$ are such that the supports of their images in $L(G_{\circ}, M, G_{\bullet})$ are contained in two disjoint bicoloured intervals on S^1 , respectively. Then we require that these two elements commute,
- the action of the group of orientation preserving circle diffeomorphisms which fix the points p and q we expect to exist on $L(G_{\circ}, M, G_{\bullet})$ should

lift to $\tilde{L}(G_\circ, M, G_\bullet)$. The same should hold for the action of (some cover of) $\text{Rot}(S^1)$ on $L(G_\circ, M, G_\bullet)$, and

- having the action of (some cover of) $\text{Rot}(S^1)$ on $\tilde{L}(G_\circ, M, G_\bullet)$ at our disposal, we are able to speak about positive energy representations of the latter group. One should be able to classify and explicitly construct the irreducible such representations up to isomorphism.

Having these properties in hand one can attempt to mimic the constructions in Examples 1.2.8 to 1.2.10 to produce a defect between the nets associated to $\tilde{L}G_\circ$ and $\tilde{L}G_\bullet$. We will elaborate on this further in a special case in Section 4.1.

Remark 1.4.1. We give a motivation for requiring $\tilde{L}(G_\circ, M, G_\bullet)$ to be disjoint-commutative that is unrelated to the axioms of conformal net defects.

For every real Lie group G there exists a procedure, named *transgression*, which produces $U(1)$ -central extensions of LG from certain geometric objects, namely *multiplicative bundle gerbes* (with connection), that are situated over G . For example, the central extensions that are discussed in Example 1.2.10 can be obtained in this way. It is shown in [Wal16] that transgression always results in a disjoint-commutative central extension. Hence, we require disjoint-commutativity also in the bicoloured situation as a prerequisite for a possible interpretation of $\tilde{L}(G_\circ, M, G_\bullet)$ in terms of finite-dimensional, ‘higher’ geometry.

1.5 Main results

The goal of this thesis is to test whether our speculative idea of bicoloured loop groups explained in Section 1.4 can be made sense of when G_\circ and G_\bullet are tori $T_\circ := \Lambda_\circ \otimes_{\mathbb{Z}} U(1)$ and $T_\bullet := \Lambda_\bullet \otimes_{\mathbb{Z}} U(1)$, where Λ_\circ and Λ_\bullet are two even, positive definite lattices of the same rank. One reason for specialising to this case is that the central extensions of unicoloured torus loop groups discussed in Example 1.2.9 can be specified through explicit 2-cocycles, as opposed to the geometric methods that are needed in the non-abelian case. A second reason is that also their representations can be built through simple, explicit means without requiring, for example, involved results on integration of Lie algebra representations. Both these reasons therefore lower the barrier of attempting to generalise these constructions. A third reason is that the wealth of examples of lattices and the breadth of their theory makes it plausible that interesting defects between lattice nets can be found as well.

Our proposal for a definition of a matching condition in this situation is as follows. We present it in a slightly simplified form and we give more details in Chapter 3.

Let Γ be an even, positive definite lattice of the same rank as Λ_\circ and Λ_\bullet and let π_\circ and π_\bullet be two lattice morphisms as in

$$\Lambda_\circ \xleftarrow{\pi_\circ} \Gamma \xrightarrow{\pi_\bullet} \Lambda_\bullet.$$

If H is the torus $\Gamma \otimes_{\mathbb{Z}} \mathrm{U}(1)$, then π_\circ and π_\bullet induce two surjective torus homomorphisms

$$T_\circ \xleftarrow{\mathrm{U}(1)\pi_\circ} H \xrightarrow{\mathrm{U}(1)\pi_\bullet} T_\bullet.$$

We now define the *bicoloured torus loop group* $L(T_\circ, H, T_\bullet)$ to be the abelian group of all triples $(\gamma_\circ, \gamma_m, \gamma_\bullet)$ of smooth maps fitting in a commutative diagram

$$\begin{array}{ccccc} \zeta & \longleftarrow & \dot{} & \longrightarrow & \eta \\ \gamma_\circ \downarrow & & \downarrow \gamma_m & & \downarrow \gamma_\bullet \\ T_\circ & \xleftarrow{\mathrm{U}(1)\pi_\circ} & H & \xrightarrow{\mathrm{U}(1)\pi_\bullet} & T_\bullet \end{array}$$

where $\dot{}$ is the subset $\{p, q\}$ of S^1 . That is, γ_m is a matching datum for the pair $(\gamma_\circ, \gamma_\bullet)$ dictating the placements of each of the pairs of points $(\gamma_\circ(p), \gamma_\bullet(p))$ and $(\gamma_\circ(q), \gamma_\bullet(q))$ relative to each other. We call $(\gamma_\circ, \gamma_m, \gamma_\bullet)$ a *bicoloured (torus) loop*.

Our inspiration for this definition has several origins:

Surface operators between Chern–Simons theories We briefly commented in Section 1.2.2 on a method by which lattice conformal nets can be used to model a class of 3d TQFT’s that are conjecturally the abelian Chern–Simons theories. Each such theory is determined by an even, positive definite lattice as well. The study of defects between abelian Chern–Simons theories was initiated in [KS11b], where these are named *surface operators*, and its authors argue that, at least at the classical level, surface operators between two theories associated to Λ_\circ and Λ_\bullet are classified by Lagrangian subgroups of the pseudo-Riemannian torus $T_\circ \oplus \overline{T}_\bullet$. Here, \overline{T}_\bullet is the torus $\overline{\Lambda}_\bullet \otimes_{\mathbb{Z}} \mathrm{U}(1)$ and $\overline{\Lambda}_\bullet$ is the negative definite lattice obtained from Λ_\bullet by negating its bi-additive form. We then note that in the case when Γ is not just contained in the intersection of Λ_\circ and Λ_\bullet but equal to it, the homomorphism

$$(\mathrm{U}(1)\pi_\circ, \mathrm{U}(1)\pi_\bullet): H \rightarrow T_\circ \oplus \overline{T}_\bullet$$

is injective with a Lagrangian image. Our definition of a bicoloured torus loop group is an attempt to translate aspects of the study done in [KS11b] to the language of loop groups.

A number-theoretical motivation With the analogy between bimodules and defects in mind, the question emerges naturally whether there exist examples of pairs of conformal nets that are ‘Morita-equivalent’, meaning that there exists an invertible defect between them. We suspect that such a defect induces a braided monoidal equivalence between the categories of sectors of the respective nets (see [GPS95, Proposition 8.6]), and hence one must only consider those nets for which these categories are (conjectured to be) braided equivalent. As we stated earlier, we expect this to be true for two lattice nets \mathcal{A}_\circ and \mathcal{A}_\bullet associated to Λ_\circ and Λ_\bullet when these lattices are in the same *genus*. (See [Höh03] for similar thoughts in the context of vertex algebras.) This is to say that they both have the same signature (which we already assumed to be true) and they become isomorphic after tensoring over \mathbb{Z} with the p -adic integers for all prime numbers p . An equivalent definition is to require that they become isomorphic after summing onto them a single copy of the indefinite hyperbolic plane lattice from Example A.1.8. This second formulation is indeed used in [KS11b] to construct an invertible surface operator between the abelian Chern–Simons theories associated to Λ_\circ and Λ_\bullet . However, their method also makes use of the theory assigned to the hyperbolic plane lattice. Because its conformal net counterpart does not exist it remains unclear how their proof can be translated to the language of conformal nets.

We abandon our hope of finding Morita-equivalent lattice nets and ask more generally whether there exists a not-necessarily invertible defect between \mathcal{A}_\circ and \mathcal{A}_\bullet when Λ_\circ and Λ_\bullet satisfy the weaker demand of becoming isomorphic after tensoring with the p -adic *rational* numbers for all p . By the *Hasse–Minkowski theorem* this is equivalent to the existence of an isomorphism $\Lambda_\circ \otimes_{\mathbb{Z}} \mathbb{Q} \cong \Lambda_\bullet \otimes_{\mathbb{Z}} \mathbb{Q}$, which is in turn the same as saying that Λ_\circ and Λ_\bullet share a common sublattice Γ of finite index.

Moduli spaces of flat connections over quilted surfaces A third influence on our definition of $L(T_\circ, H, T_\bullet)$ is the work of [LŠ15] on the study of so called moduli spaces of flat connections over *quilted surfaces*. Such surfaces are divided into regions which are each ‘coloured’ by different structure groups equipped with invariant quadratic forms on their Lie algebras. A flat connection over a quilted surface is then supposed to break down to a co-isotropic relation at the edges where the regions meet. The relevance of that work to loop groups is that the representations of centrally extended loop groups discussed in Examples 1.2.8 to 1.2.10 can be considered as geometric quantisations of moduli spaces of flat

connections over the unit disc. It is hence reasonable to ask whether there exists some bicoloured generalisation of these moduli spaces of which the group of gauge transformations gives rise to a conformal net defect. The group $L(T_\circ, H, T_\bullet)$ is an attempt at defining such a group of gauge transformations directly.

The work of [FSV14] on defects between certain types of 3d TQFT's, using notions of *relative* principal bundles over *relative* manifolds, looks to be highly related as well.

Our study of $L(T_\circ, H, T_\bullet)$ depends on the auxiliary abelian group $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ of all smooth paths $\gamma: [0, 1] \rightarrow H$ such that

$$\gamma(1) - \gamma(0) \in \frac{\Lambda_\circ - \Lambda_\bullet}{\Gamma} \subseteq H.$$

Here, we consider Λ_\circ and Λ_\bullet as submodules of the Lie algebra of H and we define

$$\Lambda_\circ - \Lambda_\bullet := \{\lambda_\circ - \lambda_\bullet \mid \lambda_\circ \in \Lambda_\circ, \lambda_\bullet \in \Lambda_\bullet\}.$$

(A more precise description of $\Lambda_\circ - \Lambda_\bullet$ can be found at the start of Section 3.1.) Many results on the unicoloured torus loop group LH easily generalise to this larger group $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$. These can be transported next to $L(T_\circ, H, T_\bullet)$ via a surjective homomorphism

$$\text{Pth}: L(T_\circ, H, T_\bullet) \twoheadrightarrow P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma) \quad (1.2)$$

with a finite kernel, which is constructed in Section 3.1.1. Among other material, we show in Section 3.1 that our definition of a bicoloured torus loop group satisfies the first list of demands in Section 1.4, namely

- (Section 3.1.3) when $\Lambda_\circ = \Lambda_\bullet = \Gamma$ and the morphisms π_\circ and π_\bullet are the identity, the group $L(H, H, H)$ is canonically isomorphic to the unicoloured torus loop group LH ,
- (Sections 3.1.2 and 3.1.5) there exist natural injective homomorphisms of abelian groups

$$L_{\subset}T_\circ \hookrightarrow L(T_\circ, H, T_\bullet) \hookleftarrow L_{\supset}T_\bullet, \quad (1.3)$$

and a notion of support for bicoloured loops that is compatible with these injections, and

- (Section 3.1.7) there exists an action of a certain covering group $\text{Diff}_+^{(n)}(S^1)$ on $L(T_\circ, H, T_\bullet)$. In particular, the orientation preserving circle diffeomorphisms which fix the points p and q and the elements of $\text{Rot}^{(n)}(S^1)$ act on $L(T_\circ, H, T_\bullet)$.

In Section 3.2 we are able to adapt the 2-cocycles on unicoloured torus loop groups discussed in Example 1.2.9 to $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ in such a way that the pullback of such a cocycle along the homomorphism Pth in (1.2) results in a $U(1)$ -central extension $\tilde{L}(T_\circ, H, T_\bullet)$ which satisfies the second list of demands in Section 1.4. That is, we show in Sections 3.2 to 3.4 that

- (Section 3.2.1) when $\Lambda_\circ = \Lambda_\bullet = \Gamma$ and the morphisms π_\circ and π_\bullet are the identity the isomorphism $LH \cong L(H, H, H)$ lifts to an isomorphism of non-abelian groups $\tilde{L}H \cong \tilde{L}(H, H, H)$,
- (Section 3.2.3) the injections (1.3) lift to homomorphisms of non-abelian groups as well,
- (Section 3.3) the action of $\text{Diff}_+^{(n)}(S^1)$ on $L(T_\circ, H, T_\bullet)$ can be extended to one on $\tilde{L}(T_\circ, H, T_\bullet)$, and
- (Section 3.4) we are able to classify and explicitly construct the irreducible, positive energy representations of $\tilde{L}(T_\circ, H, T_\bullet)$. We find that, up to isomorphism, there exist only finitely many of these.

Let us state the last of these points more precisely as our chief result:

Main theorem (Theorem 3.4.11). *Every irreducible, positive energy representation of $\tilde{L}(T_\circ, H, T_\bullet)$ such that the central subgroup $U(1)$ acts as $z \mapsto z$ is (unitarily) isomorphic to a certain such representation $\text{Ind } W_{\chi, l}$ for some characters χ and l of $(\Lambda_\circ \cap \Lambda_\bullet)/\Gamma$ and H , respectively. The isomorphism classes of such representations are therefore labelled by two parameters: one is an element of the dual group of the finite abelian group $(\Lambda_\circ \cap \Lambda_\bullet)/\Gamma$ and the other is an element of the finite abelian group $\Gamma^\vee/(\Lambda_\circ - \Lambda_\bullet)$.*

We finally outline in Section 4.1 how the construction of a defect from such a central extension $\tilde{L}(T_\circ, H, T_\bullet)$ might proceed.

1.6 Organisation of the text

The rest of this thesis is organised as follows.

We repeat that our discussion of conformal nets and their defects in this Introduction has solely been used as a motivation for the study of bicoloured loop groups. They will hence not play any role in Chapters 2 and 3, which treat the theories of uni- and bicoloured torus loop groups, respectively. Chapter 2 is mainly expository in nature. It is intended to elaborate on some claims made in Example 1.2.9 and to collect and setup results from the literature in a way that eases the step of generalisation to Chapter 3. This latter chapter starts the investigation of bicoloured torus loop groups properly and contains the proofs of the claims made in Section 1.5. It cannot be read without knowledge of Chapter 2. Both chapters assume familiarity with the notions and results in Appendix A. Finally, we make some remarks in Chapter 4 on possible further steps that can be taken in the study of bicoloured torus loop groups.

Appendices A.1 to A.3 stand alone from each other and can also be read independently from the rest of the thesis. Appendix A.4, though, on Heisenberg groups, does depend on Appendices A.2 and A.3.

1.7 Notations and conventions

Distinguishing S^1 , $U(1)$ and $\text{Rot}(S^1)$ The notation S^1 will be used for the unit circle, considered as the smooth manifold embedded in the complex plane. Its role in this thesis is that of the domain of loops (which will be valued in certain Lie groups—tori, specifically). A generic point on it is denoted by θ . We will often make use of some parametrisation of S^1 by the interval $[0, 1]$. This will always be done at unit speed and counterclockwise, but the starting point of the parametrisation will not necessarily be $1 \in S^1$. Instead, it will be some point denoted by q which is either arbitrary, or a specific one, depending on the context.

When we want to consider S^1 as an abelian topological group we will write it as $U(1)$ instead. It will show up as the group of ‘phases’ other groups we will be studying will be centrally extended by. Elements of $U(1)$ will be denoted as z or w .

The topological group of counterclockwise rotations of the manifold S^1 will be named $\text{Rot}(S^1)$. This group (or finite covers of it) will turn out to act on the loop groups and the representations of them that we will be considering. A typical element will be written as φ_θ , where $\theta \in [0, 1]$ is an angle.

Intervals and points An *interval* is defined to be an oriented, smooth manifold that is diffeomorphic to $[0, 1]$. This means in particular that it is closed, as opposed to the more common convention in the literature on conformal nets of

being open. For a subinterval $I \subseteq S^1$, we denote by I' the subinterval that is the closure of $S^1 \setminus I$.

The symbols ζ and η will be used for the closed left and right half of S^1 , respectively. When we want to emphasise the orientations they inherit if S^1 is oriented counterclockwise, we will add arrows as follows: \curvearrowright and \curvearrowleft . In Chapter 3 the two points i and $-i$, whose union \cdot forms the intersection of ζ and η , will be respectively called p and q .

Hilbert spaces, actions and representations Our convention is that a Hermitian inner product on a complex vector space is linear in its first variable and antilinear in the second one, and all group actions will be left actions.

Whenever we will speak of a *group representation* we will follow the convention stated in Appendix A.3, which is that the group is topological, the underlying vector space is a complex Hilbert space and that the representation is strongly continuous and unitary, unless stated otherwise.

Chapter 2

Unicoloured torus loop groups

Let Λ be an even, positive definite lattice and T the torus $\Lambda \otimes_{\mathbb{Z}} \mathrm{U}(1)$. In this Chapter we study a certain $\mathrm{U}(1)$ -central extension $\tilde{L}T$ of the torus loop group $LT := C^\infty(S^1, T)$ and its positive energy representations. The title of this Chapter and its sections refer to LT as being *unicoloured*, but we do not use this adjective in the text of the current Chapter. Much of this material is already covered in some form in the literature, but our aim is to provide a level of detail sufficient for the needs of Chapter 3 where results and notions in this Chapter will be used and generalised. References to the literature will be given per section and we make extensive use of material collected in Appendix A.

The study of the representation theory of the central extension $\tilde{L}T$ will require knowledge of certain subgroups and a direct sum decomposition of LT . This is treated in Section 2.1, where it is not yet necessary to assume the presence of a bi-additive form on Λ . We treat LT as an abstract group until Section 2.5, but we will refer to certain of its subsets earlier already as ‘connected components’, in anticipation. Next, the construction of $\tilde{L}T$ is described in Section 2.2 which also includes a proof of its important disjoint-commutativity property.

Sections 2.3 and 2.4 each have dual purposes. They are firstly intended to clarify to what extent $\tilde{L}T$ depends on certain extra data needed to perform its construction. One namely requires a choice of a point on S^1 and the fact shown in Section 2.3 that $\mathrm{Diff}_+(S^1)$ acts on $\tilde{L}T$ implies that this choice is irrelevant. On the other hand, we also need to pick a certain $\{\pm 1\}$ -central extension $\tilde{\Lambda}$ of Λ . We prove in Section 2.4 that it is the automorphism group of $\tilde{\Lambda}$ which acts on $\tilde{L}T$ and not that of Λ . This choice is therefore not immaterial.

A further goal of Section 2.3 is that we learn that $\tilde{L}T$ admits in particular an action of the group $\mathrm{Rot}(S^1)$, so that we may speak of its positive energy

representations. The second aim of Section 2.4, which admittedly lies outside the scope of this thesis, is that it exhibits automorphisms of the conformal net associated to Λ .

This chapter culminates with the classification and construction of the irreducible, positive energy representations (up to isomorphism) of \tilde{LT} in Section 2.5.

Modest novelties of our exposition are that we lay out in what way \tilde{LT} is a topological group, we occasionally explain how the various constructions can be made to work for odd lattices as well, and that we show how the automorphisms of $\tilde{\Lambda}$ are symmetries of the collection of irreducible, positive energy representations of \tilde{LT} . We will not return to the latter two topics in Chapter 3, though.

2.1 Unicoloured torus loop groups and their structure

Let Λ be a free \mathbb{Z} -module of finite rank. Associated to it is a torus $T := \Lambda \otimes_{\mathbb{Z}} \mathbb{U}(1)$ whose Lie algebra $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ we denote by \mathfrak{t} . In this section we explain the structure of the torus loop group $LT := C^\infty(S^1, T)$ of T . We will show that it can be decomposed (non-canonically) as a direct sum of T , a certain infinite-dimensional vector space $V\mathfrak{t}$ and Λ .

References. The structure of torus loop groups was found in [Seg81, Section 4]. We elaborate on it further by emphasising $\text{Diff}_+(S^1)$ -equivariance.

We first observe that, given a choice of a privileged point q on S^1 , LT has an alternative description in terms of paths in the Lie algebra \mathfrak{t} , namely as follows:

$$LT \cong \left\{ \xi \in C^\infty([0, 1], \mathfrak{t}) \mid \xi(1) - \xi(0) \in \Lambda, \xi^{(k)}(1) = \xi^{(k)}(0) \text{ for all } k \geq 1 \right\} / \Lambda. \quad (2.1)$$

Here we used the unit speed parametrisation of S^1 by $[0, 1]$, starting at q .

This description allows us to define for a loop $\gamma \in LT$ its *winding element* as

$$\Delta_\gamma := \xi(1) - \xi(0) \in \Lambda$$

for any path $\xi \in C^\infty([0, 1], \mathfrak{t})$ which represents γ . This is not only independent of the choice of representative, but also of our choice of point q . The winding element is therefore a canonical homomorphism $\Delta: LT \rightarrow \Lambda$ of abelian groups. It is surjective because if $\lambda \in \Lambda$, the loop $\gamma_\lambda \in LT$, defined as the projection on T of the path $[0, 1] \rightarrow \mathfrak{t}$, $\theta \mapsto \theta\lambda$, has winding element λ . The connected components of LT are the fibres of Δ .

Write $(LT)_0$ for the identity component of LT . It consists of all loops of winding element zero and it fits into a short exact sequence

$$0 \rightarrow (LT)_0 \rightarrow LT \rightarrow \Lambda \rightarrow 0.$$

(As an aside, any loop of $(LT)_0$ can be lifted to a map $S^1 \rightarrow \mathfrak{t}$, without needing to break S^1 at a point.) A choice of q splits this sequence. With it, we are namely able to define our standard choices of loops γ_λ with winding element λ . We then get an isomorphism

$$LT \xrightarrow{\sim} (LT)_0 \oplus \Lambda, \quad (2.2)$$

given by $\gamma \mapsto (\gamma - \gamma_{\Delta_\gamma}, \Delta_\gamma)$ with inverse map $(\gamma, \lambda) \mapsto \gamma + \gamma_\lambda$. This isomorphism is not ‘natural’, though, which can be made precise as follows.

The group $\text{Diff}_+(S^1)$ of orientation preserving circle diffeomorphisms acts¹ on LT by $\varphi \cdot \gamma := \varphi^* \gamma$, where $\varphi \in \text{Diff}_+(S^1)$, $\gamma \in LT$ and $\varphi^* \gamma \in LT$ is the loop defined by

$$(\varphi^* \gamma)(\theta) := \gamma(\varphi^{-1}(\theta)), \quad \theta \in S^1.$$

Because this action preserves the winding element of any loop, it restricts to each connected component of LT —in particular to $(LT)_0$. If we let $\text{Diff}_+(S^1)$ act on Λ trivially, then the isomorphism (2.2) is not $\text{Diff}_+(S^1)$ -equivariant because in general $\varphi^* \gamma_\lambda \neq \gamma_\lambda$.

Let us now study the structure of $(LT)_0$ further. Define the real vector space

$$V\mathfrak{t} := \left\{ \xi \in C^\infty(S^1, \mathfrak{t}) \mid \int_{S^1} \xi(\theta) d\theta = 0 \right\}. \quad (2.3)$$

It carries a (left) action of $\text{Rot}(S^1)$, defined similarly as for LT . The groups $V\mathfrak{t}$ and $(LT)_0$ are related via a canonical isomorphism of abelian groups

$$(LT)_0 \xrightarrow{\sim} T \oplus V\mathfrak{t}, \quad (2.4)$$

given by $\gamma \mapsto (\text{avg } \gamma, \xi - \text{avg } \xi)$, where

$$\text{avg } \gamma := \exp \text{avg } \xi, \quad \text{avg } \xi := \int_{S^1} \xi(\theta) d\theta$$

for a choice $\xi: S^1 \rightarrow \mathfrak{t}$ of lift of γ . Its inverse is given by $(x, \xi) \mapsto x + \exp \xi$ for $x \in T$ and $\xi \in V\mathfrak{t}$. It is important to note that the isomorphism (2.4)

¹Recall that by our convention all group actions are left actions.

is $\text{Rot}(S^1)$ -equivariant if we let $\text{Rot}(S^1)$ act on $T \oplus V\mathfrak{t}$ by only affecting the $V\mathfrak{t}$ -summand.

Putting (2.2) and (2.4) together, we conclude with a decomposition

$$LT \cong T \oplus V\mathfrak{t} \oplus \Lambda \tag{2.5}$$

that depends on a choice of $q \in S^1$. Despite LT being infinite-dimensional its structure therefore resembles that of a class of finite-dimensional groups, namely the compactly generated, locally compact, Hausdorff, abelian ones. Those admit a decomposition into a compact abelian group, a vector space and a free \mathbb{Z} -module of finite rank as well (see for example [HR79, Theorem 9.8]).

2.2 Central extensions associated to lattices

In this section we will explain how, to the data of an even lattice Λ together with a choice of a particular kind of $\{\pm 1\}$ -central extension of it and a 2-cocycle for this extension, we can construct a certain $U(1)$ -central extension of the loop group LT .

References. The formula for the 2-cocycle (2.7) on LT and the study of the resulting central extension originates in [Seg81]. See also [PS86, Proposition 4.8.3] for a variant of the formula, which we do not use in this thesis. However, these sources are not explicit in noting that the construction makes sense just as well for an arbitrary even lattice Λ instead of only one of *ADE*-type. To our knowledge, that generalisation was first applied in [DX06, Section 3].

Theorem 2.2.3 was stated, respectively proven, in [PS86, Proposition 13.1.3] and [DX06, Proposition 3.4] for variants of the central extensions we consider. The term *disjoint-commutativity* is due to [Wal16, Theorem 3.3.1].

Our solution to the problem of constructing central extensions associated to odd lattices in Remark 2.2.4 is inspired by the literature on vertex algebras, namely [Kac98, Corollary 5.5]. Central extensions of loop groups that are graded by $\mathbb{Z}/2\mathbb{Z}$ appeared before in [FHT11].

So let Λ be a lattice with bi-additive form $\langle \cdot, \cdot \rangle$ and let $\tilde{\Lambda}$ be a central extension of the underlying abelian group of Λ by the group $\{\pm 1\}$ with commutator map $(\lambda, \mu) \mapsto (-1)^{\langle \lambda, \mu \rangle}$. This determines $\tilde{\Lambda}$ up to non-unique isomorphism, as discussed in Appendix A.2.1. The assumption that this is a commutator map forces Λ to be even. A second piece of data we will need is a choice of a 2-cocycle $\varepsilon: \Lambda \times \Lambda \rightarrow \{\pm 1\}$ for $\tilde{\Lambda}$. Finally, choose a privileged point q on S^1 so that we can make use of the description (2.1) of LT .

Construction 2.2.1 (Central extensions of LT). We will construct the $U(1)$ -central extension $\tilde{L}T$ of LT by letting its underlying set be $LT \times U(1)$ and writing down an explicit cocycle. Let $\gamma, \rho \in LT$, $z, w \in U(1)$ and pick lifts ξ and η as in (2.1) of γ and ρ respectively. We define the multiplication on $\tilde{L}T$ by

$$(\gamma, z) \cdot (\rho, w) := (\gamma + \rho, zw \cdot c(\gamma, \rho)) \quad (2.6)$$

where c is the 2-cocycle on LT defined by

$$\begin{aligned} c(\gamma, \rho) &:= \varepsilon(\Delta_\gamma, \Delta_\rho) e^{2\pi i S(\xi, \eta)} \\ S(\xi, \eta) &:= \frac{1}{2} \int_0^1 \langle \xi'(\theta), \eta(\theta) \rangle d\theta + \frac{1}{2} \langle \Delta_\gamma, \eta(0) \rangle. \end{aligned} \quad (2.7)$$

Here, $\langle \cdot, \cdot \rangle$ stands for the bilinear extension of the form $\langle \cdot, \cdot \rangle$ on Λ to \mathfrak{t} . Notice that S is bi-additive.

We need to check that this multiplication does not depend on the choices of lifts ξ and η . If we replace ξ by $\xi + \lambda$ for a $\lambda \in \Lambda$, then obviously the value of S does not change. If we perturb η by λ , however, the following term gets added to $S(\xi, \eta)$:

$$\begin{aligned} S(\xi, \lambda) &= \frac{1}{2} \int_0^1 \langle \xi'(\theta), \lambda \rangle d\theta + \frac{1}{2} \langle \Delta_\gamma, \lambda \rangle \\ &= \frac{1}{2} \langle \Delta_\gamma, \lambda \rangle + \frac{1}{2} \langle \Delta_\gamma, \lambda \rangle \\ &= \langle \Delta_\gamma, \lambda \rangle. \end{aligned}$$

Since this is an integer, we have $e^{2\pi i S(\xi, \lambda)} = 1$. This shows that c is a well-defined function on $LT \times LT$. Lastly, it is a 2-cocycle since ε is and S is bi-additive. It is a normalised 2-cocycle if and only if ε is.

Ingredients 2.2.2. We summarise the ingredients used in the construction of the central extension $\tilde{L}T$ for clarity:

- an even lattice $(\Lambda, \langle \cdot, \cdot \rangle)$,
- a choice of a $\{\pm 1\}$ -central extension $\tilde{\Lambda}$ of Λ such that it has commutator map $(\lambda, \mu) \mapsto (-1)^{\langle \lambda, \mu \rangle}$,
- a choice of a 2-cocycle $\varepsilon: \Lambda \times \Lambda \rightarrow \{\pm 1\}$ for $\tilde{\Lambda}$ (although not needed for the construction itself, we will always choose ε to be normalised to make calculations easier),

- a choice of a privileged point q on S^1 .

Perhaps confusingly, the notation $\tilde{L}T$ omits references to any of these ingredients. We promise that whenever we will use this notation we will always explain in the surrounding context which choices were made.

This constructed central extension satisfies the following important property:

Theorem 2.2.3 (Disjoint-commutativity of central extensions). *Let (γ, z) and (ρ, w) be two elements of $\tilde{L}T$ such that the supports of γ and ρ are contained in two disjoint intervals on S^1 respectively. Then (γ, z) and (ρ, w) commute.*

Proof. This amounts to showing that $c(\gamma, \rho)c(\rho, \gamma)^{-1} = 1$. Let us first write out $c(\gamma, \rho)c(\rho, \gamma)^{-1}$ without making any assumptions about supports. We have

$$c(\gamma, \rho)c(\rho, \gamma)^{-1} = \varepsilon(\Delta_\gamma, \Delta_\rho)\varepsilon(\Delta_\rho, \Delta_\gamma)^{-1}e^{2\pi i(S(\xi, \eta) - S(\eta, \xi))},$$

where ξ and η are lifts of γ and ρ respectively as in (2.1) and

$$\begin{aligned} S(\xi, \eta) - S(\eta, \xi) &= \frac{1}{2} \int_0^1 \langle \xi'(\theta), \eta(\theta) \rangle d\theta - \frac{1}{2} \int_0^1 \langle \eta'(\theta), \xi(\theta) \rangle d\theta + \\ &\quad \frac{1}{2} \langle \Delta_\gamma, \eta(0) \rangle - \frac{1}{2} \langle \Delta_\rho, \xi(0) \rangle. \end{aligned} \tag{2.8}$$

Let us now make the assumptions about the supports as in the statement of the Theorem and write I for the interval containing the support of γ and J for the one corresponding to ρ . We distinguish two cases: either I does not contain the point $q \in S^1$, or J does not.

In the first case, we perform a partial integration on the second integral in (2.8) to get:

$$\begin{aligned} \int_0^1 \langle \eta'(\theta), \xi(\theta) \rangle d\theta &= \left[\langle \eta(\theta), \xi(\theta) \rangle \right]_0^1 - \int_0^1 \langle \eta(\theta), \xi'(\theta) \rangle d\theta \\ &= \langle \eta(1), \xi(1) \rangle - \langle \eta(0), \xi(0) \rangle - \int_0^1 \langle \eta(\theta), \xi'(\theta) \rangle d\theta. \end{aligned}$$

After substituting $\eta(1) = \Delta_\rho + \eta(0)$ and $\xi(1) = \Delta_\gamma + \xi(0)$, these first two terms become

$$\langle \eta(1), \xi(1) \rangle - \langle \eta(0), \xi(0) \rangle = \langle \Delta_\rho, \Delta_\gamma \rangle + \langle \Delta_\rho, \xi(0) \rangle + \langle \eta(0), \Delta_\gamma \rangle.$$

Putting all of this back into (2.8) gives

$$S(\xi, \eta) - S(\eta, \xi) = \int_0^1 \langle \xi'(\theta), \eta(\theta) \rangle d\theta - \frac{1}{2} \langle \Delta_\rho, \Delta_\gamma \rangle - \langle \Delta_\rho, \xi(0) \rangle. \quad (2.9)$$

Because outside of I , ξ is constant, the above integral is actually only taken over I instead of over all of $[0, 1]$. Since J is disjoint from I and $q \notin I$, η is constant with value, say, λ , in Λ on I . Therefore we may write

$$S(\xi, \eta) - S(\eta, \xi) = \langle \xi(\text{end of } I) - \xi(\text{start of } I), \lambda \rangle - \frac{1}{2} \langle \Delta_\rho, \Delta_\gamma \rangle - \langle \Delta_\rho, \xi(0) \rangle.$$

Our assumption that $q \notin I$ also implies that

$$\xi(\text{end of } I) = \xi(1) \quad \text{and} \quad \xi(\text{start of } I) = \xi(0),$$

so $\xi(\text{end of } I) - \xi(\text{start of } I) = \Delta_\gamma \in \Lambda$, and that $\xi(0) \in \Lambda$. We now see that thanks to the integrality of Λ ,

$$\langle \xi(\text{end of } I) - \xi(\text{start of } I), \lambda \rangle - \langle \Delta_\rho, \xi(0) \rangle \in \mathbb{Z}$$

and so

$$e^{2\pi i(S(\xi, \eta) - S(\eta, \xi))} = e^{-2\pi i \cdot \frac{1}{2} \langle \Delta_\rho, \Delta_\gamma \rangle} = (-1)^{\langle \Delta_\gamma, \Delta_\rho \rangle}.$$

Together with the fact that, by definition of the 2-cocycle ε ,

$$\varepsilon(\Delta_\gamma, \Delta_\rho) \varepsilon(\Delta_\rho, \Delta_\gamma)^{-1} = (-1)^{\langle \Delta_\gamma, \Delta_\rho \rangle},$$

this shows that $c(\gamma, \rho) c(\rho, \gamma)^{-1} = 1$.

In the second case where J does not contain q , we can perform a calculation similar to the one above by partially integrating the first integral in (2.8) instead. \square

Remark 2.2.4 (Central extensions associated to odd lattices). Observe that the choice of a double cover $\tilde{\Lambda}$ with commutator map $(\lambda, \mu) \mapsto (-1)^{\langle \lambda, \mu \rangle}$ is exactly what makes the disjoint-commutativity work. If Λ is odd, then $(\lambda, \mu) \mapsto (-1)^{\langle \lambda, \mu \rangle}$ is not a commutator map. However, $(\lambda, \mu) \mapsto (-1)^{\langle \lambda, \mu \rangle + \langle \lambda, \lambda \rangle \langle \mu, \mu \rangle}$ is. So if we let $\tilde{\Lambda}$ be a double cover with this commutator map instead, pick a cocycle ε for it and replace it in the definition (2.7) of the cocycle c , then the calculation in the proof of Theorem 2.2.3 shows that

$$(\gamma, z) \cdot (\rho, w) = (-1)^{p(\gamma)p(\rho)} (\rho, w) \cdot (\gamma, z)$$

if γ and ρ have support in disjoint intervals, where $p: LT \rightarrow \{0, 1\}$ is the parity homomorphism of abelian groups given by $p(\gamma) := \langle \Delta_\gamma, \Delta_\gamma \rangle \bmod 2\mathbb{Z}$. So $\tilde{L}T$ is a *graded* central extension of a *graded* abelian group which satisfies a form of disjoint *graded* commutativity. If we write $\tilde{L}T(i) \subseteq \tilde{L}T$ for $i \in \{0, 1\}$ for the pre-images of $p^{-1}(i) \subseteq LT$, then $\tilde{L}T(0)$ is a subgroup and

$$\tilde{L}T(i) \cdot \tilde{L}T(j) \subseteq \tilde{L}T(i+j),$$

where the indices are read modulo 2. We call the elements of $\tilde{L}T(0)$ *even* and those of $\tilde{L}T(1)$ *odd*.

2.2.1 The groups $(\tilde{L}T)_0$ and $\tilde{V}\mathfrak{t}$

In our study of the structure of LT in Section 2.1 two of its subgroups were singled out: its identity component $(LT)_0$ and a real vector space $V\mathfrak{t}$. We spell out what the restriction of the central extension $\tilde{L}T$ to these subgroups looks like.

Consider first the restriction $(\tilde{L}T)_0$ to $(LT)_0$. It is the identity component of $\tilde{L}T$ and its underlying set is $(LT)_0 \times U(1)$, meaning that $(\gamma, z) \in (\tilde{L}T)_0$ if and only if γ has winding element zero. Moreover, recall that such a loop γ can be lifted to a map $S^1 \rightarrow \mathfrak{t}$ without needing to break S^1 at a point. These facts imply that the cocycle c on $\tilde{L}T$ simplifies to

$$c(\gamma, \rho) := e^{2\pi i S(\xi, \eta)}, \quad S(\xi, \eta) := \frac{1}{2} \int_{S^1} \langle d\xi, \eta \rangle \quad (2.10)$$

when restricted to $(\tilde{L}T)_0$, where $\gamma, \rho \in (LT)_0$ and $\xi, \eta: S^1 \rightarrow \mathfrak{t}$ are choices of respective lifts.

Now define a $U(1)$ -central extension $\tilde{V}\mathfrak{t}$ of the underlying abelian group of $V\mathfrak{t}$ via the 2-cocycle²

$$c: V\mathfrak{t} \times V\mathfrak{t} \rightarrow U(1), \quad c(\xi, \eta) := e^{2\pi i S(\xi, \eta)}.$$

So this group has as its underlying set $V\mathfrak{t} \times U(1)$ and its multiplication is given by

$$(\xi, z) \cdot (\eta, w) := (\xi + \eta, zw \cdot c(\xi, \eta)),$$

for $\xi, \eta \in V\mathfrak{t}$ and $z, w \in U(1)$. It is then easily checked, using the fact that $S(\xi - \text{avg } \xi, \eta - \text{avg } \eta) = S(\xi, \eta)$ for all $\xi, \eta \in V\mathfrak{t}$, that the isomorphism of abelian groups (2.4) lifts to an isomorphism of groups

$$(\tilde{L}T)_0 \xrightarrow{\sim} T \times \tilde{V}\mathfrak{t} \quad (2.11)$$

²The usage of the same symbol c for the cocycles on both LT and $V\mathfrak{t}$ might cause momentary confusion, but this will clear up in a moment.

which is the identity on the central subgroup $U(1)$.

We close this section by making an important observation about the group $\tilde{V}\mathfrak{t}$. Note that the function S in (2.10) is an \mathbb{R} -valued bilinear form on the real vector space of all smooth maps $S^1 \rightarrow \mathfrak{t}$.

Proposition 2.2.5. *The bilinear form S in (2.10) is skew and, when the lattice Λ is definite, non-degenerate upon restriction to $V\mathfrak{t}$.*

Proof. To see that S is skew we return to parametrising S^1 at unit speed by $[0, 1]$ starting at the privileged point $q \in S^1$ so that we can write

$$S(\xi, \eta) = \frac{1}{2} \int_0^1 \langle \xi'(\theta), \eta(\theta) \rangle d\theta.$$

Applying partial integration, we get

$$S(\xi, \eta) = \frac{1}{2} \langle \xi(1), \eta(1) \rangle - \frac{1}{2} \langle \xi(0), \eta(0) \rangle - \frac{1}{2} \int_0^1 \langle \xi(\theta), \eta'(\theta) \rangle d\theta,$$

and since $\xi(1) = \xi(0)$ and $\eta(1) = \eta(0)$ there indeed holds $S(\xi, \eta) = -S(\eta, \xi)$.

Turning to the second claim, suppose $\xi \in V\mathfrak{t}$ is such that $S(\xi, \eta) = 0$ for all $\eta \in V\mathfrak{t}$. Because $V\mathfrak{t}$ is closed under the operation of taking derivatives we have $\xi' \in V\mathfrak{t}$, so in particular $S(\xi, \xi') = 0$. By continuity of ξ' we must have $\langle \xi'(\theta), \xi'(\theta) \rangle = 0$ for all $\theta \in [0, 1]$. The definiteness of the form $\langle \cdot, \cdot \rangle$ on \mathfrak{t} implies that $\xi'(\theta) = 0$ for all $\theta \in [0, 1]$ and so ξ is constant. Since $\int_{S^1} \xi(\theta) d\theta = 0$, it follows that ξ is identically zero. \square

This Proposition implies that when Λ is a definite lattice, $\tilde{V}\mathfrak{t}$ is the *Heisenberg group* associated to the pair $(V\mathfrak{t}, -S)$ as in Definition A.4.1 (without yet satisfying the topological requirements).

2.3 Actions of $\text{Diff}_+(S^1)$ on central extensions

We assume the setup of Section 2.2. That is, $\Lambda, \tilde{\Lambda}, \varepsilon$ and q are as in Ingredients 2.2.2, T is the torus $\Lambda \otimes_{\mathbb{Z}} U(1)$ and, finally, $\tilde{L}T$ is the $U(1)$ -central extension of $LT := C^\infty(S^1, T)$ defined by the 2-cocycle c in (2.7) on LT constructed from all this data. In this section we show that there is an action of the group $\text{Diff}_+(S^1)$ of orientation preserving diffeomorphisms of the circle on $\tilde{L}T$.

References. The observation that, in the case of an even lattice, there exists such a $\text{Diff}_+(S^1)$ -action and the formula (2.19) which achieves it are stated in [Seg81, p. 325] (as a right action).

Recall from Section 2.1 that the non-centrally extended loop group LT carries an obvious (left) action of $\text{Diff}_+(S^1)$. A naive guess for an action on $\tilde{L}T$ would be to set

$$\varphi \cdot (\gamma, z) := (\varphi^* \gamma, z), \quad (\gamma, z) \in \tilde{L}T. \quad (2.12)$$

for $\varphi \in \text{Diff}_+(S^1)$. While this does define an action on the underlying set, it respects the multiplication on $\tilde{L}T$ if and only if $c(\varphi^* \gamma, \varphi^* \rho) = c(\gamma, \rho)$ for all $\gamma, \rho \in LT$. We will see in Proposition 2.3.1 that this does not hold, which means that (2.12) is not the correct action to consider. This misbehaviour is to be expected if we recall that the definition of c involves the choice of a privileged point q on S^1 in order to define lifts $[0, 1] \rightarrow \mathfrak{t}$ of elements of LT which have non-zero winding element.

To find a correct formula for the action we adjust the guess (2.12) as follows. We propose that $\text{Diff}_+(S^1)$ should act on $\tilde{L}T$ in a way that lifts the action on LT and fixes the embedding $U(1) \hookrightarrow \tilde{L}T$. It is easily checked that the most general such action is of the form³

$$\varphi \cdot (\gamma, z) := (\varphi^* \gamma, d(\varphi, \gamma) \cdot z) \quad (2.13)$$

for some $d(\varphi, \gamma) \in U(1)$. In order for this to be an action on the underlying set of $\tilde{L}T$, that is,

$$(\psi \circ \varphi) \cdot (\gamma, z) = \psi \cdot (\varphi \cdot (\gamma, z))$$

for all $\varphi, \psi \in \text{Diff}_+(S^1)$, d should satisfy

$$d(\psi \circ \varphi, \gamma) = d(\psi, \varphi^* \gamma) d(\varphi, \gamma). \quad (2.14)$$

If (2.13) respects the multiplication on $\tilde{L}T$ we see from (2.6) that we should additionally have the following compatibility with the cocycle c :

$$d(\varphi, \gamma + \rho) c(\gamma, \rho) = d(\varphi, \gamma) d(\varphi, \rho) c(\varphi^* \gamma, \varphi^* \rho). \quad (2.15)$$

In other words, the 1-cochain $d(\varphi, \cdot): LT \rightarrow U(1)$ should exhibit the 2-cocycle $(\gamma, \rho) \mapsto c(\varphi^* \gamma, \varphi^* \rho) c(\gamma, \rho)^{-1}$ as a 2-coboundary.

To find such a function $d: \text{Diff}_+(S^1) \times LT \rightarrow U(1)$ we will calculate the difference between $c(\varphi^* \gamma, \varphi^* \rho)$ and $c(\gamma, \rho)$, that is, the failure for $\text{Diff}_+(S^1)$ to preserve c .

Whereas in Sections 2.1 and 2.2 we represented a loop $\gamma \in LT$ as a smooth map $\xi: [0, 1] \rightarrow \mathfrak{t}$ such that $\xi(1) - \xi(0) \in \Lambda$, we will now work instead with the

³That is, if we consider $\tilde{L}T \twoheadrightarrow LT$ as a principal $U(1)$ -bundle, then we are proposing that $\text{Diff}_+(S^1)$ should act by $U(1)$ -bundle automorphisms over the existing action on the base, but not in a ‘parallel’ way. This suggests that our construction of $\tilde{L}T$ as having its underlying bundle the trivial one, instead of merely a trivialisable one, is somewhat artificial.

quasi-periodic extension $\Xi: \mathbb{R} \rightarrow \mathfrak{t}$ to all of \mathbb{R} of the latter. That is, Ξ is defined as $\Xi(\theta) = \xi(\theta)$ for $\theta \in [0, 1]$ and $\Xi(\theta + 1) = \Xi(\theta) + \Delta_\gamma$ for all $\theta \in \mathbb{R}$. The definition (2.7) of the cocycle c remains unchanged if the maps ξ and η are replaced by their quasi-periodic extensions Ξ and H everywhere.

Let $\text{Diff}_+^{(\infty)}(S^1)$ be the universal covering group of $\text{Diff}_+(S^1)$. It fits in the following short exact sequence:

$$0 \rightarrow \mathbb{Z} \rightarrow \text{Diff}_+^{(\infty)}(S^1) \rightarrow \text{Diff}_+(S^1) \rightarrow 1.$$

An explicit model for $\text{Diff}_+^{(\infty)}(S^1)$ is given by

$$\begin{aligned} \text{Diff}_+^{(\infty)}(S^1) \cong \{ \Phi: \mathbb{R} \xrightarrow{\sim} \mathbb{R} \mid \Phi \text{ an orientation preserving diffeomorphism,} \\ \Phi(\theta + 1) = \Phi(\theta) + 1 \text{ for all } \theta \in \mathbb{R} \}, \end{aligned} \quad (2.16)$$

and then the homomorphism $\mathbb{Z} \rightarrow \text{Diff}_+^{(\infty)}(S^1)$ sends $1 \in \mathbb{Z}$ to the shift diffeomorphism $\theta \mapsto \theta + 1$. Using this model, it is easy to check that $\text{Diff}_+^{(\infty)}(S^1)$ acts on the abelian group of smooth maps $\Xi: \mathbb{R} \rightarrow \mathfrak{t}$ with the property that $\Xi(\theta + 1) - \Xi(\theta)$ is a constant element in Λ , via $\Phi \cdot \Xi := \Phi^* \Xi$, where

$$(\Phi^* \Xi)(\theta) := \Xi(\Phi^{-1}(\theta)), \quad \theta \in \mathbb{R}.$$

With these preliminaries in place, we can start the calculation.

Proposition 2.3.1. *Let $\gamma, \rho \in LT$ and $\Xi, H: \mathbb{R} \rightarrow \mathfrak{t}$ be the quasi-periodic extensions to \mathbb{R} of choices of lifts $\xi, \eta: [0, 1] \rightarrow \mathfrak{t}$ of γ and ρ respectively. Take $\varphi \in \text{Diff}_+(S^1)$ and make a choice of lift $\Phi \in \text{Diff}_+^{(\infty)}(S^1)$ of φ . Then*

$$c(\varphi^* \gamma, \varphi^* \rho) = \varepsilon(\Delta_\gamma, \Delta_\rho) e^{2\pi i S(\Phi^* \Xi, \Phi^* H)}$$

and

$$\begin{aligned} S(\Phi^* \Xi, \Phi^* H) = S(\Xi, H) + \frac{1}{2} \langle \Delta_\gamma, H(\Phi^{-1}(0)) - H(0) \rangle + \\ \frac{1}{2} \langle \Xi(\Phi^{-1}(0)) - \Xi(0), \Delta_\rho \rangle. \end{aligned}$$

Proof. The winding element of a torus loop does not change after precomposition with an orientation preserving circle diffeomorphism, so

$$\varepsilon(\Delta_{\varphi^* \gamma}, \Delta_{\varphi^* \rho}) = \varepsilon(\Delta_\gamma, \Delta_\rho).$$

Using the chain rule for differentiation and again the fact that $\Delta_{\varphi^*\gamma} = \Delta_\gamma$, we can write

$$S(\Phi^*\Xi, \Phi^*H) = \frac{1}{2} \int_0^1 \langle \Xi'(\Phi^{-1}(\theta)) \cdot (\Phi^{-1})'(\theta), H(\Phi^{-1}(\theta)) \rangle d\theta + \frac{1}{2} \langle \Delta_\gamma, H(\Phi^{-1}(0)) \rangle. \quad (2.17)$$

By substitution, the first term in the above can be rewritten as

$$\frac{1}{2} \int_{\Phi^{-1}(0)}^{\Phi^{-1}(1)} \langle \Xi'(\theta), H(\theta) \rangle d\theta.$$

In turn, this integral can be broken up as

$$\frac{1}{2} \int_0^1 \langle \Xi'(\theta), H(\theta) \rangle d\theta - \frac{1}{2} \int_0^{\Phi^{-1}(0)} \langle \Xi'(\theta), H(\theta) \rangle d\theta + \frac{1}{2} \int_1^{\Phi^{-1}(1)} \langle \Xi'(\theta), H(\theta) \rangle d\theta,$$

and we can replace this first term by

$$S(\Xi, H) - \frac{1}{2} \langle \Delta_\gamma, H(0) \rangle.$$

Putting everything back into (2.17) gives

$$S(\Phi^*\Xi, \Phi^*H) = S(\Xi, H) + \frac{1}{2} \langle \Delta_\gamma, H(\Phi^{-1}(0)) - H(0) \rangle - \frac{1}{2} \int_0^{\Phi^{-1}(0)} \langle \Xi'(\theta), H(\theta) \rangle d\theta + \frac{1}{2} \int_1^{\Phi^{-1}(1)} \langle \Xi'(\theta), H(\theta) \rangle d\theta. \quad (2.18)$$

We now focus on the second integral in (2.18). Its upper limit $\Phi^{-1}(1)$ is equal to $\Phi^{-1}(0) + 1$, so, using substitution first, the fourth term in (2.18)

becomes

$$\begin{aligned}
& \frac{1}{2} \int_0^{\Phi^{-1}(0)} \langle \Xi'(\theta + 1), H(\theta + 1) \rangle d\theta \\
&= \frac{1}{2} \int_0^{\Phi^{-1}(0)} \langle \Xi'(\theta), H(\theta) + \Delta_\rho \rangle d\theta \\
&= \frac{1}{2} \int_0^{\Phi^{-1}(0)} \langle \Xi'(\theta), H(\theta) \rangle d\theta + \frac{1}{2} \int_0^{\Phi^{-1}(0)} \langle \Xi'(\theta), \Delta_\rho \rangle d\theta \\
&= \frac{1}{2} \int_0^{\Phi^{-1}(0)} \langle \Xi'(\theta), H(\theta) \rangle d\theta + \frac{1}{2} \langle \Xi(\Phi^{-1}(0)) - \Xi(0), \Delta_\rho \rangle.
\end{aligned}$$

Putting this back into (2.18) now gives the desired formula for $S(\Phi^*\Xi, \Phi^*H)$. \square

From the result of this computation we see that (2.15) will be satisfied if we define

$$d(\varphi, \gamma) := e^{\pi i \langle \Xi(\Phi^{-1}(0)) - \Xi(0), \Delta_\gamma \rangle} \in U(1). \quad (2.19)$$

Thanks to the presence of the minus sign, this definition does not depend on the choice of Ξ . It also does not depend on the choice of the lift Φ of φ . A different choice would namely be of the form $\Phi + k$ for some $k \in \mathbb{Z}$. Since $(\Phi + k)^{-1} = \Phi^{-1} - k$, we calculate

$$\begin{aligned}
\langle \Xi(\Phi^{-1}(0) - k) - \Xi(0), \Delta_\gamma \rangle &= \langle \Xi(\Phi^{-1}(0)) - k\Delta_\gamma - \Xi(0), \Delta_\gamma \rangle \\
&= \langle \Xi(\Phi^{-1}(0)) - \Xi(0), \Delta_\gamma \rangle - k \langle \Delta_\gamma, \Delta_\gamma \rangle.
\end{aligned}$$

Because $\Delta_\gamma \in \Lambda$ and Λ is even, $k \langle \Delta_\gamma, \Delta_\gamma \rangle \in 2\mathbb{Z}$. This shows what we wanted.

It can furthermore be checked that d satisfies the requirement (2.14):

$$\begin{aligned}
d(\psi \circ \varphi, \gamma) &= e^{\pi i \langle \Xi(\Phi^{-1}\Psi^{-1}(0)) - \Xi(0), \Delta_\gamma \rangle} \\
&= e^{\pi i \langle \Xi(\Phi^{-1}\Psi^{-1}(0)) - \Xi(\Phi^{-1}(0)) + \Xi(\Phi^{-1}(0)) - \Xi(0), \Delta_\gamma \rangle} \\
&= e^{\pi i \langle (\Xi \circ \Phi^{-1})(\Psi^{-1}(0)) - (\Xi \circ \Phi^{-1})(0), \Delta_\gamma \rangle} e^{\pi i \langle \Xi(\Phi^{-1}(0)) - \Xi(0), \Delta_\gamma \rangle} \\
&= d(\psi, \varphi^*\gamma) d(\varphi, \gamma),
\end{aligned}$$

where $\Psi \in \text{Diff}_+^{(\infty)}(S^1)$ is a choice of a lift of $\psi \in \text{Diff}_+(S^1)$. So this d defines via (2.13) an action of $\text{Diff}_+(S^1)$ on $\tilde{L}T$.

Remark 2.3.2 (A $\text{Diff}_+^{(2)}(S^1)$ -action in the case of an odd lattice). In our above proof that the $\text{Diff}_+^{(\infty)}(S^1)$ -action descends to one of $\text{Diff}_+(S^1)$ we explicitly used that Λ is even. Recall from Remark 2.2.4 that if Λ is odd it also gives rise to a central extension $\tilde{L}T$, which is a $\mathbb{Z}/2\mathbb{Z}$ -graded group. The $\text{Diff}_+^{(\infty)}(S^1)$ -action then only descends to one of $\text{Diff}_+^{(2)}(S^1)$, a group that fits into a short exact sequence

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \text{Diff}_+^{(2)}(S^1) \rightarrow \text{Diff}_+(S^1) \rightarrow 1$$

and which can be modelled by $\text{Diff}_+^{(\infty)}(S^1)/2\mathbb{Z}$. This action obviously respects the $\mathbb{Z}/2\mathbb{Z}$ -grading on $\tilde{L}T$.

Remark 2.3.3. If we denote for an interval $I \subseteq S^1$ by $\tilde{L}_I T$ the normal subgroup of $\tilde{L}T$ of those elements (γ, z) for which $\text{supp } \gamma \subseteq I$, then

$$\varphi \cdot L_I T = L_{\varphi(I)} T \quad \text{and} \quad \varphi \cdot \tilde{L}_I T = \tilde{L}_{\varphi(I)} T.$$

The following Proposition explains the relation between the notion of support for elements of $\text{Diff}_+(S^1)$ with that for elements of $\tilde{L}T$.

Proposition 2.3.4. *If $\varphi \in \text{Diff}_+(S^1)$ has support in an interval $I \subseteq S^1$, then φ acts trivially on $\tilde{L}_{I'} T$, where I' is the subinterval that is the closure of $S^1 \setminus I$.*

Proof. Let $\gamma \in L_{I'} T$. We first claim that $\varphi^* \gamma = \gamma$. Indeed, if $\theta \in I'$, then $\varphi^{-1}(\theta) = \theta$ and so $\gamma(\varphi^{-1}(\theta)) = \gamma(\theta)$. Suppose on the other hand that $\theta \in I$, then $\varphi^{-1}(\theta) \in I$, and so $\gamma(\varphi^{-1}(\theta)) = 0_T = \gamma(\theta)$.

Our second claim is that $d(\varphi, \gamma) = 1 \in \text{U}(1)$. We distinguish two (not mutually exclusive) cases depending on the privileged point q along which we cut S^1 : either $q \in I'$, or $q \in I$. If $q \in I'$ holds, then $\varphi^{-1}(q) = q$ and so for a lift $\Phi \in \text{Diff}_+^{(\infty)}(S^1)$ of φ we must have $\Phi^{-1}(0_{\mathbb{R}}) \in \mathbb{Z}$. This means that for a lift $\Xi: \mathbb{R} \rightarrow \mathfrak{t}$,

$$\Xi(\Phi^{-1}(0_{\mathbb{R}})) - \Xi(0_{\mathbb{R}}) = \Phi^{-1}(0_{\mathbb{R}}) \cdot \Delta_{\gamma},$$

implying that $d(\varphi, \gamma) = 1$ because $\Phi^{-1}(0_{\mathbb{R}}) \cdot \langle \Delta_{\gamma}, \Delta_{\gamma} \rangle \in 2\mathbb{Z}$ since Λ is even.

Suppose now that $q \in I$. Consider the covering map $\mathbb{R} \rightarrow [0, 1] \rightarrow S^1$ that sends $0_{\mathbb{R}}$ to q , which we have been using all the time in this section, and the subintervals of \mathbb{R} that are the pre-images of I under this map. The assumption that $q \in I$ means that $0_{\mathbb{R}}$ lies in such a subinterval of \mathbb{R} . Let us call that one J . Because φ^{-1} has support in I we must have that $\Phi^{-1}(J) = J + k$ for some $k \in \mathbb{Z}$. Therefore,

$$\Xi(\Phi^{-1}(J)) = \Xi(J + k) = \Xi(J) + k \cdot \Delta_{\gamma}.$$

Since γ is identically 0_T on I this implies that Ξ is constant (with value an element of $\Lambda \subseteq \mathfrak{t}$) on J . This shows that

$$\Xi(\Phi^{-1}(0_{\mathbb{R}})) - \Xi(0_{\mathbb{R}}) = k \cdot \Delta_\gamma,$$

and so $d(\varphi, \gamma) = 1$ holds once more. \square

2.3.1 Actions of $\text{Rot}(S^1)$ on $(\tilde{L}T)_0$ and $\tilde{V}\mathfrak{t}$

Notice from the formula (2.19) that, because $\Delta_\gamma = 0$ when $\gamma \in (LT)_0$, the subtle $\text{Diff}_+(S^1)$ -action we constructed on $\tilde{L}T$ restricts to the obvious one on $(\tilde{L}T)_0$. Remember furthermore the remark made in Section 2.1 that the vector space $V\mathfrak{t}$ carries a $\text{Rot}(S^1)$ -action. The cocycle defining the central extension $\tilde{V}\mathfrak{t}$ from Section 2.2.1 is $\text{Rot}(S^1)$ -invariant, which is easily seen from an application of the chain rule followed by a substitution, and therefore the $\text{Rot}(S^1)$ -action on $V\mathfrak{t}$ lifts in the obvious way to $\tilde{V}\mathfrak{t}$. It is now immediate that the group isomorphism (2.11) is $\text{Rot}(S^1)$ -equivariant.

2.4 Actions of lifts of lattice automorphisms on central extensions

Take again the input data for the construction of a torus loop group central extension $\tilde{L}T$ in Section 2.2, as summarised in the first paragraph of Section 2.3, as given. In Section 2.3 we learned that a torus loop $S^1 \rightarrow T$ can be precomposed with an orientation preserving circle diffeomorphism, and that this lifts to an action of $\text{Diff}_+(S^1)$ on $\tilde{L}T$. One can ask with what kind of automorphism of the torus loops can be postcomposed to obtain a group automorphism of $\tilde{L}T$ also. In this section we show that this can be done for torus automorphisms induced by automorphisms of a certain $\{\pm 1\}$ -central extension of the lattice Λ .

References. Our idea that such an action on $\tilde{L}T$ should exist comes from the vertex algebra literature, namely the claim in [Bor92, Section 12] that in the case when Λ is the Leech lattice Λ_{24} the group $\text{Aut}(\tilde{\Lambda}^\varepsilon; \langle \cdot, \cdot \rangle)$ acts on the Leech lattice vertex algebra. This fact is further elaborated upon for general positive definite, even lattices in for example [DN99, Section 2.4]. The current section is simultaneously intended to clarify the first paragraph of [DX06, Section 4]. The details regarding the group $\text{Aut}(\tilde{\Lambda}^\varepsilon; \langle \cdot, \cdot \rangle)$ are taken from the start of [FLM88, Section 6.4].

An automorphism g of Λ is, as discussed in Appendix A.1, a \mathbb{Z} -module automorphism of Λ which preserves the form $\langle \cdot, \cdot \rangle$ on Λ . There is an obvious

action of $\text{Aut}(\Lambda; \langle \cdot, \cdot \rangle)$ on the non-centrally extended loop group LT given by $g \cdot \gamma := g_*\gamma$, where $\gamma \in LT$, $g_*\gamma \in LT$ is the loop defined by

$$(g_*\gamma)(\theta) := g_T(\gamma(\theta)), \quad \theta \in S^1,$$

and g_T is the induced automorphism of $T := \Lambda \otimes_{\mathbb{Z}} \mathbb{U}(1)$ by g . Just like in Section 2.3, a naive attempt at defining an action on $\tilde{L}T$ would be

$$g \cdot (\gamma, z) := (g_*\gamma, z), \quad z \in \mathbb{U}(1).$$

Again, this respects the group multiplication on $\tilde{L}T$ if and only if for the 2-cocycle c in (2.7) defining $\tilde{L}T$ there holds $c(g_*\gamma, g_*\rho) = c(\gamma, \rho)$ for all $\gamma, \rho \in LT$. Let us check whether this is true.

Recall that c is defined in terms of choices of lifts $\xi, \eta: [0, 1] \rightarrow \mathfrak{t}$ of γ and ρ respectively. We have

$$\Delta_{g_*\gamma} = (\mathbb{R}g \circ \xi)(1) - (\mathbb{R}g \circ \xi)(0) = \mathbb{R}g(\xi(1) - \xi(0)) = g\Delta_{\gamma},$$

where $\mathbb{R}g$ denotes the linear extension of g to \mathfrak{t} . Similarly, $\Delta_{g_*\rho} = g\Delta_{\rho}$. Therefore,

$$\begin{aligned} c(g_*\gamma, g_*\rho) &= \varepsilon(g\Delta_{\gamma}, g\Delta_{\rho})e^{2\pi i S(\mathbb{R}g \circ \xi, \mathbb{R}g \circ \eta)} \\ S(\mathbb{R}g \circ \xi, \mathbb{R}g \circ \eta) &= \frac{1}{2} \int_0^1 \langle (\mathbb{R}g \circ \xi)'(\theta), (\mathbb{R}g \circ \eta)(\theta) \rangle d\theta + \frac{1}{2} \langle g\Delta_{\gamma}, (g \circ \eta)(0) \rangle. \end{aligned}$$

Since $(\mathbb{R}g \circ \xi)'(\theta) = \mathbb{R}g(\xi'(\theta))$ and $\mathbb{R}g$ preserves the form $\langle \cdot, \cdot \rangle$, we find that

$$S(\mathbb{R}g \circ \xi, \mathbb{R}g \circ \eta) = S(\xi, \eta).$$

However, it is not necessarily true that $\varepsilon(g\Delta_{\gamma}, g\Delta_{\rho}) = \varepsilon(\Delta_{\gamma}, \Delta_{\rho})$. Explicit counterexamples can be found. We conclude that in general $c(g_*\gamma, g_*\rho) \neq c(\gamma, \rho)$ and that the blame lies on the extra piece of data that is the 2-cocycle ε , which is not invariant under automorphisms of Λ .

In our next attempt we will try not to let $\text{Aut}(\Lambda; \langle \cdot, \cdot \rangle)$ act on $\tilde{L}T$, but a certain bigger group instead.

Let $\tilde{\Lambda}^{\varepsilon}$ be the $\{\pm 1\}$ -central extension of Λ associated to ε . Its underlying set is $\Lambda \times \{\pm 1\}$ and its group multiplication is defined by

$$(\lambda, z) \cdot (\mu, w) := (\lambda + \mu, zw \cdot \varepsilon(\lambda, \mu)),$$

where $\lambda, \mu \in \Lambda$ and $z, w \in \{\pm 1\}$. An automorphism \tilde{g} of the group $\tilde{\Lambda}^{\varepsilon}$ induces an automorphism g of the underlying abelian group of Λ by picking for $\lambda \in \Lambda$

an arbitrary lift $\tilde{\lambda} \in \tilde{\Lambda}^\varepsilon$ and defining $g\lambda$ to be the image in Λ of $\tilde{g}\tilde{\lambda}$. This is independent of the choice of $\tilde{\lambda}$. Write $\text{Aut}(\tilde{\Lambda}^\varepsilon; \langle \cdot, \cdot \rangle)$ for the group of those automorphisms \tilde{g} of $\tilde{\Lambda}$ such that g lies in $\text{Aut}(\Lambda; \langle \cdot, \cdot \rangle)$, that is, g preserves the form $\langle \cdot, \cdot \rangle$. Notice that such a \tilde{g} fixes the embedding $\{\pm 1\} \hookrightarrow \tilde{\Lambda}^\varepsilon$, because its image $\{(0, \pm 1)\}$ consists of the only elements of $\tilde{\Lambda}^\varepsilon$ of finite order.

This group of automorphisms of $\tilde{\Lambda}$ lies in the following short exact sequence of groups:

$$1 \rightarrow \text{Hom}_{\text{Ab}}(\Lambda, \{\pm 1\}) \rightarrow \text{Aut}(\tilde{\Lambda}^\varepsilon; \langle \cdot, \cdot \rangle) \rightarrow \text{Aut}(\Lambda; \langle \cdot, \cdot \rangle) \rightarrow 1.$$

The second arrow sends $f \in \text{Hom}_{\text{Ab}}(\Lambda, \{\pm 1\})$ to the automorphism $\tilde{\lambda} \mapsto \tilde{\lambda} \cdot f(\lambda)$, where $\tilde{\lambda} \in \tilde{\Lambda}^\varepsilon$ and λ is its image in Λ . The third arrow sends $\tilde{g} \in \text{Aut}(\tilde{\Lambda}^\varepsilon; \langle \cdot, \cdot \rangle)$ to the induced automorphism $g \in \text{Aut}(\Lambda, \langle \cdot, \cdot \rangle)$. Note that a choice of a basis for Λ gives an isomorphism

$$\text{Hom}_{\text{Ab}}(\Lambda, \{\pm 1\}) \cong \{\pm 1\}^{\text{rank } \Lambda},$$

so $\text{Aut}(\tilde{\Lambda}^\varepsilon; \langle \cdot, \cdot \rangle)$ has order $2^{\text{rank } \Lambda}$ times that of $\text{Aut}(\Lambda; \langle \cdot, \cdot \rangle)$.

In Section 2.1 we learned that, given a choice of a privileged point q on S^1 (which we already made to construct \tilde{LT}), there are standard choices of loops γ_λ with winding element $\lambda \in \Lambda$. This gave an injective homomorphism of abelian groups $\Lambda \hookrightarrow LT$, $\lambda \mapsto \gamma_\lambda$. It lifts to a homomorphism $\tilde{\Lambda}^\varepsilon \hookrightarrow \tilde{LT}$, $(\lambda, z) \mapsto (\gamma_\lambda, z)$ of non-abelian groups. We would like to find an action of $\text{Aut}(\tilde{\Lambda}^\varepsilon; \langle \cdot, \cdot \rangle)$ on \tilde{LT} which restricts to the one on $\tilde{\Lambda}^\varepsilon$.

Let us therefore begin by describing the elements of $\text{Aut}(\tilde{\Lambda}^\varepsilon; \langle \cdot, \cdot \rangle)$ a bit more explicitly. If \tilde{g} is such an automorphism and $(\lambda, z) \in \tilde{\Lambda}^\varepsilon$, then

$$\tilde{g} \cdot (\lambda, z) = (g\lambda, e(\tilde{g}, \lambda) \cdot z)$$

for some $g \in \text{Aut}(\Lambda; \langle \cdot, \cdot \rangle)$ and $e(\tilde{g}, \lambda) \in \{\pm 1\}$, which does not depend on z . If we namely assume that ε is a normalised cocycle, then we can write

$$(\lambda, z) = (\lambda, z \cdot \varepsilon(\lambda, 0)) = (\lambda, 1) \cdot (0, z).$$

So therefore

$$\begin{aligned} \tilde{g}(\lambda, z) &= \tilde{g}(\lambda, 1) \cdot \tilde{g}(0, z) \\ &= (g\lambda, e(\tilde{g}, \lambda)) \cdot (0, z) \\ &= (g\lambda, e(\tilde{g}, \lambda) \cdot z \cdot \varepsilon(g\lambda, 0)) \\ &= (g\lambda, e(\tilde{g}, \lambda) \cdot z). \end{aligned}$$

Here we used in the second equality that \tilde{g} fixes the embedding $\{\pm 1\} \hookrightarrow \tilde{\Lambda}^\varepsilon$, and in the fourth equality it was used again that ε is normalised.

This e satisfies the following two identities:

$$\begin{aligned} e(\tilde{g} \circ \tilde{h}, \lambda) &= e(\tilde{g}, h\lambda)e(\tilde{h}, \lambda) \\ e(\tilde{g}, \lambda + \mu)\varepsilon(\lambda, \mu) &= e(\tilde{g}, \lambda)e(\tilde{g}, \mu)\varepsilon(g\lambda, g\mu). \end{aligned}$$

The latter should be read as $e(\tilde{g}, \cdot): \Lambda \rightarrow \{\pm 1\}$ being a 1-cochain which exhibits the 2-cocycle $(\lambda, \mu) \mapsto \varepsilon(g\lambda, g\mu)\varepsilon(\lambda, \mu)^{-1}$ as a 2-coboundary.

Let us now define an action of $\text{Aut}(\tilde{\Lambda}^\varepsilon; \langle \cdot, \cdot \rangle)$ on $\tilde{L}T$ by setting

$$\tilde{g} \cdot (\gamma, z) := (g_*\gamma, e(\tilde{g}, \Delta_\gamma) \cdot z) \quad (2.20)$$

for $\tilde{g} \in \text{Aut}(\tilde{\Lambda}^\varepsilon; \langle \cdot, \cdot \rangle)$ and $(\gamma, z) \in \tilde{L}T$. One can check that, thanks to the two identities satisfied by e above, this does respect the multiplication on $\tilde{L}T$ and that it defines a group action.

Proposition 2.4.1. *The action of $\text{Diff}_+(S^1)$ on $\tilde{L}T$ given by (2.13) and (2.19) commutes with the action of $\text{Aut}(\tilde{\Lambda}^\varepsilon; \langle \cdot, \cdot \rangle)$ given in (2.20).*

Proof. Let $(\gamma, z) \in \tilde{L}T$, $\varphi \in \text{Diff}_+(S^1)$ and $\tilde{g} \in \text{Aut}(\tilde{\Lambda}^\varepsilon; \langle \cdot, \cdot \rangle)$. We have to show that the equality

$$\varphi \cdot (\tilde{g} \cdot (\gamma, z)) = \tilde{g} \cdot (\varphi \cdot (\gamma, z))$$

holds true. The left hand side is

$$\varphi \cdot (g_*\gamma, e(\tilde{g}, \Delta_\gamma) \cdot z) = (\varphi^*g_*\gamma, d(\varphi, g_*\gamma) \cdot e(\tilde{g}, \Delta_\gamma) \cdot z),$$

while the right hand side is

$$\tilde{g} \cdot (\varphi^*\gamma, d(\varphi, \gamma) \cdot z) = (g_*\varphi^*\gamma, e(\tilde{g}, \Delta_{\varphi^*\gamma}) \cdot d(\varphi, \gamma) \cdot z).$$

We obviously have that $\varphi^*g_*\gamma = g_*\varphi^*\gamma$, since pre- and postcompositions commute. There furthermore holds that $\Delta_{\varphi^*\gamma} = \Delta_\gamma$, and

$$d(\varphi, g_*\gamma) = e^{\pi i \langle (\mathbb{R}g \circ \Xi)(\Phi^{-1}0) - (\mathbb{R}g \circ \Xi)(0), g\Delta_\gamma \rangle} = e^{\pi i \langle \Xi(\Phi^{-1}0) - \Xi(0), \Delta_\gamma \rangle} = d(\varphi, \gamma).$$

This shows what we wanted. \square

Remark 2.4.2. It is easy to see that an element $\tilde{g} \in \text{Aut}(\tilde{\Lambda}^\varepsilon; \langle \cdot, \cdot \rangle)$ restricts to an automorphism of the normal subgroup $\tilde{L}_I T$ for every interval $I \subseteq S^1$. We say that $\text{Aut}(\tilde{\Lambda}^\varepsilon; \langle \cdot, \cdot \rangle)$ acts *locally*.

2.4.1 Actions of related groups on $(\tilde{L}T)_0$ and $\tilde{V}\mathfrak{t}$

The failure for the group $\text{Aut}(\Lambda; \langle \cdot, \cdot \rangle)$ to act on $\tilde{L}T$ can, instead of being blamed on the unnatural choice of the 2-cocycle ε , alternatively be pinned on the presence of non-zero winding elements of loops. Hence, this group of lattice isomorphisms *does* act on the identity component $(\tilde{L}T)_0$.

Write $\text{Aut}(\mathfrak{t}; \langle \cdot, \cdot \rangle)$ for the compact group of automorphisms of the Lie algebra \mathfrak{t} that preserve the bilinear extension of the form $\langle \cdot, \cdot \rangle$ on Λ to \mathfrak{t} . It acts through post-composition of loops via \mathbb{R} -linear automorphisms on $V\mathfrak{t}$. Let O_* be such an automorphism for $O \in \text{Aut}(\mathfrak{t}; \langle \cdot, \cdot \rangle)$. Then O_* preserves the skew form S in (2.10) on $V\mathfrak{t}$ since $d(O \circ \xi) = O \circ d\xi$ for all $\xi \in V\mathfrak{t}$. Therefore, the action of $\text{Aut}(\mathfrak{t}; \langle \cdot, \cdot \rangle)$ lifts in the obvious way to $\tilde{V}\mathfrak{t}$.

We summarise these findings in the following figure:

$$\begin{array}{ccccc}
 \tilde{V}\mathfrak{t} & \longleftarrow & (\tilde{L}T)_0 & \longleftarrow & \tilde{L}T \\
 \circlearrowleft & & \circlearrowleft & & \circlearrowleft \\
 \text{Aut}(\mathfrak{t}; \langle \cdot, \cdot \rangle) & \longleftarrow & \text{Aut}(\Lambda; \langle \cdot, \cdot \rangle) & \longleftarrow & \text{Aut}(\tilde{\Lambda}^\varepsilon; \langle \cdot, \cdot \rangle) \\
 \mathbb{R}g & \longleftarrow & g & \longleftarrow & \tilde{g}
 \end{array}$$

2.5 Irreducible, positive energy representations

Assume once more the input data for the construction of a torus loop group central extension $\tilde{L}T$ in Section 2.2 summarised in the first paragraph of Section 2.3. For the purpose of the discussion that is about to follow we add the extra condition of being positive definite on the lattice Λ . This will be necessary for certain vector spaces we will consider to be equipped with positive definite inner products. Their Hilbert space completions will then carry group representations.⁴

In this section we will construct and classify a certain type of irreducible representations of $\tilde{L}T$, namely the *positive energy* ones (see Definition A.3.13).

References. The construction and classification we perform here is due to [Seg81]. See also [PS86, Section 9.5] for an alternative exposition. The outcomes of the calculation of the characters of these representations were learned from [FLM88, Remark 7.1.2], [Sta95, Section 4.2.1] and [Mas14, p. 356].

We saw in Section 2.3 that the group $\tilde{L}T$ carries an action of the group $\text{Diff}_+(S^1)$ and hence also of the subgroup $\text{Rot}(S^1)$. It therefore indeed makes

⁴Recall our convention that representations are always meant to be strongly continuous and unitary.

sense to talk about positive energy representations⁵ of $\tilde{L}T$, once we make $\tilde{L}T$ into a topological group. The decomposition (2.11) will be the key to the construction and classification of such representations because it allows us to do so for the identity component $(\tilde{L}T)_0$ first. We will later transfer these results to $\tilde{L}T$.

2.5.1 Irreducible representations of $\tilde{V}\mathfrak{t}$

Recall from Section 2.2.1 the definition of the subgroup $\tilde{V}\mathfrak{t}$ of $\tilde{L}T$, made in terms of the skew form S in (2.10). Because we are assuming Λ to be positive definite, the observation at the end of Section 2.2.1 applies. That is, $\tilde{V}\mathfrak{t}$ is the Heisenberg group associated to the pair $(V\mathfrak{t}, -S)$, in the non-topological sense. Our first step towards studying its representation theory will be to define a specific complex structure on $V\mathfrak{t}$ that is compatible with the skew form S , so as to turn $\tilde{V}\mathfrak{t}$ into a topological group. This complex structure will then also be used to construct a Weyl representation of $\tilde{V}\mathfrak{t}$.

A complex structure on $V\mathfrak{t}$

Because each loop $\xi: S^1 \rightarrow \mathfrak{t}$ of $V\mathfrak{t}$ is smooth, it admits a Fourier decomposition

$$\xi(\theta) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \xi_k e^{2\pi i k \theta},$$

where the Fourier coefficient ξ_k is a vector of the complexification $\mathbb{C}\mathfrak{t} := \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$. (The condition $k \neq 0$ is there because $\int_{S^1} \xi(\theta) d\theta = 0$ and so $\xi_0 = 0$.) We therefore see that the complexified vector space $\mathbb{C}(V\mathfrak{t}) := V\mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$, which can be identified with the space of smooth loops $\xi: S^1 \rightarrow \mathbb{C}\mathfrak{t}$ such that $\int_{S^1} \xi(\theta) d\theta = 0$, can be decomposed as a direct sum of two \mathbb{C} -linear subspaces

$$\mathbb{C}(V\mathfrak{t}) = V\mathfrak{t}^+ \oplus V\mathfrak{t}^-, \quad (2.21)$$

where $V\mathfrak{t}^+$ is the subspace consisting of the loops whose negative Fourier coefficients are zero and $V\mathfrak{t}^-$ is defined similarly. Equivalently, $V\mathfrak{t}^+$ is the subspace of loops which admit an extension to the closed unit disc in the complex plane such that the restriction to the open unit disc is holomorphic, and a similar condition holds for the loops of $V\mathfrak{t}^-$ with respect to the other half of the Riemann sphere.

The standard complex conjugation $\overline{\xi \otimes z} := \xi \otimes \bar{z}$ on $\mathbb{C}(V\mathfrak{t})$ interchanges $V\mathfrak{t}^+$ and $V\mathfrak{t}^-$. The image of the canonical \mathbb{R} -linear injection $V\mathfrak{t} \hookrightarrow \mathbb{C}(V\mathfrak{t})$ given

⁵We are taking the central extension $\tilde{L}T$ for the group N in Definition A.3.13, not LT .

by $\xi \mapsto \xi \otimes 1$ therefore equals $\{\xi^+ + \overline{\xi^+} \mid \xi^+ \in V\mathfrak{t}^+\}$ and so the compositions of this injection with the two \mathbb{C} -linear projections $\mathbb{C}(V\mathfrak{t}) \rightarrow V\mathfrak{t}^\pm$ are \mathbb{R} -linear isomorphisms.

Now we define a complex structure $J: V\mathfrak{t} \rightarrow V\mathfrak{t}$ by setting $V\mathfrak{t}^+$ to be the $+i$ -eigenspace of the \mathbb{C} -linear extension $\mathbb{C}J$ of J to $\mathbb{C}(V\mathfrak{t})$ and $V\mathfrak{t}^-$ the $-i$ -eigenspace, meaning that $\mathbb{C}J$ multiplies a vector of $\mathbb{C}(V\mathfrak{t})$ of the form $\xi_k e^{2\pi i k \theta}$ with $\text{sgn}(k) \cdot i$. In other words, J is determined by

$$(\mathbb{C}J)(\xi \otimes 1) := J(\xi) \otimes 1 := i\xi^+ - i\overline{\xi^+},$$

where $\xi^+ \in V\mathfrak{t}^+$ is the image of ξ under $V\mathfrak{t} \xrightarrow{\sim} V\mathfrak{t}^+$. It is then clear that J squares to $-\text{id}_{V\mathfrak{t}}$. This makes $(V\mathfrak{t})_J \xrightarrow{\sim} V\mathfrak{t}^+$ a \mathbb{C} -linear isomorphism by definition.

Compatibility of the complex structure with the skew form

Let us extend the form $\langle \cdot, \cdot \rangle$ on \mathfrak{t} complex bilinearly to $\mathbb{C}\mathfrak{t}$. Note that this extended form is still symmetric and not conjugate symmetric. In turn, this then defines a complex bilinear extension of S to $\mathbb{C}(V\mathfrak{t})$. This extension of S is again skew. We denote these extensions to $\mathbb{C}\mathfrak{t}$ and $\mathbb{C}(V\mathfrak{t})$, respectively, by $\langle \cdot, \cdot \rangle$ and S as well. If ξ and η are vectors in $V\mathfrak{t}$, then it can be checked by expanding them into their Fourier series that

$$S(\xi, \eta) = \pi i \sum_{k \in \mathbb{Z} \setminus \{0\}} k \langle \xi_k, \eta_{-k} \rangle.$$

This expression shows that the subspaces $V\mathfrak{t}^+$ and $V\mathfrak{t}^-$ are isotropic for S and that

$$S(\xi, \eta) = S(\xi^+, \eta^-) + S(\xi^-, \eta^+),$$

where $\xi = \xi^+ + \xi^-$ and $\eta = \eta^+ + \eta^-$ are their decompositions along (2.21).

The complex bilinearity and skewness of S can now be used to calculate

$$\begin{aligned} S(\xi, J\xi) &= S(\xi^+, (\mathbb{C}J)\xi^-) + S(\xi^-, (\mathbb{C}J)\xi^+) \\ &= S(\xi^+, -i\xi^-) + S(\xi^-, i\xi^+) \\ &= -iS(\xi^+, \xi^-) + iS(\xi^-, \xi^+) \\ &= -2iS(\xi^+, \xi^-) \\ &= 2\pi \sum_{k=1}^{\infty} k \langle \xi_k, \xi_{-k} \rangle. \end{aligned}$$

Because ξ takes its values in the real Lie algebra \mathfrak{t} we have $\xi_{-k} = \overline{\xi_k}$. Since the form $\langle \cdot, \cdot \rangle$ on \mathfrak{t} is positive definite, there holds $\langle \xi_k, \xi_{-k} \rangle \geq 0$. This proves that

if $\xi \neq 0$, then $S(\xi, J\xi) > 0$. It is furthermore easily checked, again using the complex bilinearity of S , that $S(J\xi, J\eta) = S(\xi, \eta)$. We conclude that J satisfies the two properties listed in Proposition A.4.5(i).

We may now proceed with the constructions described in Appendix A.4. The norm topology from $\langle \cdot, \cdot \rangle_J$ makes $V\mathfrak{t}$ a topological real vector space and the compatibility of S with J implies that S is continuous with respect to this topology. Hence, $\tilde{V}\mathfrak{t}$ is a Heisenberg group. Furthermore, $(V\mathfrak{t})_J$ is a complex pre-Hilbert space equipped with a Hermitian inner product $\langle \cdot, \cdot \rangle_J$. We may therefore form a bosonic Fock space $\mathcal{S}((V\mathfrak{t})_J)$. It carries an irreducible Weyl representation W_J of the Heisenberg group $\tilde{V}\mathfrak{t}$. The central subgroup $U(1)$ of $\tilde{V}\mathfrak{t}$ acts under W_J as $z \mapsto z$. We will abbreviate this representation and its underlying Fock space by W and \mathcal{S} , respectively, that is, we suppress the reference to the specific complex structure J .

Positivity of energy

As already noted in Section 2.3.1, the $\text{Rot}(S^1)$ -action on $V\mathfrak{t}$ by \mathbb{R} -linear operators preserves the skew form S . Moreover, the \mathbb{C} -linear extension of the $\text{Rot}(S^1)$ -action to $\mathbb{C}(V\mathfrak{t})$ preserves the decomposition (2.21) of $\mathbb{C}(V\mathfrak{t})$ because it is given on each vector of the form $\xi_k e^{2\pi i k \theta}$ simply by

$$(\mathbb{C}\varphi_{\theta'}^*)(\xi_k e^{2\pi i k \theta}) = \xi_k e^{2\pi i k(\theta - \theta')}, \quad (2.22)$$

where we write $\mathbb{C}\varphi_{\theta'}^*$ for the \mathbb{C} -linear extension of the rotation operator $\varphi_{\theta'}$ along the angle θ' . Therefore, the $\text{Rot}(S^1)$ -action on $V\mathfrak{t}$ commutes with the complex structure J .

This $\text{Rot}(S^1)$ -action is strongly continuous with respect to the norm topology on $V\mathfrak{t}$ induced by $\langle \cdot, \cdot \rangle_J$. Hence, we may apply Proposition A.4.11 to conclude that the action of $\text{Rot}(S^1)$ on $\tilde{V}\mathfrak{t}$ is strongly continuous, and that there is a representation R of $\text{Rot}(S^1)$ on \mathcal{S} which extends the action on the subspace $(V\mathfrak{t})_J = \text{Sym}^1((V\mathfrak{t})_J) \subseteq \mathcal{S}$ and such that the intertwining property

$$R(\varphi_{\theta})W(\xi, z)R(\varphi_{\theta})^* = W(\varphi_{\theta} \cdot (\xi, z)) \quad (2.23)$$

is satisfied for all $\varphi_{\theta} \in \text{Rot}(S^1)$ and $(\xi, z) \in \tilde{V}\mathfrak{t}$.

Let us calculate the *character* of R (see Definition A.3.20). For this purpose we might as well temporarily redefine \mathcal{S} to be $\mathcal{S}(V\mathfrak{t}^+)$ instead of $\mathcal{S}((V\mathfrak{t})_J)$ because the \mathbb{C} -linear isomorphism $(V\mathfrak{t})_J \xrightarrow{\sim} V\mathfrak{t}^+$ intertwines the respective $\text{Rot}(S^1)$ -actions. Recall that $V\mathfrak{t}^+$ is the complex vector space of smooth loops $S^1 \rightarrow \mathbb{C}\mathfrak{t}$ that are of the form

$$\sum_{k=1}^{\infty} \xi_k e^{2\pi i k \theta}, \quad \theta \in S^1.$$

We then learn from the expression (2.22) for the rotation action on $V\mathfrak{t}^+$, together with the fact that R is defined in the proof of Proposition A.4.11 by extending the rotation action on $V\mathfrak{t}^+$ to monomials factor-wise, that the k -th energy eigenspace $\mathcal{S}(k)$ is spanned by monomials of the form

$$\xi_{1r} e^{2\pi i k_1 \theta} \xi_{2r} e^{2\pi i k_2 \theta} \dots \xi_{rr} e^{2\pi i k_r \theta},$$

where $r \geq 0$, $\xi_{ir} \in \mathbb{C}\mathfrak{t}$, $k_i \geq 1$ and $k_1 + \dots + k_r = k$. Hence, \mathcal{S} contains no vectors of negative energy:

Proposition 2.5.1. *The intertwining $\text{Rot}(S^1)$ -action R on the Weyl representation W of $\tilde{V}\mathfrak{t}$ is of positive energy.*

By picking a basis for the Lie algebra $\mathbb{C}\mathfrak{t}$ we see that the dimension of $\mathcal{S}(k)$ equals the coefficient of q^k in the power series

$$\left(\sum_{k=0}^{\infty} p(k) q^k \right)^{\dim \mathfrak{t}},$$

where $p(k)$ is the *Euler partition function*, that is, $p(k)$ for $k \geq 1$ denotes the number of ways to write k as the sum of positive integers, and $p(0) := 1$ by convention. (This coefficient can also be understood as the number of *(dim \mathfrak{t})-coloured partitions* of k .) In turn, L. Euler found that the generating function of the partition function named after him admits an expression as an infinite product:

$$\sum_{k=0}^{\infty} p(k) q^k = \prod_{j=1}^{\infty} (1 - q^j)^{-1},$$

so if we set

$$\eta(q) := q^{1/24} \prod_{j=1}^{\infty} (1 - q^j),$$

then we obtain the formula

$$\text{ch}_R(q) = q^{\dim \mathfrak{t}/24} \eta(q)^{-\dim \mathfrak{t}}.$$

Our reason for writing $\text{ch}_R(q)$ in this way is that if the formal variable q is substituted by $e^{2\pi iz}$ for $z \in \mathbb{C}$ with $\text{Im } z > 0$, then $\eta(q)$ is known as *Dedekind's eta function*. It is a *holomorphic modular form* for the full group $\text{SL}(2, \mathbb{Z})$ of weight $1/2$ (with a highly non-trivial multiplier) (see [Apo90, Theorem 3.4]). While $\text{ch}_R(q)$ does not satisfy the necessary transformation behaviour to be a modular form as well, the normalised character

$$Z_W(q) := q^{-\dim \mathfrak{t}/24} \text{ch}_R(q) = \eta(q)^{-\dim \mathfrak{t}} \quad (2.24)$$

does.

Having confirmed that W is of positive energy, the unicity result Theorem A.4.13 now tells us that

Theorem 2.5.2. *Every irreducible, positive energy representation of $\tilde{V}\mathfrak{t}$ such that the central subgroup $U(1) \subseteq \tilde{V}\mathfrak{t}$ acts as $z \mapsto z$ is isomorphic to W .*

The intertwining action of Lie algebra isometries

We exhibit, on top of the representations of $\tilde{V}\mathfrak{t}$ and $\text{Rot}(S^1)$, a third piece of structure that the Hilbert space \mathcal{S} possesses.

Recall from Section 2.4.1 that the group $\text{Aut}(\mathfrak{t}; \langle \cdot, \cdot \rangle)$ acts through \mathbb{R} -linear automorphisms on $V\mathfrak{t}$ in a way that preserves the skew form S in (2.10). Furthermore, if $O \in \text{Aut}(\mathfrak{t}; \langle \cdot, \cdot \rangle)$ and O_* is the corresponding operator on $V\mathfrak{t}$, then the \mathbb{C} -linear extension $\mathbb{C}(O_*)$ of O_* to $\mathbb{C}(V\mathfrak{t})$ preserves each of the subspaces $V\mathfrak{t}^\pm$ because it is given on each vector of the form $\xi_k e^{2\pi i k \theta}$ by

$$\mathbb{C}(O_*)(\xi_k e^{2\pi i k \theta}) = (\mathbb{C}O)(\xi_k) e^{2\pi i k \theta},$$

where $\mathbb{C}O$ is the \mathbb{C} -linear extension of O to $\mathbb{C}\mathfrak{t}$. Therefore, O_* commutes with the complex structure J on $V\mathfrak{t}$.

It is strongly continuous with respect to the norm topology on $V\mathfrak{t}$ induced by $\langle \cdot, \cdot \rangle_J$. Hence the demands of Proposition A.4.11 are met and $\text{Aut}(\mathfrak{t}; \langle \cdot, \cdot \rangle)$ acts strongly continuously on $\tilde{V}\mathfrak{t}$ (we denote the translate of $(\xi, z) \in \tilde{V}\mathfrak{t}$ by O as $O \cdot (\xi, z)$) and there is a representation U of $\text{Aut}(\mathfrak{t}; \langle \cdot, \cdot \rangle)$ on \mathcal{S} which extends the action on $(V\mathfrak{t})_J$ such that the intertwining property

$$U(O)W(\xi, z)U(O)^* = W(O \cdot (\xi, z)) \quad (2.25)$$

is satisfied. This action fixes the vacuum vector.

It is obvious from looking at their actions on $\mathbb{C}(V\mathfrak{t})$ that the actions of $\text{Rot}(S^1)$ and $\text{Aut}(\mathfrak{t}; \langle \cdot, \cdot \rangle)$ on $V\mathfrak{t}$ commute. The same then holds for their respective extensions R and U to \mathcal{S} , given the way they are defined in the proof of Proposition A.4.11.

2.5.2 Irreducible representations of $(\tilde{L}T)_0$

In Section 2.5.1 we made $V\mathfrak{t}$ and $\tilde{V}\mathfrak{t}$ into topological groups. By giving $T \oplus V\mathfrak{t}$ and $T \times \tilde{V}\mathfrak{t}$ the product topologies, $(LT)_0$ and $(\tilde{L}T)_0$ become topological groups as well through the decompositions (2.4) and (2.11), respectively.

The representation W of $\tilde{V}\mathfrak{t}$ now allows us to easily build irreducible, positive energy representations of $(\tilde{L}T)_0$. We namely construct for every character

$l \in \text{Hom}_{\text{Lie}}(T, \text{U}(1))$ a representation W_l of $(\tilde{L}T)_0$ on the Hilbert space tensor product $\mathcal{S}_l := \mathbb{C}_l \otimes \mathcal{S}$, where \mathbb{C}_l denotes a copy of \mathbb{C} , as follows. Let $(\gamma, z) \in (\tilde{L}T)_0$ and consider its image

$$(\text{avg } \gamma, (\xi - \text{avg } \xi, z)) \in T \times \tilde{V}\mathfrak{t}$$

under the isomorphism (2.11). Here, $\xi: S^1 \rightarrow \mathfrak{t}$ is any choice of lift of γ . Then make $\text{avg } \gamma$ act on \mathbb{C}_l via l , and let $(\xi - \text{avg } \xi, z)$ act on \mathcal{S} via W . In other words, W_l is the tensor product representation of l and W . It is irreducible because l and W are.

Remark 2.5.3 (Characters of T as elements of the dual lattice of Λ). We will often use the isomorphism of \mathbb{Z} -modules

$$\text{Hom}_{\text{Lie}}(T, \text{U}(1)) \xrightarrow{\sim} \Lambda^\vee := \text{Hom}_{\text{Ab}}(\Lambda, \mathbb{Z}).$$

It is given by first differentiating a character $l: T \rightarrow \text{U}(1)$ and then restricting to $\Lambda \subseteq \mathfrak{t} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. The inverse isomorphism takes the tensor product of a homomorphism $f: \Lambda \rightarrow \mathbb{Z}$ with $\text{U}(1)$. We will use this identification to give meaning to expressions like $\langle l, l \rangle$, or $\langle l, \alpha \rangle$ if $\alpha \in \mathfrak{t}$.

Next, we equip such a representation W_l of $(\tilde{L}T)_0$ with an intertwining rotation action. Unlike what we did for \mathcal{S} , we will not let $\text{Rot}(S^1)$ itself act on \mathcal{S}_l , but a certain finite cover of $\text{Rot}(S^1)$ instead.⁶ Let m be the smallest positive integer such that $m\langle l, l \rangle \in 2\mathbb{Z}$. We define a representation R_l of $\text{Rot}^{(m)}(S^1)$ on \mathcal{S}_l through the character $[\Phi_\theta] \mapsto e^{\pi i \langle l, l \rangle \theta}$ on \mathbb{C}_l and the action R of $\text{Rot}(S^1)$ on \mathcal{S} via the covering homomorphism $\text{Rot}^{(m)}(S^1) \rightarrow \text{Rot}(S^1)$. That is, R_l is the tensor product representation of this character with R . Then, because R intertwines in the manner (2.23) with W , it can be checked that also R_l does so with W_l . To prove this one uses that if $\xi: S^1 \rightarrow \mathfrak{t}$ is a lift of $\gamma \in (LT)_0$, then $[\Phi_\theta]^* \xi$ is a lift of $[\Phi_\theta]^* \gamma$.

There is a Hilbert space isomorphism

$$f_l: \mathcal{S}_l \xrightarrow{\sim} \mathcal{S}, \quad 1 \otimes v \mapsto v,$$

which obviously intertwines the representations $W_l|_{\tilde{V}\mathfrak{t}}$ and W of $\tilde{V}\mathfrak{t}$, but does not do so for R_l and R . Instead, for $\Phi_\theta \in \text{Rot}^{(\infty)}(S^1)$ the following square

⁶See the beginning of Appendix A.3.2 for the models and corresponding notations we use for the covering groups of $\text{Rot}(S^1)$.

commutes:⁷

$$\begin{array}{ccc} \mathcal{S}_l & \xrightarrow[\sim]{f_l} & \mathcal{S} \\ R_l[\Phi_\theta] \downarrow & & \downarrow R[\Phi_\theta] \\ \mathcal{S}_l & \xrightarrow[\sim]{e^{-\pi i \langle l, l \rangle \theta} f_l} & \mathcal{S}. \end{array}$$

Therefore, f_l restricts for each $a \in (1/m)\mathbb{Z}$ on the energy eigenspace $\mathcal{S}(a)$ to an isomorphism

$$f_l|_{\mathcal{S}(a)}: \mathcal{S}_l(a) \xrightarrow{\sim} \mathcal{S}(a - \langle l, l \rangle / 2)$$

of finite-dimensional Hilbert spaces. That is, the energy of \mathcal{S}_l is shifted from that of \mathcal{S} by a fractional value. Because $\mathcal{S}(a - \langle l, l \rangle / 2) = \{0\}$ for $a < \langle l, l \rangle / 2$ and there holds $\langle l, l \rangle \geq 0$ by the positive definiteness of Λ , this shows

Proposition 2.5.4. *The intertwining $\text{Rot}^{(m)}(S^1)$ -action R_l on the representation W_l of $(\tilde{L}T)_0$ is of positive energy.*

Using the isomorphisms $f_l|_{\mathcal{S}_l(a)}$ we can even calculate the character of R_l knowing that of R :

$$\begin{aligned} \text{ch}_{R_l}(q) &= \sum_{a \in (1/m)\mathbb{Z}} \dim(\mathcal{S}_l(a)) q^a \\ &= \sum_{a \in \langle l, l \rangle / 2 + \mathbb{Z}_{\geq 0}} \dim(\mathcal{S}(a - \langle l, l \rangle / 2)) q^a \\ &= \sum_{k=0}^{\infty} \dim(\mathcal{S}(k)) q^{k + \langle l, l \rangle / 2} \\ &= q^{\langle l, l \rangle / 2} \text{ch}_R(q) \\ &= q^{\dim T / 24} q^{\langle l, l \rangle / 2} \eta(q)^{-\dim T}. \end{aligned}$$

Just as we did for the triple (W, \mathcal{S}, R) in (2.24), this result becomes nicer if we shift the energy downwards a bit:

$$Z_{W_l}(q) := q^{-\dim T / 24} \text{ch}_{R_l}(q) = q^{\langle l, l \rangle / 2} \eta(q)^{-\dim T}.$$

These representations W_l are clearly mutually non-isomorphic since this holds for their restrictions to T . What is more, all irreducible representations of $(\tilde{L}T)_0$ of the type we are concerned with are of this form. More precisely,

⁷In writing $R[\Phi_\theta]$, the notation $[\Phi_\theta]$ stands for the image of Φ_θ in $\text{Rot}(S^1)$, while $R_l[\Phi_\theta]$ in $R_l[\Phi_\theta]$ means its image in $\text{Rot}^{(m)}(S^1)$.

Theorem 2.5.5. *Every irreducible, positive energy representation of $(\tilde{L}T)_0$ such that the central subgroup $U(1)$ acts as $z \mapsto z$ is isomorphic to W_l for some character l of T .*

Proof. Let Q be such a representation on a Hilbert space \mathcal{H} and consider it as a representation of $T \times \tilde{V}\mathfrak{t}$ via the isomorphism (2.11). Because $Q|_T$ commutes with Q , $Q|_T$ is a character, say, l , by Schur's lemma. This implies that $Q|_{\tilde{V}\mathfrak{t}}$ is irreducible. A closed linear subspace of \mathcal{H} which is stable under $\tilde{V}\mathfrak{t}$ then namely is stable also under $T \times \tilde{V}\mathfrak{t}$. Furthermore, $Q|_{\tilde{V}\mathfrak{t}}$ is again of positive energy. Let R' namely be a positive energy representation of $\text{Rot}^{(n)}(S^1)$, for some n , which intertwines in the manner (A.8) with Q , now considered as a representation of $(\tilde{L}T)_0$ again. As we already noted in Section 2.3.1, the isomorphism (2.11) is $\text{Rot}(S^1)$ -equivariant. Hence R' also intertwines with $Q|_{\tilde{V}\mathfrak{t}}$. By the unicity result Theorem 2.5.2 for $\tilde{V}\mathfrak{t}$, this proves that $Q|_{\tilde{V}\mathfrak{t}}$ is isomorphic to W . \square

Having discussed these representations of $(\tilde{L}T)_0$ and $\text{Rot}^{(m)}(S^1)$ on \mathcal{S}_l , we finally demonstrate a third piece of structure on the collection of all these Hilbert spaces.

Recall from Section 2.5.1 that there is a representation U of the group $\text{Aut}(\mathfrak{t}; \langle \cdot, \cdot \rangle)$ of Lie algebra isometries on \mathcal{S} which intertwines in a certain way with W . Because we learned in Section 2.4.1 that the smaller group $\text{Aut}(\Lambda; \langle \cdot, \cdot \rangle)$ of lattice automorphisms acts on $(\tilde{L}T)_0$, one might initially expect a representation of $\text{Aut}(\Lambda; \langle \cdot, \cdot \rangle)$ on \mathcal{S}_l to exist which similarly intertwines with W_l . However, this turns out not to be the case since $\text{Aut}(\Lambda; \langle \cdot, \cdot \rangle)$ translates the parameter l also, as noted in Appendix A.1.1. The situation is instead described by

Proposition 2.5.6. *Let l be a character of T and g a lattice automorphism of Λ . Then the Hilbert space isomorphism*

$$U_l(g): \mathcal{S}_l \xrightarrow{\sim} \mathcal{S}_{g \cdot l}, \quad 1 \otimes v \mapsto 1 \otimes U(\mathbb{R}g)(v).$$

satisfies the intertwining properties

$$U_l(g)W_l(\gamma, z)U_l(g)^* = W_{g \cdot l}(g \cdot (\gamma, z)) \quad (2.26)$$

for all $(\gamma, z) \in (\tilde{L}T)_0$, and

$$U_l(g)R_l[\Phi_\theta]U_l(g)^* = R_{g \cdot l}[\Phi_\theta] \quad (2.27)$$

for all $[\Phi_\theta] \in \text{Rot}^{(m)}(S^1)$.

Note that the smallest positive integer m such that $m\langle l, l \rangle \in 2\mathbb{Z}$ is also the smallest positive integer m such that $m\langle g \cdot l, g \cdot l \rangle \in 2\mathbb{Z}$ because $\langle g \cdot l, g \cdot l \rangle = \langle l, l \rangle$. Hence, R_l and $R_{g \cdot l}$ are both representations of the same covering group $\text{Rot}^{(m)}(S^1)$. Moreover, they act by the same character $[\Phi_\theta] \mapsto e^{\pi i \langle l, l \rangle \theta} = e^{\pi i \langle g \cdot l, g \cdot l \rangle \theta}$ on the tensor factors \mathbb{C}_l and $\mathbb{C}_{g \cdot l}$ of \mathcal{S}_l and $\mathcal{S}_{g \cdot l}$, respectively.

Proof. On the one hand, there holds

$$\begin{aligned} U_l(g)W_l(\gamma, z)U_l(g)^*(1 \otimes v) &= U_l(g)W_l(\gamma, z)(1 \otimes U(\mathbb{R}g)^*(v)) \\ &= U_l(g)(l(\text{avg } \gamma) \otimes W(\xi - \text{avg } \xi, z)U(\mathbb{R}g)^*(v)) \\ &= l(\text{avg } \gamma) \otimes U(\mathbb{R}g)W(\xi - \text{avg } \xi, z)U(\mathbb{R}g)^*(v), \end{aligned}$$

and thanks to the intertwining property (2.25) of U with W we can write

$$\begin{aligned} U(\mathbb{R}g)W(\xi - \text{avg } \xi, z)U(\mathbb{R}g)^* &= W((\mathbb{R}g) \cdot (\xi - \text{avg } \xi, z)) \\ &= W((\mathbb{R}g)_* \xi - \text{avg}(\mathbb{R}g)_* \xi, z). \end{aligned}$$

On the other hand,

$$\begin{aligned} W_{g \cdot l}(g \cdot (\gamma, z))(1 \otimes v) &= W_{g \cdot l}(g_* \gamma, z)(1 \otimes v) \\ &= (g \cdot l)(\text{avg } g_* \gamma) \otimes W((\mathbb{R}g)_* \xi - \text{avg}(\mathbb{R}g)_* \xi, z), \end{aligned}$$

because if $\xi: S^1 \rightarrow \mathfrak{t}$ is a lift of $\gamma \in (\tilde{L}T)_0$, then $(\mathbb{R}g)_* \xi$ is a lift of $g_* \gamma$. Now observe that $(g \cdot l)(\text{avg } g_* \gamma) = l(\text{avg } \gamma)$. This proves (2.26).

The intertwining property (2.27) is now easily seen to be equivalent to U commuting with R . This is, in turn, a fact we already noted in Section 2.5.1. \square

2.5.3 Irreducible representations of $\tilde{L}T$

Having given $(LT)_0$ the structure of a topological group in Section 2.5.2, the full group LT now acquires by Lemma A.2.4 a unique structure of a topological group such that $(LT)_0$ is open in LT .

Proposition 2.5.7. *There exists a unique structure of a topological group on the central extension $\tilde{L}T$ such that $(\tilde{L}T)_0$ is open in $\tilde{L}T$.*

Proof. We employ Corollary A.2.5 with $G := LT$, $A := U(1)$ and $G_0 := (LT)_0$. What needs to be checked is whether for every fixed loop $\gamma \in LT$ the map $(LT)_0 \rightarrow U(1)$ given by $\rho \mapsto c(\gamma, \rho)c(\gamma + \rho, -\gamma)$ is continuous.

Because of the way the topology on $(LT)_0$ is defined (namely as a product topology) it is convenient to work with elements of LT via the right hand side of (2.5). That is, we write γ as a triple

$$([\xi_0], \xi, \Delta_\gamma) \in T \oplus V\mathfrak{t} \oplus \Lambda.$$

Here, $[\xi_0] \in T \cong \mathfrak{t}/\Lambda$ is the equivalence class of an element $\xi_0 \in \mathfrak{t}$. Similarly, we will work with a triple $([\eta_0], \eta, 0)$ in the place of ρ . Lifts $[0, 1] \rightarrow \mathfrak{t}$ of γ and ρ are then given by $\theta \mapsto \xi(\theta) + \xi_0 + \Delta_\gamma\theta$ and $\theta \mapsto \eta(\theta) + \eta_0$, respectively.

Filling in these lifts into $c(\gamma, \rho)c(\gamma + \rho, -\gamma)$ and using that $\Delta_\rho = 0$, we quickly realise that we need to verify whether the real number

$$\begin{aligned} & \frac{1}{2} \int_0^1 \langle \xi'(\theta) + \Delta_\gamma, \eta(\theta) + \eta_0 \rangle d\theta + \frac{1}{2} \langle \Delta_\gamma, \eta(0) + \eta_0 \rangle - \\ & \frac{1}{2} \int_0^1 \langle \xi'(\theta) + \eta'(\theta) + \Delta_\gamma, \xi(\theta) + \xi_0 + \Delta_\gamma\theta \rangle d\theta - \frac{1}{2} \langle \Delta_\gamma, \Delta_\gamma \rangle \end{aligned} \quad (2.28)$$

depends continuously on η and η_0 . The first term in (2.28) is equal to

$$\frac{1}{2} \int_0^1 \langle \xi'(\theta), \eta(\theta) \rangle d\theta + \frac{1}{2} \langle \Delta_\gamma, \eta_0 \rangle = S(\xi, \eta) + \frac{1}{2} \langle \Delta_\gamma, \eta_0 \rangle$$

and we already noted in Section 2.5.1 that the restriction of S to $V\mathfrak{t}$ is continuous. The third term in (2.28) can be expanded to

$$\begin{aligned} & \frac{1}{2} \int_0^1 \langle \xi'(\theta) + \Delta_\gamma, \xi(\theta) + \xi_0 + \Delta_\gamma\theta \rangle d\theta + \\ & \frac{1}{2} \int_0^1 \langle \eta'(\theta), \xi(\theta) \rangle d\theta + \frac{1}{2} \left[\langle \eta(\theta), \Delta_\gamma\theta \rangle \right]_0^1 \end{aligned} \quad (2.29)$$

using partial integration. Now note that the third term in (2.29) is equal to $\frac{1}{2} \langle \eta(0), \Delta_\gamma \rangle$ and hence cancels against the same term in (2.28). This concludes the proof. \square

With the irreducible, positive energy representations W_l of the identity component $(\tilde{L}T)_0$ in hand, we will be able to construct and classify the same class of representations of the full group $\tilde{L}T$, starting as follows. Let us take such a W_l for a character l of T and consider the induced representation

$$\text{Ind}_{(\tilde{L}T)_0}^{\tilde{L}T} W_l$$

of $\tilde{L}T$, which we will shorten to $\text{Ind } W_l$.

We refer to Appendix A.3.1 for generalities regarding induced group representations, and content ourselves here with spelling out some details of this particular one. The underlying Hilbert space of $\text{Ind } W_l$ is the completion of an algebraic direct sum indexed over the left cosets σ of $(\tilde{L}T)_0$ in $\tilde{L}T$:

$$\text{Ind}_{(\tilde{L}T)_0}^{\tilde{L}T} \mathcal{S}_l := \overline{\bigoplus_{\sigma \in \tilde{L}T/(\tilde{L}T)_0} \mathcal{S}_l^\sigma}. \quad (2.30)$$

We will abbreviate it as $\text{Ind } \mathcal{S}_l$. Here, the summand \mathcal{S}_l^σ is the Hilbert space

$$\mathcal{S}_l^\sigma := \sigma \times_{(\tilde{L}T)_0} \mathcal{S}_l.$$

A vector of \mathcal{S}_l^σ is an equivalence class $[(\gamma, z), v]$ of pairs $((\gamma, z), v)$ with $(\gamma, z) \in \sigma$ and $v \in \mathcal{S}_l$, and the relations are given by

$$((\gamma, z) \cdot (\rho, w), v) \sim ((\gamma, z), W_l(\rho, w)(v))$$

for $(\rho, w) \in (\tilde{L}T)_0$. A general vector of $\text{Ind } \mathcal{S}_l$ is an infinite tuple of vectors

$$\left([(\gamma^\sigma, z^\sigma), v^\sigma] \right)_{\sigma \in \tilde{L}T/(\tilde{L}T)_0}, \quad (2.31)$$

where $[(\gamma^\sigma, z^\sigma), v^\sigma] \in \mathcal{S}_l^\sigma$, satisfying a square-integrability condition. The action $\text{Ind } W_l$ of $\tilde{L}T$ on $\text{Ind } \mathcal{S}_l$ is given by setting for an element $(\gamma, z) \in \tilde{L}T$

$$(\text{Ind } W_l)(\gamma, z) \cdot \left([(\gamma^\sigma, z^\sigma), v^\sigma] \right)_\sigma := \left(\left[(\gamma, z) \cdot (\gamma^{(\gamma, z)^{-1}\sigma}, z^{(\gamma, z)^{-1}\sigma}), v^{(\gamma, z)^{-1}\sigma} \right] \right)_\sigma.$$

If we again let m be the smallest positive integer such that $m\langle l, l \rangle \in 2\mathbb{Z}$, then there is an action, which we denote by $\text{Ind } R_l$, of $\text{Rot}^{(m)}(S^1)$ on $\text{Ind } \mathcal{S}_l$ affecting each summand \mathcal{S}_l^σ individually. Namely, if $\Phi_\theta \in \text{Rot}^{(\infty)}(S^1)$ and $[(\gamma, z), v] \in \mathcal{S}_l^\sigma$, then we set

$$(\text{Ind } R_l)[\Phi_\theta] \cdot [(\gamma, z), v] := [[\Phi_\theta] \cdot (\gamma, z), R_l[\Phi_\theta](v)].$$

It can be checked that $\text{Ind } R_l$ is well-defined, linear, unitary and that it satisfies the intertwining property (A.8) with respect to $\text{Ind } W_l$ because R_l does so with W_l . The reason why this $\text{Rot}^{(m)}(S^1)$ -action on $\text{Ind } \mathcal{S}_l$ is the correct one to take, given the one we already defined on \mathcal{S}_l , is that it is the one which appears when we consider $\text{Ind } W_l$ as a representation induced up from the semidirect product $(\tilde{L}T)_0 \rtimes \text{Rot}^{(m)}(S^1)$ to $\tilde{L}T \rtimes \text{Rot}^{(m)}(S^1)$.

Note that the pre-image in $\tilde{L}T$ of any subgroup of LT is a normal subgroup. In particular, $(\tilde{L}T)_0$ is normal in $\tilde{L}T$, being the pre-image of the identity component $(LT)_0$ of LT . Therefore, the representations conjugate to W_l are again representations of $(\tilde{L}T)_0$. We start our study of $\text{Ind } W_l$ by calculating these conjugate representations:

Lemma 2.5.8. *Let (γ, z) be an element of $\tilde{L}T$ that is not contained in the (normal) subgroup $(\tilde{L}T)_0$ and consider the representation $W_l^{(\gamma, z)}$ of $(\tilde{L}T)_0$ conjugate to W_l , defined by*

$$W_l^{(\gamma, z)}(\rho, w) := W_l((\gamma, z)^{-1}(\rho, w)(\gamma, z))$$

for $(\rho, w) \in (\tilde{L}T)_0$. Then $W_l^{(\gamma, z)}$ is the tensor product representation of W_l and the character

$$(\tilde{L}T)_0 \rightarrow (LT)_0 \rightarrow \mathrm{U}(1), \quad (\rho, w) \mapsto \rho \mapsto c(\rho, \gamma)c(\gamma, \rho)^{-1}, \quad (2.32)$$

where $\mathrm{U}(1)$ acts on \mathbb{C} as $z \mapsto z$. In turn, for any lifts $\xi, \eta: [0, 1] \rightarrow \mathfrak{t}$ of γ and ρ , respectively, there holds

$$c(\rho, \gamma)c(\gamma, \rho)^{-1} = e^{2\pi i(S(\eta, \xi) - S(\xi, \eta))}, \quad (2.33)$$

where

$$S(\eta, \xi) - S(\xi, \eta) = - \int_0^1 \langle \xi'(\theta), \eta(\theta) \rangle d\theta. \quad (2.34)$$

Proof. Using the definition of the multiplication in $\tilde{L}T$ in terms of the cocycle c and the standing assumption that c is normalised, the conjugated element $(\gamma, z)^{-1}(\rho, w)(\gamma, z)$ can be simplified and the first claim follows easily.

Since $\rho \in (LT)_0$, we have $\Delta_\rho = 0 \in \Lambda$ and because the cocycle ε for $\tilde{\Lambda}$ is assumed to be normalised there holds $\varepsilon(\Delta_\gamma, \Delta_\rho) = \varepsilon(\Delta_\rho, \Delta_\gamma) = 1$. The equations (2.33) and (2.34) now follow from the expression (2.9) for $S(\eta, \xi) - S(\xi, \eta)$. \square

Proposition 2.5.9. *The induced representation $\mathrm{Ind} W_l$ of $\tilde{L}T$ is irreducible.*

Proof. Since W_l is irreducible and $(\tilde{L}T)_0$ is a normal subgroup of $\tilde{L}T$, it is by Mackey's irreducibility criterion Theorem A.3.12 sufficient to show that all the conjugate representations $W_l^{(\gamma, z)}$ as in Lemma 2.5.8 are not isomorphic to W_l . To do this, we will examine the restriction of $W_l^{(\gamma, z)}$ to the subgroup of $(\tilde{L}T)_0$ consisting of the elements of the form $(\rho, 1)$ where ρ is a constant loop. Because this subgroup is canonically isomorphic to T , we will denote it as such. Say that $\rho = \exp \alpha$ for some $\alpha \in \mathfrak{t}$. According to (2.34) we then have

$$S(\eta, \xi) - S(\xi, \eta) = -\langle \Delta_\gamma, \alpha \rangle.$$

So (2.32) has T acting by the character $-\Delta_\gamma$, which implies that $W_l^{(\gamma, z)}$ is letting T act by $l - \Delta_\gamma$. Because $\Delta_\gamma \neq 0$, we have $l - \Delta_\gamma \neq l$ and therefore $W_l^{(\gamma, z)}$ and W_l are not isomorphic. \square

So we constructed a countably infinite family of irreducible representations of $\tilde{L}T$; one for every element l of $\text{Hom}_{\text{Lie}}(T, \text{U}(1)) \cong \Lambda^\vee$. These are far from mutually non-isomorphic, though. It will turn out that they are partitioned into finitely many isomorphism classes. In order to prove this we will first determine the restriction of an induced representation $\text{Ind } W_l$ back to the identity component $(\tilde{L}T)_0$.

Recall that \mathcal{S}_l and $\mathcal{S}_{l'}$, where l and l' are characters of T , carry different representations of $(\tilde{L}T)_0$ but are identical as Hilbert spaces. We may therefore speak of the identity map of Hilbert spaces $\mathcal{S}_l \xrightarrow{\sim} \mathcal{S}_{l'}$, given by $v \mapsto v$. Note furthermore that $(\tilde{L}T)_0$ being a normal subgroup of $\tilde{L}T$ implies that the restriction of $\text{Ind } W_l$ to $(\tilde{L}T)_0$ restricts to each subspace \mathcal{S}_l^σ for all σ .

Theorem 2.5.10 (Restriction of $\text{Ind } W_l$ from $\tilde{L}T$ to $(\tilde{L}T)_0$). *Fix a character l of T , a lattice element $\lambda \in \Lambda$ and let σ be the (left) coset of $(\tilde{L}T)_0$ in $\tilde{L}T$ consisting of all elements (γ, z) such that γ has winding element λ . Then the composite unitary map⁸*

$$f_l^\sigma : \mathcal{S}_l^\sigma \xrightarrow{\sim} \mathcal{S}_l \xrightarrow{\sim} \mathcal{S}_{l-\lambda}, \quad [(\gamma_\lambda, 1), v] \mapsto v \mapsto v$$

intertwines the representations $(\text{Res Ind } W_l)|_{\mathcal{S}_l^\sigma}$ and $W_{l-\lambda}$ of $(\tilde{L}T)_0$ and the representations $\text{Ind } R_l|_{\mathcal{S}_l^\sigma}$ and $R_{l-\lambda}$ of $\text{Rot}^{(m)}(S^1)$.

We defined an action of $\text{Rot}^{(m)}(S^1)$ on \mathcal{S}_l and it might not be immediately clear which action is meant, for this same integer m , on $\mathcal{S}_{l-\lambda}$. We will explain this in the proof of this Theorem.

Proof. To show the first claim of the Theorem, we start by using Lemma A.3.11 to observe that the first map in the composition f_l^σ is an isomorphism from the restriction of $\text{Res Ind } W_l$ to \mathcal{S}_l^σ to the conjugate representation $W_l^{(\gamma_\lambda, 1)}$. The latter was calculated partially in Lemma 2.5.8. To obtain a more precise result we substitute the lift $\theta \mapsto \theta\lambda$ of γ_λ into (2.34), where $(\rho, w) \in (\tilde{L}T)_0$ and $\eta : S^1 \rightarrow \mathfrak{t}$ is a lift of ρ . This gives

$$S(\eta, \xi) - S(\xi, \eta) = -\langle \lambda, \text{avg } \eta \rangle$$

and therefore,

$$c(\rho, \gamma_\lambda)c(\gamma_\lambda, \rho)^{-1} = e^{-2\pi i \langle \lambda, \text{avg } \eta \rangle}.$$

⁸The loop γ_λ was defined in Section 2.1. It is the standard choice of representative of σ given by the projection on T of the path $[0, 1] \rightarrow \mathfrak{t}$, $\theta \mapsto \theta\lambda$.

The crucial observation now is that⁹

$$e^{2\pi i \langle -\lambda, \text{avg } \eta \rangle} \cdot W_l(\rho, w)(v) = W_{l-\lambda}(\rho, w)(v)$$

for all v . We conclude that the second map in the composition f_l^σ is an isomorphism from $W_l^{(\gamma_\lambda, 1)}$ to $W_{l-\lambda}$.

For the second claim of the Theorem, observe that because $\langle l, \lambda \rangle \in \mathbb{Z}$ and $\langle \lambda, \lambda \rangle \in 2\mathbb{Z}$, the smallest positive integer m such that $m\langle l, \lambda \rangle \in 2\mathbb{Z}$ is also the smallest positive integer m such that $m\langle l-\lambda, l-\lambda \rangle \in 2\mathbb{Z}$. Therefore, $\text{Rot}^{(m)}(S^1)$ acts on both \mathcal{S}_l and $\mathcal{S}_{l-\lambda}$. However, the two characters $[\Phi_\theta] \mapsto e^{\pi i \langle l, \lambda \rangle \theta}$ and $[\Phi_\theta] \mapsto e^{\pi i \langle l-\lambda, l-\lambda \rangle \theta}$ by which we defined it to act on their respective tensor factors \mathbb{C}_l and $\mathbb{C}_{l-\lambda}$ are different: they differ by the character $e^{2\pi i \langle l, \lambda \rangle \theta} e^{-\pi i \langle \lambda, \lambda \rangle \theta}$.

For $[\Phi_\theta] \in \text{Rot}^{(m)}(S^1)$ and $[(\gamma_\lambda, 1), v] \in \mathcal{S}_l^\sigma$ we have

$$\begin{aligned} f_l^\sigma \left((\text{Ind } R_l)[\Phi_\theta] \cdot [(\gamma_\lambda, 1), v] \right) &= f_l^\sigma \left[[\Phi_\theta] \cdot (\gamma_\lambda, 1), R_l[\Phi_\theta](v) \right] \\ &= f_l^\sigma \left[([\Phi_\theta]^* \gamma_\lambda, d([\Phi_\theta], \gamma_\lambda)), R_l[\Phi_\theta](v) \right]. \end{aligned} \tag{2.35}$$

Because by definition $\gamma_\lambda(\theta') = \exp(\lambda\theta')$ for all $\theta' \in [0, 1]$, there holds $[\Phi_\theta]^* \gamma_\lambda = \gamma_\lambda + \exp(-\lambda\theta)$. That is, $[\Phi_\theta]$ shifts γ_λ by the constant loop $\exp(-\lambda\theta) \in T$. Furthermore, we can calculate from the definition of d in (2.19) that $d([\Phi_\theta], \gamma_\lambda) = e^{-\pi i \langle \lambda, \lambda \rangle \theta}$. Hence,

$$([\Phi_\theta]^* \gamma_\lambda, d([\Phi_\theta], \gamma_\lambda)) = (\gamma_\lambda + \exp(-\lambda\theta), e^{-\pi i \langle \lambda, \lambda \rangle \theta}).$$

In turn, because we can compute that $c(\gamma_\lambda, \exp(-\lambda\theta)) = e^{-2\pi i \langle \lambda, \lambda \rangle \theta}$, the above can be written as a product

$$(\gamma_\lambda + \exp(-\lambda\theta), e^{-\pi i \langle \lambda, \lambda \rangle \theta}) = (\gamma_\lambda, 1) \cdot (\exp(-\lambda\theta), e^{\pi i \langle \lambda, \lambda \rangle \theta}).$$

Filling this back into (2.35) gives

$$\begin{aligned} f_l^\sigma \left((\text{Ind } R_l)[\Phi_\theta] \cdot [(\gamma_\lambda, 1), v] \right) &= f_l^\sigma \left[(\gamma_\lambda, 1) \cdot (\exp(-\lambda\theta), e^{\pi i \langle \lambda, \lambda \rangle \theta}), R_l[\Phi_\theta](v) \right] \\ &= f_l^\sigma \left[(\gamma_\lambda, 1), W_l(\exp(-\lambda\theta), e^{\pi i \langle \lambda, \lambda \rangle \theta}) \cdot R_l[\Phi_\theta](v) \right] \\ &= W_l(\exp(-\lambda\theta), e^{\pi i \langle \lambda, \lambda \rangle \theta}) \cdot R_l[\Phi_\theta](v). \end{aligned}$$

⁹On a pedantic note: on the left hand side of this equation we are considering v as a vector in \mathcal{S}_l , while on the right hand side we see it as lying in $\mathcal{S}_{l-\lambda}$ again.

Now on the one hand, $W_l(\exp(-\lambda\theta), e^{\pi i\langle\lambda,\lambda\rangle\theta})$ acts as multiplication by the scalar

$$e^{-2\pi i\langle l,\lambda\rangle\theta} e^{\pi i\langle\lambda,\lambda\rangle\theta},$$

while, on the other hand, our earlier argument in this proof claimed that¹⁰

$$R_l[\Phi_\theta](v) = e^{2\pi i\langle l,\lambda\rangle\theta} e^{-\pi i\langle\lambda,\lambda\rangle\theta} R_{l-\lambda}[\Phi_\theta](v).$$

We therefore see that all scalar factors cancel against each other and we conclude that

$$f_l^\sigma\left((\text{Ind } R_l)[\Phi_\theta] \cdot [(\gamma_\lambda, 1), v]\right) = R_{l-\lambda}[\Phi_\theta](v). \quad \square$$

Summarising, Theorem 2.5.10, together with the winding element isomorphism $\tilde{L}T/(\tilde{L}T)_0 \xrightarrow{\sim} \Lambda$ describes how $\text{Ind } W_l$ combined with the intertwining $\text{Rot}^{(m)}(S^1)$ -action breaks up into irreducible subrepresentations after restriction to $(\tilde{L}T)_0 \rtimes \text{Rot}^{(m)}(S^1)$. We namely have a unitary isomorphism

$$\bigoplus_{\lambda \in \Lambda} f_l^{\sigma_\lambda} : \text{Res}_{\tilde{L}T}^{(\tilde{L}T)_0} \text{Ind}_{(\tilde{L}T)_0}^{\tilde{L}T} W_l \xrightarrow{\sim} \bigoplus_{\lambda \in \Lambda} W_{l-\lambda}$$

of representations¹¹ of $(\tilde{L}T)_0 \rtimes \text{Rot}^{(m)}(S^1)$, where σ_λ is the coset associated to λ as in the statement of Theorem 2.5.10.

It is now purely formal to prove

Theorem 2.5.11. *Two representations $\text{Ind } W_l$ and $\text{Ind } W_{l'}$ of $\tilde{L}T$, where l and l' are characters of T , are (unitarily) isomorphic if and only if $l' = l - \lambda$ for some $\lambda \in \Lambda$.*

Proof. For the ‘if’ claim, write σ for the coset of $(\tilde{L}T)_0$ in $\tilde{L}T$ corresponding to λ and let us postcompose the unitary $(\tilde{L}T)_0$ -intertwiner $(f_l^\sigma)^{-1}$ from Theorem 2.5.10 with the isometric $(\tilde{L}T)_0$ -intertwining inclusion $\mathcal{S}_l^\sigma \hookrightarrow \text{Ind } \mathcal{S}_l$. By imitating the proof of Proposition A.3.8 it can be shown that $\text{Ind } \mathcal{S}_l$ is large enough for this composition to lift to a unique isometric $\tilde{L}T$ -intertwiner from $\text{Ind } \mathcal{S}_{l-\lambda}$ to $\text{Ind } \mathcal{S}_l$. It is obviously non-zero and must therefore be a unitary isomorphism by Schur’s lemma after we learned in Proposition 2.5.9 that $\text{Ind } W_{l-\lambda}$ and $\text{Ind } W_l$ are irreducible.

Conversely, assume that for two characters l and l' of T there is an isomorphism $f : \text{Ind } \mathcal{S}_{l'} \xrightarrow{\sim} \text{Ind } \mathcal{S}_l$. Let $\sigma_0 := (\tilde{L}T)_0$ —in other words: the coset in $\tilde{L}T$ consisting of all elements (ρ, w) for which ρ has winding

¹⁰See Footnote 9 for a similar remark on notation.

¹¹As usual, this direct sum of representations has by definition as underlying Hilbert space the completion of an algebraic direct sum of Hilbert spaces.

element $0 \in \Lambda$. Then the restriction $f|_{\mathcal{S}_{l'}^{\sigma_0}} : \mathcal{S}_{l'}^{\sigma_0} \hookrightarrow \text{Ind } \mathcal{S}_l$ is a non-zero $(\tilde{L}T)_0$ -intertwiner. Because we learned from Theorem 2.5.10 that all the subspaces \mathcal{S}_l^σ of $\text{Ind } \mathcal{S}_l$ for different cosets σ are mutually non-isomorphic and irreducible as representations of $(\tilde{L}T)_0$, by Schur's lemma we must have an isomorphism $f|_{\mathcal{S}_{l'}^{\sigma_0}} : \mathcal{S}_{l'}^{\sigma_0} \xrightarrow{\sim} \mathcal{S}_l^\sigma$ for some coset σ . Say that σ consists of the elements (γ, z) for which γ has winding element some $\lambda \in \Lambda$. What is more, we learned that $\mathcal{S}_{l'}^{\sigma_0}$ and \mathcal{S}_l^σ are isomorphic to $\mathcal{S}_{l'}$ and $\mathcal{S}_{l-\lambda}$, respectively. Restriction to the subgroup T of $(\tilde{L}T)_0$ now shows that $l' = l - \lambda$. \square

Because $\text{Ind } R_l$ preserves each of the subspaces $\mathcal{S}_l^\sigma \subseteq \text{Ind } \mathcal{S}_l$, it makes sense to talk about their energy eigenspaces $\mathcal{S}_l^\sigma(a)$ for $a \in (1/m)\mathbb{Z}$. They sum to the energy eigenspaces of $\text{Ind } \mathcal{S}_l$ (see (A.10)):

$$(\text{Ind } \mathcal{S}_l)(a) = \overline{\bigoplus_{\sigma \in \tilde{L}T/(\tilde{L}T)_0} \mathcal{S}_l^\sigma(a)}. \quad (2.36)$$

We are now able to calculate the character of $\text{Ind } R_l$. Let namely a fixed element $\lambda \in \Lambda$ and its corresponding coset $\sigma \subseteq \tilde{L}T$ be as in Theorem 2.5.10. Then that Theorem gives us the character of the restriction of $\text{Ind } R_l$ to \mathcal{S}_l^σ because we learned about that of $R_{l-\lambda}$ already in Section 2.5.2:¹²

$$\text{ch}_{\mathcal{S}_l^\sigma}(q) = \text{ch}_{R_{l-\lambda}}(q) = q^{\text{rank } \Lambda/24} q^{(l-\lambda, l-\lambda)/2} \eta(q)^{-\text{rank } \Lambda},$$

since $\text{rank } \Lambda = \dim T$. Next, we allow our fixed λ and so also its corresponding coset σ to vary in order to calculate the character of $\text{Ind } R_l$ on all of $\text{Ind } \mathcal{S}_l$. Using the decomposition (2.36), we get

$$\begin{aligned} \text{ch}_{\text{Ind } R_l}(q) &= \sum_{\lambda \in \Lambda} \text{ch}_{\mathcal{S}_l^\sigma}(q) \\ &= q^{\text{rank } \Lambda/24} \sum_{\lambda \in \Lambda} q^{(l-\lambda, l-\lambda)/2} \eta(q)^{-\text{rank } \Lambda} \\ &= q^{\text{rank } \Lambda/24} \theta_{l+\Lambda}(q) \cdot \eta(q)^{-\text{rank } \Lambda}, \end{aligned}$$

where $\theta_{l+\Lambda}(q)$ is the *theta series* of the translated lattice $l + \Lambda$ that is the coset of l in Λ^\vee (see Appendix A.1.4). This also proves

Proposition 2.5.12. *The intertwining $\text{Rot}^{(m)}(S^1)$ -action $\text{Ind } R_l$ on the representation $\text{Ind } W_l$ of $\tilde{L}T$ is of positive energy.*

¹²In the first following equation we put a Hilbert space as a subscript in $\text{ch}_{\mathcal{S}_l^\sigma}$, contrary to the usual notation, to emphasise that we are only considering the restriction of $\text{Ind } R_l$ to \mathcal{S}_l^σ .

Finally, it is again appropriate to apply a small energy correction by defining

$$Z_{\text{Ind } W_l}(q) := q^{-\text{rank } \Lambda / 24} \text{ch}_{\text{Ind } R_l}(q) = \theta_{l+\Lambda}(q) \cdot \eta(q)^{-\text{rank } \Lambda}.$$

Upon substituting q by $e^{2\pi iz}$, for $z \in \mathbb{C}$ with $\text{Im } z > 0$, this is a meromorphic modular form of weight 0 for some congruence subgroup of $\text{SL}(2, \mathbb{Z})$ (the full group $\text{SL}(2, \mathbb{Z})$ if Λ is unimodular) [MT10, Exercise 8.12].

Theorem 2.5.13. *Every irreducible, positive energy representation of $\tilde{L}T$ such that the central subgroup $U(1)$ acts as $z \mapsto z$ is (unitarily) isomorphic to $\text{Ind } W_l$ for some character l of T .*

Proof. Let Q be such a representation on a Hilbert space \mathcal{H} and restrict it to $(\tilde{L}T)_0$. Combining Proposition A.3.18 and Lemma A.3.17 tells us that $Q|_{(\tilde{L}T)_0}$ contains an irreducible, positive energy subrepresentation of $(\tilde{L}T)_0$. By the classification result Theorem 2.5.5 the latter must be isomorphic to W_l for some character l of T . Suppose we can show that the $(\tilde{L}T)_0$ -intertwining inclusion $i: W_l \hookrightarrow Q|_{(\tilde{L}T)_0}$ lifts to a unique morphism $\text{Ind } W_l \rightarrow Q$ of $\tilde{L}T$ -representations. Because we learned in Proposition 2.5.9 that $\text{Ind } W_l$ is irreducible and we assumed the same for Q , this morphism is then an isomorphism by Schur's lemma.

As explained in Appendix A.3.1, the question of whether the lift of i to $\text{Ind } W_l$ exists comes down to asking whether for every vector of the form (2.31) of $\text{Ind } \mathcal{A}_l$ the series

$$\sum_{\sigma \in \tilde{L}T/(\tilde{L}T)_0} Q((\gamma^\sigma, z^\sigma)) i(v^\sigma)$$

converges in \mathcal{H} . Note that the representation $Q|_{(\tilde{L}T)_0}$ on the subspace

$$Q((\gamma^\sigma, z^\sigma))(i(\mathcal{A}_l))$$

of \mathcal{H} is isomorphic to the conjugate representation $W_l^{(\gamma^\sigma, z^\sigma)}$ of $(\tilde{L}T)_0$. We learned in Theorem 2.5.10 that for different cosets σ these conjugate representations are mutually non-isomorphic and hence orthogonal. The squared norm of the above series is therefore equal to $\sum_\sigma \|v^\sigma\|_{\mathcal{A}_l}^2$ and this converges by assumption. \square

We conclude from Theorems 2.5.11 and 2.5.13 that $\tilde{L}T$ possesses only finitely many irreducible, positive energy representations up to isomorphism if we fix the character by which $U(1)$ acts. The isomorphism classes are labelled by the elements of the finite abelian *discriminant group* $D_\Lambda := \Lambda^\vee / \Lambda$ of the

lattice Λ . There is exactly one isomorphism class, represented by $\text{Ind } W_0$, if and only if Λ is *unimodular*.

Moreover, Proposition A.3.19 tells us that arbitrary positive energy representations of $\tilde{L}T$ are now understood as well. They are the direct sums of the irreducible ones we constructed in this section.

Remark 2.5.14 (The representation theory in the case of an odd lattice). If Λ is an odd lattice, the construction of the representations $\text{Ind } W_l$ is identical to the one above in the case that Λ is even. Recall, however, from Section 2.3.1 and Remark 2.3.2 that if Λ is odd, then $\text{Diff}_+(S^1)$ does act on $(\tilde{L}T)_0$ because the cocycle defining this central extension is preserved by $\text{Diff}_+(S^1)$, but only $\text{Diff}_+^{(2)}(S^1)$ acts on $\tilde{L}T$. In particular, there is only a $\text{Rot}^{(2)}(S^1)$ -action on $\tilde{L}T$ which intertwines with a $\text{Rot}^{(m)}(S^1)$ -action $\text{Ind } R_l$ on the Hilbert space $\text{Ind } \mathcal{S}_l$. Here, m is the smallest integer such that both $m\langle l, l \rangle \in 2\mathbb{Z}$ and $m \geq 2$. The calculation of its graded character and its outcome is identical to the case in which Λ is even, which therefore shows that this $\text{Rot}^{(m)}(S^1)$ -action $\text{Ind } R_l$ is of positive energy. Unlike in the odd case, $\text{ch}_{\text{Ind } R_0}(q)$ is now a series in half-integral powers of q .

The grading (2.30) on $\text{Ind } \mathcal{S}_l$ over the cosets σ of $(\tilde{L}T)_0$ in $\tilde{L}T$ now refines to a further $\mathbb{Z}/2\mathbb{Z}$ -grading

$$\text{Ind } \mathcal{S}_l = \overline{\bigoplus_{\sigma \text{ even}} \mathcal{S}_l^\sigma} \oplus \overline{\bigoplus_{\sigma \text{ odd}} \mathcal{S}_l^\sigma},$$

where we call a coset σ *even* if $\sigma \subseteq \tilde{L}T(0)$ and *odd* if $\sigma \subseteq \tilde{L}T(1)$. (See Remark 2.2.4 for the definitions of the $\tilde{L}T(i)$.) Every coset is either even or odd. We denote the two subspaces in the above by $(\text{Ind } \mathcal{S}_l)_0$ and $(\text{Ind } \mathcal{S}_l)_1$ and call their vectors *even* and *odd*, respectively. By the definition of $\text{Ind } W_l$ it is clear that even elements of $\tilde{L}T$ preserve the parity of vectors, while odd ones reverse them.

We close this chapter by lifting the structure on the Hilbert spaces \mathcal{S}_l described in Proposition 2.5.6 to the induced spaces $\text{Ind } \mathcal{S}_l$.

The winding element isomorphism $\tilde{L}T/(\tilde{L}T)_0 \xrightarrow{\sim} \Lambda$ implies that the group $\text{Aut}(\Lambda; \langle \cdot, \cdot \rangle)$ of automorphisms of the lattice Λ acts on $\tilde{L}T/(\tilde{L}T)_0$ also. The translate $g\sigma$ of a coset σ by an element $g \in \text{Aut}(\Lambda; \langle \cdot, \cdot \rangle)$ consists, by definition, of all elements (γ, z) such that γ has winding element $g\lambda$ if σ corresponds similarly to λ .

Proposition 2.5.15. (Compare with [Shi06, Lemma 1.5(2)].) *Let l be a character of T and $\tilde{g} \in \text{Aut}(\tilde{\Lambda}^\varepsilon; \langle \cdot, \cdot \rangle)$. Then the function*

$$(\text{Ind } U_l)(\tilde{g}): \text{Ind } \mathcal{S}_l \xrightarrow{\sim} \text{Ind } \mathcal{S}_{g \cdot l}$$

given by

$$(\text{Ind } U_l)(\tilde{g}) \cdot \left([(\gamma^\sigma, z^\sigma), v^\sigma] \right)_\sigma := \left(\left[\tilde{g} \cdot (\gamma^{g^{-1}\sigma}, z^{g^{-1}\sigma}), U_l(g)(v^{g^{-1}\sigma}) \right] \right)_\sigma \quad (2.37)$$

is well-defined, linear, unitary and satisfies the intertwining properties

$$(\text{Ind } U_l)(\tilde{g})(\text{Ind } W_l)(\gamma, z)(\text{Ind } U_l)(\tilde{g})^* = (\text{Ind } W_{g \cdot l})(\tilde{g} \cdot (\gamma, z)) \quad (2.38)$$

for all $(\gamma, z) \in \tilde{L}T$, and

$$(\text{Ind } U_l)(\tilde{g})(\text{Ind } R_l)[\Phi_\theta](\text{Ind } U_l)(\tilde{g})^* = (\text{Ind } R_{g \cdot l})[\Phi_\theta] \quad (2.39)$$

for all $[\Phi_\theta] \in \text{Rot}^{(m)}(S^1)$.

We refer to the proof of Proposition 2.5.6 for an explanation of why $\text{Ind } R_{g \cdot l}$ is a representation of the same covering group $\text{Rot}^{(m)}(S^1)$ as $\text{Ind } R_l$ is.

Proof. Verifying well-definedness comes down to checking whether, if we choose for every coset σ of $(\tilde{L}T)_0$ in $\tilde{L}T$ an arbitrary element $(\rho^\sigma, w^\sigma) \in (\tilde{L}T)_0$, implying that

$$\left([(\gamma^\sigma, z^\sigma), v^\sigma] \right)_\sigma = \left([(\gamma^\sigma, z^\sigma) \cdot (\rho^\sigma, w^\sigma)^{-1}, W_l(\rho^\sigma, w^\sigma)(v^\sigma)] \right)_\sigma,$$

then the left hand side of (2.37) equals

$$(\text{Ind } U_l)(\tilde{g}) \cdot \left([(\gamma^\sigma, z^\sigma) \cdot (\rho^\sigma, w^\sigma)^{-1}, W_l(\rho^\sigma, w^\sigma)(v^\sigma)] \right)_\sigma. \quad (2.40)$$

If we fill in the definition of $(\text{Ind } U_l)(\tilde{g})$ in (2.40) this gives

$$\begin{aligned} & \left(\left[\tilde{g} \cdot (\gamma^{g^{-1}\sigma}, z^{g^{-1}\sigma}) \cdot g \cdot (\rho^{g^{-1}\sigma}, w^{g^{-1}\sigma})^{-1}, U_l(g)W_l(\rho^{g^{-1}\sigma}, w^{g^{-1}\sigma})(v^{g^{-1}\sigma}) \right] \right)_\sigma \\ &= \left(\left[\tilde{g} \cdot (\gamma^{g^{-1}\sigma}, z^{g^{-1}\sigma}), \right. \right. \\ & \quad \left. \left. W_{g \cdot l}(g \cdot (\rho^{g^{-1}\sigma}, w^{g^{-1}\sigma}))^* U_l(g)W_l(\rho^{g^{-1}\sigma}, w^{g^{-1}\sigma})(v^{g^{-1}\sigma}) \right] \right)_\sigma, \end{aligned}$$

and there indeed holds

$$W_{g \cdot l}(g \cdot (\rho^{g^{-1}\sigma}, w^{g^{-1}\sigma}))^* U_l(g)W_l(\rho^{g^{-1}\sigma}, w^{g^{-1}\sigma}) = U_l(g)$$

thanks to the intertwining property (2.26) of U_l , which settles the question.

Showing linearity and unitarity can similarly be reduced to an application of the intertwining property of U_l . We therefore turn to proving the intertwining relation (2.38) for $\text{Ind } U_l$.

Applying the left hand side of (2.38) to a vector $([(\gamma^\sigma, z^\sigma), \nu^\sigma])_\sigma \in \text{Ind } \mathcal{S}_l$ gives

$$\begin{aligned}
& (\text{Ind } U_l)(\tilde{g})(\text{Ind } W_l)(\gamma, z) \cdot \left([\tilde{g}^{-1} \cdot (\gamma^{g\sigma}, z^{g\sigma}), U_l(g)^*(\nu^{g\sigma})] \right)_\sigma \\
&= (\text{Ind } U_l)(\tilde{g}) \cdot \left([(\gamma, z) \cdot \tilde{g}^{-1} \cdot (\gamma^{g(\gamma, z)^{-1}\sigma}, z^{g(\gamma, z)^{-1}\sigma}), U_l(g)^*(\nu^{g(\gamma, z)^{-1}\sigma})] \right)_\sigma \\
&= \left([\tilde{g} \cdot (\gamma, z) \cdot \tilde{g}^{-1} \cdot (\gamma^{g(\gamma, z)^{-1}g^{-1}\sigma}, z^{g(\gamma, z)^{-1}g^{-1}\sigma}), \nu^{g(\gamma, z)^{-1}g^{-1}\sigma}] \right)_\sigma \\
&= \left([(\tilde{g} \cdot (\gamma, z)) \cdot (\gamma^{g(\gamma, z)^{-1}g^{-1}\sigma}, z^{g(\gamma, z)^{-1}g^{-1}\sigma}), \nu^{g(\gamma, z)^{-1}g^{-1}\sigma}] \right)_\sigma.
\end{aligned}$$

The crucial insight is now that the coset $g(\gamma, z)^{-1}g^{-1}\sigma$ of $(\tilde{L}T)_0$ in $\tilde{L}T$ can equivalently be written as $(\tilde{g} \cdot (\gamma, z))^{-1}\sigma$. If σ is namely the coset associated to a lattice element $\lambda \in \Lambda$, then $g(\gamma, z)^{-1}g^{-1}\sigma$ is associated to $g(g^{-1}\lambda - \Delta_\gamma)$, which can be simplified to $\lambda - g\Delta_\gamma$. And, indeed, the image of $(\tilde{g} \cdot (\gamma, z))^{-1}$ in LT has winding element $-g\Delta_\gamma$.

The intertwining property (2.39) has an analogous, but easier proof which uses Proposition 2.4.1 and the fact stated in Proposition 2.5.6 that U_l intertwines R_l and $R_{g \cdot l}$. \square

This result implies in particular that $\text{Aut}(\tilde{\Lambda}^\varepsilon; \langle \cdot, \cdot \rangle)$ acts on the Hilbert space $\text{Ind } \mathcal{S}_0$, intertwining with the representation $\text{Ind } W_0$ in the same way as $\text{Ind } R_0$ and commuting with $\text{Ind } R_0$. This has been proved at the beginning of [DX06, Section 4] for a projective action of the larger group $\text{Diff}_+(S^1)$ instead of only for the honest action of $\text{Rot}(S^1)$ that we studied here.

Chapter 3

Bicoloured torus loop groups

In this Chapter we introduce and study our new notion of a bicoloured torus loop group. Since this is a generalisation of a unicoloured torus loop group as studied in Chapter 2, the format of this Chapter will largely mirror that of the previous one. We assume that the reader is familiar with the notations from Section 1.5 for the next two paragraphs.

We start by laying out the structure of a bicoloured torus loop group $L(T_\circ, H, T_\bullet)$ in Section 3.1 and comparing it to various related unicoloured torus loop groups. We do not regard it as a topological group until Section 3.4, but we will nevertheless speak freely in advance of its connected components. Two differences between that section and Section 2.1 are that we introduce another group $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ as an auxiliary tool to analyse the group of actual interest $L(T_\circ, H, T_\bullet)$, and that certain constructions, such as our description of an action of a covering group of $\text{Diff}_+(S^1)$ on $L(T_\circ, H, T_\bullet)$, are much more involved than in the unicoloured situation. All this material does not yet require Λ_\circ , Λ_\bullet and Γ to be endowed with bi-additive forms.

Next, we construct in Section 3.2 from the data of the lattices Λ_\circ , Λ_\bullet and Γ a $U(1)$ -central extension $\tilde{L}(T_\circ, H, T_\bullet)$ of $L(T_\circ, H, T_\bullet)$ which we show to be disjoint-commutative. We also again relate it to the central extensions of unicoloured torus loop groups defined in Chapter 2. The action of a covering group of $\text{Diff}_+(S^1)$ on $L(T_\circ, H, T_\bullet)$ is proved to lift to $\tilde{L}(T_\circ, H, T_\bullet)$ in Section 3.3, opening the door to the study of its positive energy representations. The irreducible such representations are finally classified and constructed in Section 3.4.

3.1 Bicoloured torus loop groups and their structure

Consider the following setup. Let

$$\Lambda_\circ \xleftrightarrow{\pi_\circ} \Gamma \xleftrightarrow{\pi_\bullet} \Lambda_\bullet \quad (3.1)$$

be a *span* of free \mathbb{Z} -modules, by which we mean that Λ_\circ , Λ_\bullet and Γ share the same finite rank, and that the $\pi_{\circ/\bullet}$ are injective module homomorphisms.¹ Write $T_{\circ/\bullet} := \Lambda_{\circ/\bullet} \otimes_{\mathbb{Z}} \mathrm{U}(1)$ and $H := \Gamma \otimes_{\mathbb{Z}} \mathrm{U}(1)$ for the three associated tori and $\mathfrak{t}_{\circ/\bullet} := \Lambda_{\circ/\bullet} \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathfrak{h} := \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ for their respective Lie algebras. The homomorphisms $\pi_{\circ/\bullet}$ induce surjective Lie group homomorphisms $\mathrm{U}(1)\pi_{\circ/\bullet}: H \twoheadrightarrow T_{\circ/\bullet}$ and Lie algebra isomorphisms $\mathbb{R}\pi_{\circ/\bullet}: \mathfrak{h} \xrightarrow{\sim} \mathfrak{t}_{\circ/\bullet}$. Denote the closed left half of S^1 by ζ , the closed right half by η and their intersection $\{i, -i\}$ as \cdot . From now on we will call the points i and $-i$, p and q respectively.²

We record the following notations for later use:³

$$\begin{aligned} \Lambda_\circ \cap \Lambda_\bullet &:= (\mathbb{R}\pi_\circ)^{-1}(\Lambda_\circ) \cap (\mathbb{R}\pi_\bullet)^{-1}(\Lambda_\bullet) \\ \Lambda_\circ - \Lambda_\bullet &:= (\mathbb{R}\pi_\circ)^{-1}(\Lambda_\circ) - (\mathbb{R}\pi_\bullet)^{-1}(\Lambda_\bullet) \\ &:= \{(\mathbb{R}\pi_\circ)^{-1}(\lambda_\circ) - (\mathbb{R}\pi_\bullet)^{-1}(\lambda_\bullet) \in \mathfrak{h} \mid \lambda_\circ \in \Lambda_\circ, \lambda_\bullet \in \Lambda_\bullet\}. \end{aligned}$$

These are both free \mathbb{Z} -submodules of \mathfrak{h} , of the same rank as $\Lambda_{\circ/\bullet}$ and Γ . There are inclusions

$$\Gamma \subseteq \Lambda_\circ \cap \Lambda_\bullet \quad \begin{array}{c} \supseteq \\ \subseteq \end{array} \quad \begin{array}{c} (\mathbb{R}\pi_\circ)^{-1}(\Lambda_\circ) \\ \supseteq \\ (\mathbb{R}\pi_\bullet)^{-1}(\Lambda_\bullet) \end{array} \quad \begin{array}{c} \supseteq \\ \subseteq \end{array} \quad \Lambda_\circ - \Lambda_\bullet \hookrightarrow \Gamma^\vee.$$

Associated to the span (3.1), we define n to be the smallest positive integer such that $n(\Lambda_\circ - \Lambda_\bullet) \subseteq \Gamma$.

Definition 3.1.1. The *bicoloured torus loop group* $L(T_\circ, H, T_\bullet)$ associated to the quintuple

$$(\Lambda_\circ, \Gamma, \Lambda_\bullet, \pi_\circ, \pi_\bullet)$$

¹A statement containing the subscript \circ/\bullet should be read as being valid when this symbol is replaced everywhere in the sentence with either \circ or \bullet .

²This alludes to the fact that the particular cutting of S^1 into two arcs we use is not important.

³The reason why we write $\Lambda_\circ - \Lambda_\bullet$ instead of $\Lambda_\circ + \Lambda_\bullet$ is that this notation suggests the correct way to map $\Lambda_\circ \oplus \Lambda_\bullet$ into $\Lambda_\circ - \Lambda_\bullet$, as we will see in Proposition 3.1.9.

is the abelian group of all triples $(\gamma_\circ, \gamma_m, \gamma_\bullet)$, where $\gamma_\circ: \mathbb{C} \rightarrow T_\circ$ and $\gamma_\bullet: \mathbb{C} \rightarrow T_\bullet$ are smooth maps and $\gamma_m: \mathbb{C} \rightarrow H$ is a function⁴ such that the following diagram commutes:

$$\begin{array}{ccccc}
 \mathbb{C} & \longleftarrow & \mathbb{C} & \longrightarrow & \mathbb{C} \\
 \gamma_\circ \downarrow & & \downarrow \gamma_m & & \downarrow \gamma_\bullet \\
 T_\circ & \xleftarrow{U(1)\pi_\circ} & H & \xrightarrow{U(1)\pi_\bullet} & T_\bullet
 \end{array} \tag{3.2}$$

and such that for all $k \geq 1$ the left and right derivatives of γ_\circ and γ_\bullet respectively at p and q agree:

$$\begin{aligned}
 (\mathbb{R}\pi_\circ)^{-1}(\gamma_\circ^{(k)}(p)) &= (\mathbb{R}\pi_\bullet)^{-1}(\gamma_\bullet^{(k)}(p)), \\
 (\mathbb{R}\pi_\circ)^{-1}(\gamma_\circ^{(k)}(q)) &= (\mathbb{R}\pi_\bullet)^{-1}(\gamma_\bullet^{(k)}(q)).
 \end{aligned} \tag{3.3}$$

(In this notation we silently identify tangent spaces of the tori $T_{\circ/\bullet}$ with the Lie algebras $\mathfrak{t}_{\circ/\bullet}$ via the translation homomorphisms, so that we can say that $\gamma_{\circ/\bullet}^{(k)}(p), \gamma_{\circ/\bullet}^{(k)}(q) \in \mathfrak{t}_{\circ/\bullet}$.) Such triples $(\gamma_\circ, \gamma_m, \gamma_\bullet)$ are called *bicoloured (torus) loops*.

In order to reduce clutter, the notation $L(T_\circ, H, T_\bullet)$ does not mention the homomorphisms $\pi_{\circ/\bullet}$, although they are part of the data needed to construct the group.

Paralleling the above terminology, we will often refer to the elements of an ordinary torus loop group LT as *unicoloured (torus) loops*.

3.1.1 Bicoloured loops as unicoloured loops with a discontinuity

We now offer an alternative point of view on the group $L(T_\circ, H, T_\bullet)$. Let $\gamma = (\gamma_\circ, \gamma_m, \gamma_\bullet)$ be a bicoloured loop. There is a unique lift $\hat{\gamma}_{\circ/\bullet}$ to H of $\gamma_{\circ/\bullet}$ along $U(1)\pi_{\circ/\bullet}$ such that $\hat{\gamma}_{\circ/\bullet}(p) = \gamma_m(p)$. Glue these two lifts together at p by defining a smooth map $\text{Pth}(\gamma): [0, 1] \rightarrow H$ as

$$\text{Pth}(\gamma)(\theta) := \begin{cases} \hat{\gamma}_\circ(\theta) & \text{if } \theta \in [1/2, 1], \\ \hat{\gamma}_\bullet(\theta) & \text{if } \theta \in [0, 1/2], \end{cases}$$

where we used the unit speed parametrisations of \mathfrak{C} and \mathfrak{C} by $[0, 1/2]$ and $[1/2, 1]$ respectively. Then, because $\text{Pth}(\gamma)(1)$ and $\text{Pth}(\gamma)(0)$ differ from $\gamma_m(q)$

⁴The subscript ‘m’ stands for ‘matching datum’ or ‘middle’.

by an element of the subgroups $(\mathbb{R}\pi_\circ)^{-1}(\Lambda_\circ)/\Gamma$ and $(\mathbb{R}\pi_\bullet)^{-1}(\Lambda_\bullet)/\Gamma$ of H respectively, $\text{Pth}(\gamma)$ lies in the following group of paths on H :

$$P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma) := \left\{ \gamma \in C^\infty([0, 1], H) \left| \begin{array}{l} \gamma(1) - \gamma(0) \in \frac{\Lambda_\circ - \Lambda_\bullet}{\Gamma}, \\ \gamma^{(k)}(1) = \gamma^{(k)}(0) \text{ for all } k \geq 1 \end{array} \right. \right\}. \quad (3.4)$$

This gives a homomorphism

$$\text{Pth}: L(T_\circ, H, T_\bullet) \rightarrow P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma), \quad \gamma \mapsto \text{Pth}(\gamma)$$

of abelian groups. It is surjective because if, conversely, a path γ lies in (3.4), then $(\gamma_\circ, \gamma_m, \gamma_\bullet)$ is in $L(T_\circ, H, T_\bullet)$, where γ_\circ is defined as the composition $U(1)\pi_\circ \circ \gamma|_{[1/2, 1]}$, γ_\bullet is $U(1)\pi_\bullet \circ \gamma|_{[0, 1/2]}$, $\gamma_m(p) := \gamma(1/2)$ and $\gamma_m(q)$ is an arbitrary element constructed as follows. Choose a decomposition $\gamma(1) - \gamma(0) \equiv \lambda_\circ - \lambda_\bullet \pmod{\Gamma}$ for some $\lambda_{\circ/\bullet} \in (\mathbb{R}\pi_{\circ/\bullet})^{-1}(\Lambda_{\circ/\bullet}) \subseteq \mathfrak{h}$. Then $\gamma(1) - \lambda_\circ \equiv \gamma(0) - \lambda_\bullet \pmod{\Gamma}$, so setting $\gamma_m(q) := \gamma(1) - \lambda_\circ \pmod{\Gamma}$ makes γ_m fit into a commutative diagram (3.2).

Note that the datum $\gamma_m(q)$ is not used by Pth and, moreover, that this is the only information about γ that is thrown away. Said more precisely, the kernel of Pth is the finite abelian group consisting of triples $(\gamma_\circ, \gamma_m, \gamma_\bullet)$ where the $\gamma_{\circ/\bullet}$ are identically $0_{T_{\circ/\bullet}}$, $\gamma_m(p) = 0_H$ and $\gamma_m(q) \in (\Lambda_\circ \cap \Lambda_\bullet)/\Gamma$. We conclude that $L(T_\circ, H, T_\bullet)$ is part of a short exact sequence of abelian groups

$$0 \rightarrow \frac{\Lambda_\circ \cap \Lambda_\bullet}{\Gamma} \rightarrow L(T_\circ, H, T_\bullet) \xrightarrow{\text{Pth}} P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma) \rightarrow 0. \quad (3.5)$$

In other words, bicoloured loops are almost ordinary, unicoloured loops $S^1 \rightarrow H$, except for two differences. First, a bicoloured loop has a specific kind of discontinuity at the point $q \in S^1$, namely one which takes values in the finite abelian group $(\Lambda_\circ - \Lambda_\bullet)/\Gamma \subseteq H$. Second, there is a small piece of extra data consisting of a ‘reference point’ in $(\Lambda_\circ \cap \Lambda_\bullet)/\Gamma \subseteq H$ with respect to which the two loose ends of the loop at the discontinuity jump.

3.1.2 Support of a bicoloured loop

Bicoloured loops have the following natural notion of support:

Definition 3.1.2. The *support* of a bicoloured loop $\gamma = (\gamma_\circ, \gamma_m, \gamma_\bullet) \in L(T_\circ, H, T_\bullet)$ is the closed subset of S^1 given by

$$\text{supp } \gamma := \text{supp } \gamma_\circ \cup \text{supp } \gamma_\bullet.$$

Note that a path γ in $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ has a notion of support also, namely the closure of the points on S^1 where it is not 0_H . If such a path γ has support in an interval, then any bicoloured loop in its pre-image under Pth has support there as well in the sense of Definition 3.1.2, but the converse does not necessarily hold. It will be crucial in Section 3.2.4 that, even if $\Gamma = \Lambda_\circ \cap \Lambda_\bullet$ in which case Pth is an isomorphism, we use the Definition 3.1.2 of support for bicoloured loops.

3.1.3 Unicoloured loop groups are a special case

Consider the case where $\Lambda_\circ = \Lambda_\bullet = \Gamma$ and the homomorphisms $\pi_{\circ/\bullet}: \Gamma \hookrightarrow \Lambda_{\circ/\bullet}$ are the identity. Then of course $T_\circ = T_\bullet = H$ and the homomorphisms $U(1)\pi_{\circ/\bullet}: H \rightarrow H$ are the identity. We claim that the bicoloured torus loop group $L(H, H, H)$ is then merely a complicated way to describe the unicoloured torus loop group LH . Indeed, in this case $(\Lambda_\circ \cap \Lambda_\bullet)/\Gamma = 0$, so the homomorphism Pth in (3.5) is an isomorphism, and we furthermore now have $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma) \cong LH$.

Let us describe the identification more directly. There is a homomorphism of abelian groups $\text{Bi}: LH \rightarrow L(H, H, H)$ given by $\gamma \mapsto (\gamma|_{\zeta}, \gamma|_{\{p,q\}}, \gamma|_{\beth})$. It can be thought of as ‘bicolourising’ unicoloured loops in a canonical way. Its inverse Pth sends a bicoloured loop $\gamma = (\gamma_\circ, \gamma_m, \gamma_\bullet)$ to $\text{Pth}(\gamma)$, where for $\theta \in S^1$

$$\text{Pth}(\gamma)(\theta) := \begin{cases} \gamma_\circ(\theta) & \text{if } \theta \in \zeta, \\ \gamma_\bullet(\theta) & \text{if } \theta \in \beth. \end{cases}$$

This unicoloured loop $\text{Pth}(\gamma)$ is well-defined at the points $\theta = p$ and $\theta = q$ because of the commutativity of the diagram (3.2). Smoothness at those two points is satisfied thanks to the conditions (3.3) on the (higher) derivatives of γ_\circ and γ_\bullet .

3.1.4 The inclusion of LH

Generalising the observation made in Section 3.1.3, we may see unicoloured loops with values in H as certain bicoloured loops in $L(T_\circ, H, T_\bullet)$. Indeed, there is a homomorphism $\text{Bi}: LH \rightarrow L(T_\circ, H, T_\bullet)$ given by $\gamma \mapsto (\gamma_\circ, \gamma|_{\{p,q\}}, \gamma_\bullet)$, where γ_\circ is the composition $U(1)\pi_\circ \circ \gamma|_{\zeta}$ and γ_\bullet is $U(1)\pi_\bullet \circ \gamma|_{\beth}$. This homomorphism is injective.

Suppose namely that $\text{Bi}(\gamma) = (0_{T_\circ}, 0_H, 0_{T_\bullet})$. This means that the image of $\gamma|_{\zeta}$ is in the kernel $(\mathbb{R}\pi_\circ)^{-1}(\Lambda_\circ)/\Gamma \subseteq H$ of $U(1)\pi_\circ$ and the image of $\gamma|_{\beth}$ is in the kernel $(\mathbb{R}\pi_\bullet)^{-1}(\Lambda_\bullet)/\Gamma \subseteq H$ of $U(1)\pi_\bullet$. Since these kernels are finite abelian

groups, $\gamma|_{\zeta}$ and $\gamma|_{\gamma}$ must both be constant. However, we also assumed that $\gamma|_{\{p,q\}}$ vanishes. So $\gamma|_{\zeta}$ and $\gamma|_{\gamma}$ are identically 0_H as well, proving our claim.

The image of Bi consists of those bicoloured loops $(\gamma_{\circ}, \gamma_{\mathfrak{m}}, \gamma_{\bullet})$ for which the unique lifts $\hat{\gamma}_{\circ/\bullet}$ to H of the $\gamma_{\circ/\bullet}$ that match at p also match with each other, and with $\gamma_{\mathfrak{m}}$, at q .

3.1.5 Isotony with respect to unicoloured loop groups

Write $L_{\zeta}T_{\circ}$ for those loops in LT_{\circ} which have support in ζ and define $L_{\gamma}T_{\bullet}$ similarly. Then there are injective homomorphisms of abelian groups

$$L_{\zeta}T_{\circ} \hookrightarrow L(T_{\circ}, H, T_{\bullet}) \hookrightarrow L_{\gamma}T_{\bullet}, \quad (3.6)$$

where the homomorphism from $L_{\zeta}T_{\circ}$ is given by $\gamma_{\circ} \mapsto (\gamma_{\circ}, 0_H, 0_{T_{\bullet}})$ and the one from $L_{\gamma}T_{\bullet}$ by $\gamma_{\bullet} \mapsto (0_{T_{\circ}}, 0_H, \gamma_{\bullet})$. These triples indeed make the diagram (3.2) commute, since $\gamma_{\circ/\bullet}(p) = \gamma_{\circ/\bullet}(q) = 0_{T_{\circ/\bullet}}$.

3.1.6 The special case when $\Gamma = \Lambda_{\circ} \cap \Lambda_{\bullet}$

Write H^{\cap} for the torus $(\Lambda_{\circ} \cap \Lambda_{\bullet}) \otimes_{\mathbb{Z}} U(1)$ associated to the \mathbb{Z} -module $\Lambda_{\circ} \cap \Lambda_{\bullet}$. The two obvious injections $\Lambda_{\circ} \cap \Lambda_{\bullet} \hookrightarrow \Lambda_{\circ/\bullet}$, which we will denote by $\pi_{\circ/\bullet}^{\cap}$, allow us to define in particular a bicoloured torus loop group $L(T_{\circ}, H^{\cap}, T_{\bullet})$. By the short exact sequence (3.5) it is isomorphic to the group of paths

$$P(H^{\cap}, (\Lambda_{\circ} - \Lambda_{\bullet}) / (\Lambda_{\circ} \cap \Lambda_{\bullet})).$$

An equivalent way to formulate this special feature in this case is to observe that the homomorphism

$$(U(1)\pi_{\circ}^{\cap}, U(1)\pi_{\bullet}^{\cap}): H^{\cap} \rightarrow T_{\circ} \oplus T_{\bullet}$$

is injective. This implies that for maps $\gamma_{\circ}: \zeta \rightarrow T_{\circ}$ and $\gamma_{\bullet}: \gamma \rightarrow T_{\bullet}$ the existence of a function $\gamma_{\mathfrak{m}}: \cdot \rightarrow H^{\cap}$ making the diagram (3.2) (with H^{\cap} in place of H) commute determines $\gamma_{\mathfrak{m}}$ uniquely. That is, $\gamma_{\mathfrak{m}}$ is not genuine extra data. Its existence is merely a property of the pair $(\gamma_{\circ}, \gamma_{\bullet})$.⁵ The definition of $L(T_{\circ}, H^{\cap}, T_{\bullet})$ therefore simplifies to

$$L(T_{\circ}, H^{\cap}, T_{\bullet}) = \left\{ (\gamma_{\circ}, \gamma_{\bullet}) \in C^{\infty}(\zeta, T_{\circ}) \times C^{\infty}(\gamma, T_{\bullet}) \mid \right. \\ \left. (\gamma_{\circ}(p), \gamma_{\bullet}(p)), (\gamma_{\circ}(q), \gamma_{\bullet}(q)) \in (U(1)\pi_{\circ}^{\cap}, U(1)\pi_{\bullet}^{\cap})(H^{\cap}) \right\}.$$

⁵This fact was already used in Section 3.1.3, where we considered a special case of the situation we are in now.

(Here we omitted the conditions on the derivatives of γ_\circ and γ_\bullet at p and q for brevity.)

If we return to the general case considered at the beginning of Section 3.1 when Γ is a possibly-properly included submodule of $\Lambda_\circ \cap \Lambda_\bullet$, then its associated bicoloured torus loop group $L(T_\circ, H, T_\bullet)$ is related to $L(T_\circ, H^\cap, T_\bullet)$ via a short exact sequence of abelian groups

$$0 \rightarrow \left(\frac{\Lambda_\circ \cap \Lambda_\bullet}{\Gamma} \right)^{\oplus 2} \rightarrow L(T_\circ, H, T_\bullet) \rightarrow L(T_\circ, H^\cap, T_\bullet) \rightarrow 0.$$

The second arrow sends $([\nu_p], [\nu_q])$ to $(0, \gamma_m, 0)$, where $\gamma_m(p) := [\nu_p]$ and $\gamma_m(q) := [\nu_q]$. The third arrow is given by $(\gamma_\circ, \gamma_m, \gamma_\bullet) \mapsto (\gamma_\circ, \gamma_\bullet)$. This ‘forgetting’ of γ_m can also be seen as postcomposing γ_m with the surjective homomorphism $H \twoheadrightarrow H^\cap$ induced by the inclusion $\Gamma \subseteq \Lambda_\circ \cap \Lambda_\bullet$.

3.1.7 Actions of covers of $\text{Diff}_+(S^1)$

Recall from Section 2.1 that a unicoloured torus loop group has an obvious (left) action of $\text{Diff}_+(S^1)$ on it. Similarly, $L(T_\circ, H, T_\bullet)$ is naturally acted upon by the circle diffeomorphisms which fix the points p and q and preserve \lrcorner and \rceil . However, using (3.5) we can prove something more. Denote for any integer $m \geq 1$ by $\text{Diff}_+^{(m)}(S^1)$ the m -fold covering group of $\text{Diff}_+(S^1)$. It fits in a short exact sequence

$$0 \rightarrow \mathbb{Z}/m\mathbb{Z} \rightarrow \text{Diff}_+^{(m)}(S^1) \rightarrow \text{Diff}_+(S^1) \rightarrow 1$$

and can be modelled by the quotient group $\text{Diff}_+^{(\infty)}(S^1)/m\mathbb{Z}$, where in turn we use the model (2.16) for the universal covering group $\text{Diff}_+^{(\infty)}(S^1)$.

Remember the definition of the integer n on Page 72.

Proposition 3.1.3. *There is a (left) action of $\text{Diff}_+^{(n)}(S^1)$ on $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ which lifts to one on $L(T_\circ, H, T_\bullet)$ in a way that fixes the subgroup $(\Lambda_\circ \cap \Lambda_\bullet)/\Gamma$ of $L(T_\circ, H, T_\bullet)$.*

Proof. We start by explaining how $\text{Diff}_+^{(n)}(S^1)$ acts on $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$. If $\Phi \in \text{Diff}_+^{(\infty)}(S^1)$ and a path γ lies in $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$, then the path $\Phi^*\gamma: [0, 1] \rightarrow H$ is again in $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$. Here, $\Phi^*\gamma$ is defined by first extending γ quasi-periodically to all of \mathbb{R} , precomposing this extension (which we will also

denote by γ) with Φ^{-1} next and finally restricting to $[0, 1]$ again. Indeed, there holds

$$\begin{aligned} (\Phi^*\gamma)(1) - (\Phi^*\gamma)(0) &= \gamma(\Phi^{-1}(1)) - \gamma(\Phi^{-1}(0)) \\ &= \gamma(\Phi^{-1}(0) + 1) - \gamma(\Phi^{-1}(0)) \\ &= \gamma(\Phi^{-1}(0)) + \gamma(1) - \gamma(0) - \gamma(\Phi^{-1}(0)) \\ &= \gamma(1) - \gamma(0) \in (\Lambda_\circ - \Lambda_\bullet)/\Gamma. \end{aligned}$$

This defines a left action $\Phi \cdot \gamma := \Phi^*\gamma$ of $\text{Diff}_+^{(\infty)}(S^1)$ on $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$. The shift diffeomorphism $\theta \mapsto \theta + n$ then acts trivially, though. Therefore, this action of $\text{Diff}_+^{(\infty)}(S^1)$ descends to $\text{Diff}_+^{(\infty)}(S^1)/n\mathbb{Z}$, where \mathbb{Z} is the central subgroup of $\text{Diff}_+^{(\infty)}(S^1)$ generated by $\theta \mapsto \theta + 1$. This quotient group is a model for $\text{Diff}_+^{(n)}(S^1)$.

Let us now construct the lift of this $\text{Diff}_+^{(n)}(S^1)$ -action to $L(T_\circ, H, T_\bullet)$. Take $\Phi \in \text{Diff}_+^{(\infty)}(S^1)$ and a bicoloured loop $\gamma = (\gamma_\circ, \gamma_m, \gamma_\bullet) \in L(T_\circ, H, T_\bullet)$. We already just defined the translate $\Phi \cdot (\text{Pth } \gamma)$ for the path $\text{Pth}(\gamma)$, so in order to define $\Phi \cdot \gamma$ in the discrete pre-image under Pth over $\Phi \cdot (\text{Pth } \gamma)$ all we need to do is prescribe $(\Phi \cdot \gamma)_m(q)$. That is, we partially define $\Phi \cdot \gamma$ by the demand that $\text{Pth}(\Phi \cdot \gamma) = \Phi \cdot (\text{Pth } \gamma)$. We set

$$(\Phi \cdot \gamma)_m(q) := \gamma_m(q) - (\text{Pth } \gamma)(0) + (\Phi^*(\text{Pth } \gamma))(0).$$

Since by definition of the homomorphism Pth we already know that

$$\begin{aligned} \gamma_m(q) - (\text{Pth } \gamma)(1) &\in (\mathbb{R}\pi_\circ)^{-1}(\Lambda_\circ)/\Gamma, \\ \gamma_m(q) - (\text{Pth } \gamma)(0) &\in (\mathbb{R}\pi_\bullet)^{-1}(\Lambda_\bullet)/\Gamma, \end{aligned}$$

it follows that again $(\Phi \cdot \gamma)_m(q) - (\Phi^*(\text{Pth } \gamma))(0) \in (\mathbb{R}\pi_\bullet)^{-1}(\Lambda_\bullet)/\Gamma$ and that

$$\begin{aligned} (\Phi \cdot \gamma)_m(q) - (\Phi^*(\text{Pth } \gamma))(1) &= \gamma_m(q) - (\text{Pth } \gamma)(0) - (\text{Pth } \gamma)(1) + (\text{Pth } \gamma)(0) \\ &= \gamma_m(q) - (\text{Pth } \gamma)(1) \in (\mathbb{R}\pi_\circ)^{-1}(\Lambda_\circ)/\Gamma \end{aligned}$$

since

$$(\Phi^*(\text{Pth } \gamma))(0) - (\Phi^*(\text{Pth } \gamma))(1) = -(\text{Pth } \gamma)(1) + (\text{Pth } \gamma)(0).$$

This shows that $\Phi \cdot \gamma \in L(T_\circ, H, T_\bullet)$.

Using the fact that Pth is a homomorphism and Φ acts as an automorphism on $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ it is easily checked that Φ also acts as an automorphism on $L(T_\circ, H, T_\bullet)$. Furthermore, it can be shown that this is compatible with

the composition of the elements of $\text{Diff}_+^{(\infty)}(S^1)$. Therefore the $\text{Diff}_+^{(\infty)}(S^1)$ -action on $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ lifts to one on $L(T_\circ, H, T_\bullet)$. It also descends to $\text{Diff}_+^{(n)}(S^1)$ because $(n \cdot \gamma)_m(q) = \gamma_m(q)$. By definition of this $\text{Diff}_+^{(n)}(S^1)$ -action, the homomorphism Pth is $\text{Diff}_+^{(n)}(S^1)$ -equivariant. \square

Remark 3.1.4. The action of $\text{Diff}_+^{(n)}(S^1)$ on $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ can alternatively be understood by seeing elements of the latter group as a certain type of maps to the torus H from an n -fold cover of S^1 .

Remark 3.1.5 ($\text{Diff}_+^{(n)}(S^1)$ -equivariance with respect to LH). Note that the isomorphism $\text{Bi}: LH \xrightarrow{\sim} L(H, H, H)$ defined in Section 3.1.3 is equivariant with respect to the $\text{Diff}_+(S^1)$ -actions on both groups.

Regarding the inclusion $\text{Bi}: LH \hookrightarrow L(T_\circ, H, T_\bullet)$ from Section 3.1.4; breaking S^1 at the point q allows us to embed LH into $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ as well. We can then namely identify LH with

$$\left\{ \gamma \in C^\infty([0, 1], H) \mid \gamma(1) = \gamma(0), \gamma^{(k)}(1) = \gamma^{(k)}(0) \text{ for all } k \geq 1 \right\}. \quad (3.7)$$

This inclusion homomorphism ι is equivariant with respect to the standard $\text{Diff}_+(S^1)$ -action on LH described in Section 2.1 and the $\text{Diff}_+^{(n)}(S^1)$ -action on $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ constructed in the proof of Proposition 3.1.3. The following triangle commutes:

$$\begin{array}{ccc} L(T_\circ, H, T_\bullet) & \xrightarrow{\text{Pth}} & P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma) \\ & \swarrow \text{Bi} & \nearrow \iota \\ & LH & \end{array} . \quad (3.8)$$

Together with the equivariance of Pth this makes it clear that

$$\text{Pth}([\Phi] \cdot \text{Bi}(\gamma)) = \text{Pth}(\text{Bi}([\Phi]^* \gamma)),$$

where $\gamma \in LH$ and $\Phi \in \text{Diff}_+^{(\infty)}(S^1)$. It is then easily checked that also

$$([\Phi] \cdot \text{Bi}(\gamma))_m(q) = \text{Bi}([\Phi]^* \gamma)_m(q)$$

and therefore $[\Phi] \cdot \text{Bi}(\gamma) = \text{Bi}([\Phi]^* \gamma)$. In other words, Bi is equivariant with respect to the $\text{Diff}_+(S^1)$ -action on LH and the $\text{Diff}_+^{(n)}(S^1)$ -action on $L(T_\circ, H, T_\bullet)$.

Remark 3.1.6. It might seem unnatural to consider the action of the full group $\text{Diff}_+^{(n)}(S^1)$ on $L(T_\circ, H, T_\bullet)$, instead of for example the smaller group of equivalence classes of elements of $\text{Diff}_+^{(\infty)}(S^1)$ that preserve the intervals $[k, k + 1/2]$

for all $k \in \mathbb{Z}$. The former group is namely not compatible with the notion of support of bicoloured loops in a way similar to Remark 2.3.3 about the unicoloured situation. Instead, $\text{Diff}_+^{(n)}(S^1)$ ‘mixes’ the left and right halves of S^1 together.

The reason for our interest in $\text{Diff}_+^{(n)}(S^1)$ is that it contains the subgroup $\text{Rot}^{(n)}(S^1)$, and the action of the latter will allow us to speak about *positive energy* representations of (central extensions of) $L(T_\circ, H, T_\bullet)$.

3.1.8 The connected components of $L(T_\circ, H, T_\bullet)$

Recall from (2.1) how a choice of a privileged point on S^1 made it possible to alternatively describe the elements of a unicoloured torus loop group as Lie algebra valued paths. It allowed us to define the winding element of a unicoloured loop and, eventually, to understand the structure of the group. The definition of a bicoloured torus loop group already carries two privileged points p and q . We will choose the point q and carry out in this and the coming sections similar steps as in the unicoloured case.

The following Lemma explains that, given a triple $\gamma = (\gamma_\circ, \gamma_m, \gamma_\bullet)$ in $L(T_\circ, H, T_\bullet)$, which are torus valued maps, we may lift them to Lie algebra valued maps which ‘glue’ to a map that is continuous at the point p . This is essentially done by first applying Pth to γ , then taking a lift $[0, 1] \rightarrow \mathfrak{h}$ of $\text{Pth}(\gamma)$, writing this as a pair of maps $\zeta \rightarrow \mathfrak{t}_\circ$ and $\zeta \rightarrow \mathfrak{t}_\bullet$ and finally putting the datum of a lift of γ_m back in. We will write the proof in a ‘bicoloured fashion’, though, without referring to the homomorphism P .

We first observe that the group $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ has an alternative description in terms of paths in the Lie algebra \mathfrak{t} , namely as follows:

$$P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma) \cong \left\{ \hat{\xi} \in C^\infty([0, 1], \mathfrak{h}) \mid \begin{array}{l} \hat{\xi}(1) - \hat{\xi}(0) \in \Lambda_\circ - \Lambda_\bullet, \\ \hat{\xi}^{(k)}(1) = \hat{\xi}^{(k)}(0) \text{ for all } k \geq 1 \end{array} \right\} / \Gamma. \quad (3.9)$$

Lemma 3.1.7. *Let $\gamma = (\gamma_\circ, \gamma_m, \gamma_\bullet) \in L(T_\circ, H, T_\bullet)$ be a bicoloured loop. Then*

(i) *there exists a triple $\xi = (\xi_\circ, \xi_m, \xi_\bullet)$ of smooth maps*

$$\xi_\circ: \zeta \rightarrow \mathfrak{t}_\circ, \quad \xi_m: \dot{\cdot} \rightarrow \mathfrak{h} \quad \text{and} \quad \xi_\bullet: \zeta \rightarrow \mathfrak{t}_\bullet$$

such that

- (a) $\xi_{\circ/\bullet}$ is a lift of $\gamma_{\circ/\bullet}$ and ξ_m is a lift of γ_m , that is, $\exp_{\circ/\bullet} \circ \xi_{\circ/\bullet} = \gamma_{\circ/\bullet}$ and $\exp_H \circ \xi_m = \gamma_m$, where $\exp_{\circ/\bullet} : \mathfrak{t}_{\circ/\bullet} \rightarrow T_{\circ/\bullet}$ and $\exp_H : \mathfrak{h} \rightarrow H$,
- (b) $(\mathbb{R}\pi_{\circ})^{-1}(\xi_{\circ}(p)) = \xi_m(p) = (\mathbb{R}\pi_{\bullet})^{-1}(\xi_{\bullet}(p))$,

(ii) if $\eta = (\eta_{\circ}, \eta_m, \eta_{\bullet})$ is another triple of maps satisfying (i)(a) and (i)(b) above, then

$$\eta = \xi + (\pi_{\circ}(\mu_p), (p, q) \rightarrow (\mu_p, \mu_q), \pi_{\bullet}(\mu_p))$$

for some $\mu_p, \mu_q \in \Gamma$,

(iii) the map $\hat{\xi} : [0, 1] \rightarrow \mathfrak{h}$ defined by

$$\hat{\xi}(\theta) := \begin{cases} (\mathbb{R}\pi_{\circ})^{-1}(\xi_{\circ}(\theta)) & \text{if } \theta \in [1/2, 1], \\ (\mathbb{R}\pi_{\bullet})^{-1}(\xi_{\bullet}(\theta)) & \text{if } \theta \in [0, 1/2], \end{cases}$$

where we used the unit speed parametrisations of \curvearrowright and \curvearrowleft by $[0, 1/2]$ and $[1/2, 1]$ respectively again, is a lift of $P(\gamma)$ as in (3.9).

Proof. (i): Pick an arbitrary triple $(\xi_{\circ}, \xi_m, \xi_{\bullet})$ which satisfies (i)(a). Then the diagram

$$\begin{array}{ccccc} \textcircled{\leftarrow} & \longleftarrow & \textcircled{\cdot} & \longrightarrow & \textcircled{\rightarrow} \\ \xi_{\circ} \downarrow & & \downarrow \xi_m & & \downarrow \xi_{\bullet} \\ \mathfrak{t}_{\circ} & \xleftarrow{\sim \mathbb{R}\pi_{\circ}} & \mathfrak{h} & \xrightarrow{\sim \mathbb{R}\pi_{\bullet}} & \mathfrak{t}_{\bullet} \end{array}$$

does not necessarily commute. Instead, we have

$$\begin{aligned} (\xi_{\circ}(p), \xi_{\bullet}(p)) &= (\lambda_{\circ}, \lambda_{\bullet}) + (\mathbb{R}\pi_{\circ}, \mathbb{R}\pi_{\bullet})(\xi_m(p)) \\ (\xi_{\circ}(q), \xi_{\bullet}(q)) &= (\lambda'_{\circ}, \lambda'_{\bullet}) + (\mathbb{R}\pi_{\circ}, \mathbb{R}\pi_{\bullet})(\xi_m(q)) \end{aligned}$$

for some $\lambda_{\circ/\bullet}, \lambda'_{\circ/\bullet} \in \Lambda_{\circ/\bullet}$. It follows that the triple $(\xi_{\circ} - \lambda_{\circ}, \xi_m, \xi_{\bullet} - \lambda_{\bullet})$ now satisfies both (i)(a) and (i)(b).

(ii): Since η also satisfies (i)(a), we have

$$\eta = \xi + (\lambda_{\circ}, (p, q) \rightarrow (\mu_p, \mu_q), \lambda_{\bullet})$$

for some $\lambda_{\circ/\bullet} \in \Lambda_{\circ/\bullet}$ and $\mu_p, \mu_q \in \Gamma$. In particular, $\eta_{\circ/\bullet}(p) = \xi_{\circ/\bullet}(p) + \lambda_{\circ/\bullet}$. But using property (i)(b) for both η and ξ ,

$$(\mathbb{R}\pi_{\circ/\bullet})^{-1}(\eta_{\circ/\bullet}(p) - \xi_{\circ/\bullet}(p)) = \eta_m(p) - \xi_m(p) = \mu_p,$$

so $\lambda_{\circ/\bullet} = \pi_{\circ/\bullet}(\mu_p)$.

(iii): This claim is obvious given the definition of the homomorphism P . \square

Because we will use the Lie algebra valued lifts constructed in the previous Lemma often, we give them their own name:

Definition 3.1.8. For a bicoloured loop $\gamma = (\gamma_\circ, \gamma_m, \gamma_\bullet) \in L(T_\circ, H, T_\bullet)$ a triple of maps $\xi = (\xi_\circ, \xi_m, \xi_\bullet)$ as in Lemma 3.1.7(i) is called a *glued lift* of γ .

We will denote the quotient of the direct sum $\Lambda_\circ \oplus \Lambda_\bullet$ by the image of Γ under the homomorphism (π_\circ, π_\bullet) as $(\Lambda_\circ \oplus \Lambda_\bullet)/\Gamma$.

Proposition 3.1.9. *There are surjective homomorphisms*

$$\begin{aligned} \Delta' : P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma) &\twoheadrightarrow \Lambda_\circ - \Lambda_\bullet, \\ \Delta : L(T_\circ, H, T_\bullet) &\twoheadrightarrow \frac{\Lambda_\circ \oplus \Lambda_\bullet}{\Gamma} \end{aligned} \quad (3.10)$$

of abelian groups which make the following diagram commute:

$$\begin{array}{ccccccc} 0 & \rightarrow & \frac{\Lambda_\circ \cap \Lambda_\bullet}{\Gamma} & \rightarrow & L(T_\circ, H, T_\bullet) & \xrightarrow{\text{Pth}} & P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma) \rightarrow 0 \\ & & \downarrow -1 & & \downarrow \Delta & & \downarrow \Delta' \\ 0 & \rightarrow & \frac{\Lambda_\circ \cap \Lambda_\bullet}{\Gamma} & \rightarrow & \frac{\Lambda_\circ \oplus \Lambda_\bullet}{\Gamma} & \xrightarrow{(\mathbb{R}\pi_\circ)^{-1} - (\mathbb{R}\pi_\bullet)^{-1}} & \Lambda_\circ - \Lambda_\bullet \rightarrow 0, \end{array} \quad (3.11)$$

where the top row is the short exact sequence (3.5) and the bottom row is exact as well.

Proof. The homomorphism Δ' is defined in exactly the same way as in the unicoloured case, namely by picking for a path $\gamma \in P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ a lift $\hat{\xi} : [0, 1] \rightarrow \mathfrak{h}$ as in (3.9). The element $\Delta'_\gamma := \hat{\xi}(1) - \hat{\xi}(0) \in \Lambda_\circ - \Lambda_\bullet$ is then independent of the choice of $\hat{\xi}$ and this clearly defines a homomorphism. It is surjective because $\lambda_\circ - \lambda_\bullet \in \Lambda_\circ - \Lambda_\bullet$ has as pre-image for example the element of $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ defined as the projection on H of the Lie algebra-valued path $[0, 1] \rightarrow \mathfrak{h}$, $\theta \mapsto \theta(\lambda_\circ - \lambda_\bullet)$.

To define the homomorphism Δ , let $\gamma = (\gamma_\circ, \gamma_m, \gamma_\bullet) \in L(T_\circ, H, T_\bullet)$ be a bicoloured loop and pick a glued lift $\xi = (\xi_\circ, \xi_m, \xi_\bullet)$ of it. As explained in the proof of Lemma 3.1.7(i), there are then elements $\lambda_{\circ/\bullet} \in \Lambda_{\circ/\bullet}$ such that

$$(\xi_\circ(q), \xi_\bullet(q)) = (\lambda_\circ, \lambda_\bullet) + (\mathbb{R}\pi_\circ, \mathbb{R}\pi_\bullet)(\xi_m(q)).$$

If $\eta = (\eta_\circ, \eta_m, \eta_\bullet)$ is another such lift, then by Lemma 3.1.7(ii),

$$(\eta_\circ(q), \eta_\bullet(q)) = (\lambda_\circ, \lambda_\bullet) + (\pi_\circ(\mu), \pi_\bullet(\mu)) + (\mathbb{R}\pi_\circ, \mathbb{R}\pi_\bullet)(\eta_m(q))$$

for some $\mu \in \Gamma$. Therefore, setting Δ_γ to be the equivalence class $[\lambda_\circ, \lambda_\bullet]$ of $(\lambda_\circ, \lambda_\bullet)$ gives a well-defined map (3.10). We show that Δ is surjective. Let $(\lambda_\circ, \lambda_\bullet)$ be an element in $\Lambda_\circ \oplus \Lambda_\bullet$. Choose $\xi_\circ: \mathbb{C} \rightarrow \mathfrak{t}_\circ$ to be any map satisfying $\xi_\circ(p) = 0_{\mathfrak{t}_\circ}$ and $\xi_\circ(q) = \lambda_\circ$, pick $\xi_\bullet: \mathbb{C} \rightarrow \mathfrak{t}_\bullet$ to be any map with $\xi_\bullet(p) = 0_{\mathfrak{t}_\bullet}$ and $\xi_\bullet(q) = \lambda_\bullet$ and define $\xi_m: \mathbb{C} \rightarrow \mathfrak{h}$ as $\xi_m(p) = \xi_m(q) = 0_{\mathfrak{h}}$. Then the triple $(\xi_\circ, \xi_m, \xi_\bullet)$ is a glued lift of a bicoloured loop in $L(T_\circ, H, T_\bullet)$ and one has

$$(\xi_\circ(q), \xi_\bullet(q)) = (\lambda_\circ, \lambda_\bullet) + (\mathbb{R}\pi_\circ, \mathbb{R}\pi_\bullet)(0),$$

which shows what we wanted.

The second map in the bottom row of (3.11) is given by the obvious inclusion if we identify $(\Lambda_\circ \cap \Lambda_\bullet)/\Gamma$ with the quotient group

$$\frac{\{(\mathbb{R}\pi_\circ(\nu), \mathbb{R}\pi_\bullet(\nu)) \mid \nu \in \Lambda_\circ \cap \Lambda_\bullet\}}{\{(\pi_\circ(\mu), \pi_\bullet(\mu)) \mid \mu \in \Gamma\}}, \quad (3.12)$$

and the third map stands for

$$[\lambda_\circ, \lambda_\bullet] \mapsto (\mathbb{R}\pi_\circ)^{-1}(\lambda_\circ) - (\mathbb{R}\pi_\bullet)^{-1}(\lambda_\bullet),$$

where $[\lambda_\circ, \lambda_\bullet]$ is the equivalence class of $(\lambda_\circ, \lambda_\bullet)$ in $(\Lambda_\circ \oplus \Lambda_\bullet)/\Gamma$. This third map indeed has (3.12) as its kernel.

To explain the commutativity of the first square in (3.11), let $[\nu] \in (\Lambda_\circ \cap \Lambda_\bullet)/\Gamma$. Then its image in $(\Lambda_\circ \oplus \Lambda_\bullet)/\Gamma$ when travelling along the lower left corner of the square is the equivalence class $-(\mathbb{R}\pi_\circ, \mathbb{R}\pi_\bullet)(\nu)$. On the other hand, if we travel along the upper right corner, recall that the image of $[\nu]$ in $L(T_\circ, H, T_\bullet)$ is the bicoloured loop $\gamma = (\gamma_\circ, \gamma_m, \gamma_\bullet)$, where the $\gamma_{\circ/\bullet}$ are identically $0_{T_{\circ/\bullet}}$, $\gamma_m(p) = 0_H$ and $\gamma_m(q) = [\nu]$. A possible glued lift $\xi = (\xi_\circ, \xi_m, \xi_\bullet)$ of γ has the maps $\xi_{\circ/\bullet}$ identically $0_{\mathfrak{t}_{\circ/\bullet}}$, $\xi_m(p) = 0_{\mathfrak{h}}$ and $\xi_m(q) = \nu$. This satisfies

$$(\xi_\circ(q), \xi_\bullet(q)) = (0, 0) = -(\mathbb{R}\pi_\circ, \mathbb{R}\pi_\bullet)(\nu) + (\mathbb{R}\pi_\circ, \mathbb{R}\pi_\bullet)(\nu),$$

as desired, given the way we defined Δ .

The commutativity of the second square in (3.11) follows from the claim of Lemma 3.1.7(iii). \square

The homomorphism Δ serves the same role in our bicoloured situation as the winding element homomorphism $LT \rightarrow \Lambda$ does in the unicoloured case in the sense that its fibres are exactly the connected components of the group. The exactness of the bottom row in (3.11) implies that the group of connected components $(\Lambda_\circ \oplus \Lambda_\bullet)/\Gamma$ of $L(T_\circ, H, T_\bullet)$ is torsion-free if and only if $\Gamma = \Lambda_\circ \cap \Lambda_\bullet$.

Remark 3.1.10. It follows from Lemma 3.1.7(iii) that for a bicoloured loop $\gamma \in L(T_\circ, H, T_\bullet)$ we have

$$\Delta'_{\text{pth}\gamma} = (\mathbb{R}\pi_\circ)^{-1}(\xi_\circ(q)) - (\mathbb{R}\pi_\bullet)^{-1}(\xi_\bullet(q))$$

for any glued lift $(\xi_\circ, \xi_m, \xi_\bullet)$ of γ .

3.1.9 The structure of $L(T_\circ, H, T_\bullet)$

Since we learned in Section 3.1.4 that $L(T_\circ, H, T_\bullet)$ contains a canonical copy of LH via an injective ‘bicolouring’ homomorphism Bi , there is in particular an inclusion of the identity component $(LH)_0$ of the latter group.

Proposition 3.1.11. *The subgroup $\text{Bi}((LH)_0)$ of $L(T_\circ, H, T_\bullet)$ is the kernel of the homomorphism Δ , and it therefore also equals the identity component of $L(T_\circ, H, T_\bullet)$.*

Proof. Recall first that $(LH)_0$ can be canonically identified with the group of all smooth maps $S^1 \rightarrow \mathfrak{h}$, modulo Γ . Now suppose that γ is a bicoloured loop in the kernel of Δ . According to the definition of Δ in the proof of Proposition 3.1.9 this means that if $\xi = (\xi_\circ, \xi_m, \xi_\bullet)$ is a glued lift of γ , then there exists a $\mu \in \Gamma$ such that

$$(\xi_\circ(q), \xi_\bullet(q)) = (\pi_\circ(\mu), \pi_\bullet(\mu)) + (\mathbb{R}\pi_\circ, \mathbb{R}\pi_\bullet)(\xi_m(q)).$$

Therefore,

$$(\mathbb{R}\pi_\circ)^{-1}(\xi_\circ(q)) = (\mathbb{R}\pi_\bullet)^{-1}(\xi_\bullet(q)),$$

which together with Lemma 3.1.7(i)(b) implies that the two maps $(\mathbb{R}\pi_{\circ\bullet})^{-1} \circ \xi_{\circ\bullet}$ glue together to a smooth map $S^1 \rightarrow \mathfrak{h}$. Its value at q differs from that of ξ_m by μ . By the characterisation of the image of Bi mentioned at the end of Section 3.1.4 we see that γ lies in $\text{Bi}((LH)_0)$. \square

We are now able to understand the structure of $L(T_\circ, H, T_\bullet)$ by showing that the short exact sequence

$$0 \rightarrow (LH)_0 \xrightarrow{\text{Bi}} L(T_\circ, H, T_\bullet) \xrightarrow{\Delta} \frac{\Lambda_\circ \oplus \Lambda_\bullet}{\Gamma} \rightarrow 0. \quad (3.13)$$

admits a splitting, which we construct as follows. Let $(\lambda_\circ, \lambda_\bullet) \in \Lambda_\circ \oplus \Lambda_\bullet$. Define the \mathfrak{h} -valued straight line segment

$$\hat{\xi}: [0, 1] \rightarrow \mathfrak{h}, \quad \theta \mapsto (\mathbb{R}\pi_\bullet)^{-1}(\lambda_\bullet) + \theta((\mathbb{R}\pi_\circ)^{-1}(\lambda_\circ) - (\mathbb{R}\pi_\bullet)^{-1}(\lambda_\bullet))$$

from $(\mathbb{R}\pi_\bullet)^{-1}(\lambda_\bullet)$ to $(\mathbb{R}\pi_\circ)^{-1}(\lambda_\circ)$, and use it to define three Lie algebra valued maps ξ_\circ , ξ_m and ξ_\bullet in turn by restricting to the relevant parts of the interval $[0, 1]$:

$$\begin{aligned}\xi_\circ &:= \mathbb{R}\pi_\circ \circ \hat{\xi}|_{[1/2, 1]}: \mathbb{C} \rightarrow \mathfrak{t}_\circ \\ \xi_m(p) &:= \hat{\xi}(1/2) = \frac{1}{2}((\mathbb{R}\pi_\circ)^{-1}(\lambda_\circ) + (\mathbb{R}\pi_\bullet)^{-1}(\lambda_\bullet)) \in \mathfrak{h} \\ \xi_m(q) &:= 0_{\mathfrak{h}} \in \mathfrak{h} \\ \xi_\bullet &:= \mathbb{R}\pi_\bullet \circ \hat{\xi}|_{[0, 1/2]}: \mathbb{C} \rightarrow \mathfrak{t}_\bullet.\end{aligned}$$

Because a different choice of representative from the equivalence class $[\lambda_\circ, \lambda_\bullet]$ in $(\Lambda_\circ \oplus \Lambda_\bullet)/\Gamma$ of $(\lambda_\circ, \lambda_\bullet)$ would merely translate $\hat{\xi}$ by an element of $\Gamma \subseteq \mathfrak{h}$, the following torus valued maps γ_\circ , γ_m and γ_\bullet only depend on $[\lambda_\circ, \lambda_\bullet]$:

$$\begin{aligned}\gamma_\circ &:= \exp_\circ \circ \xi_\circ: \mathbb{C} \rightarrow T_\circ \\ \gamma_m(p) &:= \exp_H(\xi_m(p)) \in H \\ \gamma_m(q) &:= \exp_H(\xi_m(q)) = 0_H \in H \\ \gamma_\bullet &:= \exp_\bullet \circ \xi_\bullet: \mathbb{C} \rightarrow T_\bullet.\end{aligned}$$

This triple $\gamma_{[\lambda_\circ, \lambda_\bullet]} := (\gamma_\circ, \gamma_m, \gamma_\bullet)$ forms a bicoloured loop. The triple $(\xi_\circ, \xi_m, \xi_\bullet)$ is then a glued lift of $\gamma_{[\lambda_\circ, \lambda_\bullet]}$ and

$$(\xi_\circ(q), \xi_\bullet(q)) = (\mathbb{R}\pi_\circ(\hat{\xi}(1)), \mathbb{R}\pi_\bullet(\hat{\xi}(0))) = (\lambda_\circ, \lambda_\bullet) + (\mathbb{R}\pi_\circ, \mathbb{R}\pi_\bullet)(\xi_m(q))$$

shows that $\Delta_{\gamma_{[\lambda_\circ, \lambda_\bullet]}} = [\lambda_\circ, \lambda_\bullet]$. In other words, the homomorphism

$$\frac{\Lambda_\circ \oplus \Lambda_\bullet}{\Gamma} \rightarrow L(T_\circ, H, T_\bullet), \quad [\lambda_\circ, \lambda_\bullet] \mapsto \gamma_{[\lambda_\circ, \lambda_\bullet]} \quad (3.14)$$

that we just constructed is a section of Δ . This in particular implies that (3.14) is injective. The splitting (3.14) gives an isomorphism

$$L(T_\circ, H, T_\bullet) \xrightarrow{\sim} (LH)_0 \oplus \left(\frac{\Lambda_\circ \oplus \Lambda_\bullet}{\Gamma} \right)$$

sending a bicoloured loop $\gamma = (\gamma_\circ, \gamma_m, \gamma_\bullet)$ to $(\gamma - \gamma_{\Delta_\gamma}, \Delta_\gamma)$. Its inverse takes a pair $(\gamma, [\lambda_\circ, \lambda_\bullet])$ to the bicoloured loop $\gamma + \gamma_{[\lambda_\circ, \lambda_\bullet]}$. Recalling the isomorphism (2.4) we conclude with a decomposition

$$L(T_\circ, H, T_\bullet) \cong H \oplus V\mathfrak{h} \oplus \left(\frac{\Lambda_\circ \oplus \Lambda_\bullet}{\Gamma} \right),$$

which explains the structure of $L(T_\circ, H, T_\bullet)$.

Remark 3.1.12. In the proof of Proposition 3.1.9 it was shown that the homomorphism Δ is surjective by sketching how to find some bicoloured loop in the pre-image of every element $[\lambda_\circ, \lambda_\bullet]$ of $(\Lambda_\circ \oplus \Lambda_\bullet)/\Gamma$. One could ask whether that construction could be made more precise so as to obtain a simpler splitting of Δ than the more complicated one we built in the current section. However, the former construction *does* depend on the choice of representative $(\lambda_\circ, \lambda_\bullet)$ from $[\lambda_\circ, \lambda_\bullet]$, so this plan seems unreasonable.

3.2 Central extensions associated to spans of lattices

The bicoloured torus loop group $L(T_\circ, H, T_\bullet)$ and the group $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ defined in Section 3.1 can in particular be defined if the \mathbb{Z} -modules $\Lambda_{\circ/\bullet}$ and Γ come with the extra structure of bi-additive forms that make them even lattices. In this section we give, using these lattice structures, constructions of central extensions $\tilde{L}(T_\circ, H, T_\bullet)$ and $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ of $L(T_\circ, H, T_\bullet)$ and $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ that are analogous to the central extensions of unicoloured torus loop groups described in Section 2.2. As we will see in Section 3.2.1, they will in fact reduce exactly to those unicoloured extensions in the special case that $\Lambda_\circ = \Lambda_\bullet = \Gamma$ and the homomorphisms $\pi_{\circ/\bullet}: \Gamma \hookrightarrow \Lambda_{\circ/\bullet}$ are the identity. We will construct $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ through an explicit 2-cocycle. The formula for it will be nearly identical to the one for unicoloured torus loop groups. The central extension of $L(T_\circ, H, T_\bullet)$ will then simply be defined as the pullback of $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ along the homomorphism Pth . That is, it will be trivial on the kernel of Pth .

So let $\Lambda_{\circ/\bullet}$ and Γ be as in Section 3.1 but assume additionally that they are even lattices, denoting their forms by $\langle \cdot, \cdot \rangle_{\circ/\bullet}$ and $\langle \cdot, \cdot \rangle_\Gamma$ respectively, and that the $\pi_{\circ/\bullet}$ are lattice morphisms. We will use the same notation for the forms that are extended bilinearly to the Lie algebras $\mathfrak{t}_{\circ/\bullet}$ and \mathfrak{h} .

Remark 3.2.1. Special cases of spans of lattices have been considered before in the literature. If namely $\Gamma = \Lambda_\circ \cap \Lambda_\bullet$ and $[\Lambda_\circ : \Gamma] = p = [\Lambda_\bullet : \Gamma]$ for some prime number p , then Λ_\circ and Λ_\bullet have been called *p-neighbours* (see [Kne02, Section (28.2)]).

If one has the freedom to choose the middle even lattice Γ beforehand, then it is easy to construct and classify spans of lattices via the technique of *discriminant groups*. The even lattices $\Lambda_{\circ/\bullet}$ namely correspond to the q -isotropic subgroups of the finite abelian discriminant group $D_\Gamma := \Gamma^\vee/\Gamma$ of Γ , as explained in Appendix A.1.2.

Remark 3.2.2. The restriction of the \mathbb{R} -valued form $\langle \cdot, \cdot \rangle_\Gamma$ on \mathfrak{h} to $\Lambda_\circ \cap \Lambda_\bullet \subseteq \mathfrak{h}$ is \mathbb{Z} -valued, making $\Lambda_\circ \cap \Lambda_\bullet$ an even lattice, while the restriction to $\Lambda_\circ - \Lambda_\bullet \subseteq \mathfrak{h}$

is in general \mathbb{Q} -valued, making this a rational lattice.

Just like in the unicoloured case, the construction of central extensions of $L(T_\circ, H, T_\bullet)$ and $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ will require a little bit more data than just the lattices $\Lambda_{\circ/\bullet}$ and Γ and the two embeddings $\pi_{\circ/\bullet}$ between them. We will namely need a certain central extension of the underlying abelian group of the rational lattice $\Lambda_\circ - \Lambda_\bullet$ to ensure that the central extension of $L(T_\circ, H, T_\bullet)$ will be disjoint-commutative.

Construction 3.2.3 (A central extension of $\Lambda_\circ - \Lambda_\bullet$). Let us define a function⁶

$$\begin{aligned} b &: (\Lambda_\circ - \Lambda_\bullet) \times (\Lambda_\circ - \Lambda_\bullet) \rightarrow \text{U}(1), \\ b(\lambda, \mu) &:= e^{2\pi i b_0(\lambda, \mu)}, \quad b_0(\lambda, \mu) := \frac{1}{2} \langle \mu, \lambda \rangle_\Gamma + \langle \mu, \lambda_\bullet \rangle_\Gamma \in \mathbb{Q}, \end{aligned} \quad (3.15)$$

where $\lambda, \mu \in \Lambda_\circ - \Lambda_\bullet$ and λ_\bullet is defined via a choice of decomposition $\lambda = \lambda_\circ - \lambda_\bullet$, with $\lambda_{\circ/\bullet} \in (\mathbb{R}\pi_{\circ/\bullet})^{-1}(\Lambda_{\circ/\bullet})$. Notice that $b_0(\lambda, \mu) = -\frac{1}{2} \langle \mu, \lambda \rangle_\Gamma + \langle \mu, \lambda_\circ \rangle_\Gamma$ also, since we can add $-\langle \mu, \lambda \rangle_\Gamma + \langle \mu, \lambda_\circ \rangle_\Gamma$ to $b_0(\lambda, \mu)$, so we can use this as an alternative expression for b_0 if we so please.

The definition of b is independent of the choice of decomposition of λ used. Suppose namely that also $\lambda = \lambda'_\circ - \lambda'_\bullet$. Then $\lambda'_\bullet = \lambda_\bullet + \nu$ for some $\nu \in \Lambda_\circ \cap \Lambda_\bullet$. This implies that

$$\frac{1}{2} \langle \mu, \lambda \rangle_\Gamma + \langle \mu, \lambda'_\bullet \rangle_\Gamma = b_0(\lambda, \mu) + \langle \mu, \nu \rangle_\Gamma \equiv b_0(\lambda, \mu) \pmod{\mathbb{Z}}.$$

It is clear that b is bi-additive and we furthermore have that $b(\lambda, \lambda) = 1$ for all $\lambda \in \Lambda_\circ - \Lambda_\bullet$ since

$$b_0(\lambda, \lambda) = \frac{1}{2} \langle \lambda_\circ - \lambda_\bullet, \lambda_\circ - \lambda_\bullet \rangle_\Gamma + \langle \lambda_\circ - \lambda_\bullet, \lambda_\bullet \rangle_\Gamma = \frac{1}{2} \langle \lambda_\circ, \lambda_\circ \rangle_\Gamma - \frac{1}{2} \langle \lambda_\bullet, \lambda_\bullet \rangle_\Gamma,$$

and $\langle \lambda_{\circ/\bullet}, \lambda_{\circ/\bullet} \rangle_\Gamma / 2 \in \mathbb{Z}$ because the lattices $\Lambda_{\circ/\bullet}$ are even. Together with the bi-additivity, this property of b implies by the discussion in Appendix A.2.1 that there exists a $\text{U}(1)$ -central extension $(\Lambda_\circ - \Lambda_\bullet)^\sim$ of $\Lambda_\circ - \Lambda_\bullet$ which has b as a commutator map. It is determined up to non-unique isomorphism.

We repeat that the formula (3.15) for the commutator map b , which looks somewhat ad hoc, is designed specifically to make Theorem 3.2.10 true.

⁶More precisely, b takes values only in the finite cyclic subgroup of $\text{U}(1)$ of order

$$2 \cdot [\Lambda_\circ - \Lambda_\bullet : \Lambda_\circ \cap \Lambda_\bullet].$$

Construction 3.2.4 (Central extensions of $P(H, (\Lambda_o - \Lambda_\bullet)/\Gamma)$). We will construct a $U(1)$ -central extension $\tilde{P}(H, (\Lambda_o - \Lambda_\bullet)/\Gamma)$ of $P(H, (\Lambda_o - \Lambda_\bullet)/\Gamma)$ by letting its underlying set be $P(H, (\Lambda_o - \Lambda_\bullet)/\Gamma) \times U(1)$ and writing down an explicit 2-cocycle. We fix a choice of a $U(1)$ -central extension $(\Lambda_o - \Lambda_\bullet)^\sim$ of $\Lambda_o - \Lambda_\bullet$ with the commutator map (3.15) and a choice of a 2-cocycle

$$\varepsilon : (\Lambda_o - \Lambda_\bullet) \times (\Lambda_o - \Lambda_\bullet) \rightarrow U(1)$$

for it. Let $\gamma, \rho \in P(H, (\Lambda_o - \Lambda_\bullet)/\Gamma)$, $z, w \in U(1)$ and pick lifts $\xi, \eta : [0, 1] \rightarrow \mathfrak{h}$ of γ and ρ respectively using the isomorphism (3.9). We define the multiplication on $\tilde{P}(H, (\Lambda_o - \Lambda_\bullet)/\Gamma)$ by

$$(\gamma, z) \cdot (\rho, w) := (\gamma + \rho, zw \cdot c'(\gamma, \rho))$$

where c' is the 2-cocycle on $P(H, (\Lambda_o - \Lambda_\bullet)/\Gamma)$ given by

$$\begin{aligned} c'(\gamma, \rho) &:= \varepsilon(\Delta'_\gamma, \Delta'_\rho) e^{2\pi i S'(\xi, \eta)}, \\ S'(\xi, \eta) &:= \frac{1}{2} \int_0^1 \langle \xi'(\theta), \eta(\theta) \rangle_\Gamma d\theta + \frac{1}{2} \langle \Delta'_\gamma, \eta(0) \rangle_\Gamma. \end{aligned} \quad (3.16)$$

Notice that S' is bi-additive.

Since ξ and η are well-defined up to an element of Γ and $\langle \Delta'_\gamma, \mu \rangle_\Gamma \in \mathbb{Z}$ for all $\mu \in \Gamma$ the proof of the well-definedness of c' is identical to the one in the unicoloured case.

Remark 3.2.5 (Failure of disjoint-commutativity for $\tilde{P}(H, (\Lambda_o - \Lambda_\bullet)/\Gamma)$). One could try to imitate for the central extension $\tilde{P}(H, (\Lambda_o - \Lambda_\bullet)/\Gamma)$ we just constructed the calculation in the proof of Theorem 2.2.3 which shows disjoint-commutativity for central extensions of unicoloured torus loop groups. A possible adjustment, needed to even make sense of the statement of that Theorem, would be to consider subintervals of $[0, 1]$ which are allowed to contain the endpoints $\{0, 1\}$ and to use the notion of support for elements of $P(H, (\Lambda_o - \Lambda_\bullet)/\Gamma)$ already mentioned in Section 3.1.2. The intermediate result would then be that

$$e^{2\pi i (S'(\xi, \eta) - S'(\eta, \xi))} = e^{-\pi i \langle \Delta'_\rho, \Delta'_\gamma \rangle_\Gamma}. \quad (3.17)$$

On the other hand, we have that

$$\varepsilon(\Delta'_\gamma, \Delta'_\rho) \varepsilon(\Delta'_\rho, \Delta'_\gamma)^{-1} = b(\Delta'_\gamma, \Delta'_\rho) = e^{\pi i \langle \Delta'_\rho, \Delta'_\gamma \rangle_\Gamma} e^{2\pi i \langle \Delta'_\rho, \lambda_\bullet \rangle_\Gamma},$$

where $\Delta'_\gamma = \lambda_o - \lambda_\bullet$ with $\lambda_{o_\bullet} \in (\mathbb{R}\pi_{o_\bullet})^{-1}(\Lambda_{o_\bullet})$. If $\lambda_\bullet \in \Lambda_o \cap \Lambda_\bullet$, which happens for example if $\text{supp } \gamma$ does not contain $1 \in [0, 1]$, then $\langle \Delta'_\rho, \lambda_\bullet \rangle_\Gamma \in \mathbb{Z}$ and so

$\varepsilon(\Delta'_\gamma, \Delta'_\rho)\varepsilon(\Delta'_\rho, \Delta'_\gamma)^{-1}$ would absorb (3.17). Therefore (γ, z) and (ρ, w) would indeed commute. However, it need not be true in general that $\lambda_\bullet \in \Lambda_\circ \cap \Lambda_\bullet$ when $1 \in \text{supp } \gamma$. We conclude that with the definitions we made $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ is not disjoint-commutative.

We do not know how to adjust the group $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ in such a way that it becomes disjoint-commutative also for intervals that do contain the point 1, except by inheriting via the homomorphism Pth the notion of support from $L(T_\circ, H, T_\bullet)$ as in Definition 3.1.2. The central extension of $L(T_\circ, H, T_\bullet)$ that we will construct in a moment namely *will* turn out to have the desired disjoint-commutativity property.

Construction 3.2.6 (Central extensions of $L(T_\circ, H, T_\bullet)$). Given the central extension $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ from Construction 3.2.4, we define a $U(1)$ -central extension $\tilde{L}(T_\circ, H, T_\bullet)$ of $L(T_\circ, H, T_\bullet)$ as the pullback of $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ along the homomorphism Pth . That is, the 2-cocycle c defining $\tilde{L}(T_\circ, H, T_\bullet)$ is set to be

$$c(\gamma, \rho) := c'(\text{Pth } \gamma, \text{Pth } \rho) \quad (3.18)$$

for two bicoloured loops $\gamma, \rho \in L(T_\circ, H, T_\bullet)$.

By construction there is a short exact sequence of groups

$$0 \rightarrow \frac{\Lambda_\circ \cap \Lambda_\bullet}{\Gamma} \rightarrow \tilde{L}(T_\circ, H, T_\bullet) \xrightarrow{\widetilde{\text{Pth}}} \tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma) \rightarrow 1,$$

where $\widetilde{\text{Pth}}$ is the obvious lift of the homomorphism of abelian groups Pth , sending $(\gamma, z) \in \tilde{L}(T_\circ, H, T_\bullet)$ to $(\text{Pth } \gamma, z)$. The second arrow in this sequence sends an equivalence class $[v] \in (\Lambda_\circ \cap \Lambda_\bullet)/\Gamma$ to the element

$$\left((0_{T_\circ}, (p, q) \rightarrow (0_H, [v]), 0_{T_\bullet}), 1 \right) \in \tilde{L}(T_\circ, H, T_\bullet).$$

Having defined $\widetilde{\text{Pth}}$, we can now state that we have a pullback diagram in the category of groups:

$$\begin{array}{ccc} \tilde{L}(T_\circ, H, T_\bullet) & \xrightarrow{\widetilde{\text{Pth}}} & \tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma) \\ \downarrow & \lrcorner & \downarrow \\ L(T_\circ, H, T_\bullet) & \xrightarrow{\text{Pth}} & P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma). \end{array} \quad (3.19)$$

The cocycle c can also be written in a ‘bicoloured fashion’ without directly using the group $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$, namely by picking glued lifts $\xi = (\xi_\circ, \xi_m, \xi_\bullet)$

and $\eta = (\eta_\circ, \eta_m, \eta_\bullet)$ of γ and ρ respectively. If we then split the integral in (3.16) in the middle, appeal to Lemma 3.1.7(iii), which relates ξ and η to lifts of $\text{Pth}(\gamma)$ and $\text{Pth}(\rho)$, and finally use the unit speed parametrisations of ζ and γ by $[0, 1/2]$ and $[1/2, 1]$, we get

$$c(\gamma, \rho) := \varepsilon(\Delta'_{\text{Pth}\gamma}, \Delta'_{\text{Pth}\rho})e^{2\pi i S(\xi, \eta)},$$

$$S(\xi, \eta) := \frac{1}{2} \left[\int_{\zeta} \langle d\xi_\circ, \eta_\circ \rangle_\circ + \int_{\gamma} \langle d\xi_\bullet, \eta_\bullet \rangle_\bullet + \left\langle \Delta'_{\text{Pth}\gamma}, (\mathbb{R}\pi_\bullet)^{-1}(\eta_\bullet(q)) \right\rangle_\Gamma \right]. \quad (3.20)$$

Using this description of c instead, its well-definedness can alternatively be checked via Lemma 3.1.7(ii) that explains the ambiguity in the choices of the glued lifts ξ and η .

Ingredients 3.2.7. We summarise the ingredients used in the construction of the central extensions $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ and $\tilde{L}(T_\circ, H, T_\bullet)$ for clarity:

- three even lattices $(\Lambda_\circ, \langle \cdot, \cdot \rangle_\circ)$, $(\Lambda_\bullet, \langle \cdot, \cdot \rangle_\bullet)$ and $(\Gamma, \langle \cdot, \cdot \rangle_\Gamma)$ of the same rank,
- two lattice morphisms $\pi_{\circ/\bullet}: \Gamma \hookrightarrow \Lambda_{\circ/\bullet}$,
- a choice of a $U(1)$ -central extension $(\Lambda_\circ - \Lambda_\bullet)^\sim$ of $\Lambda_\circ - \Lambda_\bullet$ such that it has commutator map (3.15),
- a choice of a 2-cocycle ε for $(\Lambda_\circ - \Lambda_\bullet)^\sim$ (we will always choose ε to be normalised to make calculations easier),
- (for the central extension $\tilde{L}(T_\circ, H, T_\bullet)$) a choice of one of the two points p or q on S^1 as being privileged.

Just like for a unicoloured central extension $\tilde{L}\tilde{T}$, also the notations $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ and $\tilde{L}(T_\circ, H, T_\bullet)$ do not refer to these ingredients, so it will be important in the sequel to explain this separately when necessary.

3.2.1 Unicoloured central extensions are a special case

Recall from Section 3.1.3 that if $\Lambda_\circ = \Lambda_\bullet = \Gamma$ and the morphisms $\pi_{\circ/\bullet}: \Gamma \hookrightarrow \Lambda_{\circ/\bullet}$ are the identity, there exists an isomorphism of abelian groups $\text{Bi}: LH \xrightarrow{\sim} L(H, H, H)$ given by $\gamma \mapsto (\gamma|_C, \gamma|_{\{p, q\}}, \gamma|_D)$. In this case the \mathbb{Z} -module $\Lambda_\circ - \Lambda_\bullet \subseteq \mathfrak{h}$ equals Γ , and the commutator map b on $\Lambda_\circ - \Lambda_\bullet = \Gamma$ defined in (3.15) is nothing but the commutator map $(\lambda, \mu) \mapsto (-1)^{\langle \lambda, \mu \rangle_\Gamma}$ since Γ is an integral lattice. Now note that a choice of a $\{\pm 1\}$ -central extension $\tilde{\Gamma}$ of Γ with this as commutator map, a choice of 2-cocycle $\varepsilon: \Gamma \times \Gamma \rightarrow \{\pm 1\}$ for it and the choice of the point q

among p and q , as needed in Construction 3.2.6, is also exactly the data we needed in Section 2.2 to construct a $U(1)$ -central extension $\widetilde{L}H$ of LH . We now claim that the obvious lift to the underlying sets

$$\widetilde{L}H \xrightarrow{\sim} \widetilde{L}(H, H, H), \quad (\gamma, z) \mapsto (\text{Bi}(\gamma), z) \quad (3.21)$$

of Bi is an isomorphism of (non-abelian) groups. It amounts to showing that for all $\gamma, \rho \in LH$,

$$c(\text{Bi}(\gamma), \text{Bi}(\rho)) = c_H(\gamma, \rho), \quad (3.22)$$

where we write the cocycle defining $\widetilde{L}H$ as c_H and still write c for the one defining $\widetilde{L}(H, H, H)$. We will also denote by S_H the map in the definition of c_H that takes two \mathfrak{h} -valued maps as input.

The first observation to make is that if $\xi: [0, 1] \rightarrow \mathfrak{h}$ is a lift of γ as in (2.1), made by cutting S^1 at the privileged point q , then we may choose the triple of maps

$$(\xi_\circ, \xi_m, \xi_\bullet) := (\xi|_{[1/2, 1]}, (p, q) \rightarrow (\xi(1/2), \xi(0)), \xi|_{[0, 1/2]})$$

as a glued lift of $\text{Bi}(\gamma)$. Knowing this, we see that⁷

$$\Delta'_{\text{Pth}(\text{Bi}\gamma)} = \xi_\circ(q) - \xi_\bullet(q) = \xi(1) - \xi(0) = \Delta_\gamma.$$

One chooses a glued lift $(\eta_\circ, \eta_m, \eta_\bullet)$ for $\text{Bi}(\rho)$ similarly using a lift $\eta: [0, 1] \rightarrow \mathfrak{h}$ of ρ . This gives us

$$\varepsilon(\Delta'_{\text{Pth}(\text{Bi}\gamma)}, \Delta'_{\text{Pth}(\text{Bi}\rho)}) = \varepsilon(\Delta_\gamma, \Delta_\rho). \quad (3.23)$$

Next, using again that $\eta_\bullet(q) = \eta(0)$ by our definition of η_\bullet , we observe that the second line in the definition (3.20) of the bicoloured cocycle now reads

$$\begin{aligned} & S((\xi_\circ, \xi_m, \xi_\bullet), (\eta_\circ, \eta_m, \eta_\bullet)) \\ &= \frac{1}{2} \left[\int_{\curvearrowright} \langle d\xi_\circ, \eta_\circ \rangle_\Gamma + \int_{\curvearrowleft} \langle d\xi_\bullet, \eta_\bullet \rangle_\Gamma + \langle \Delta'_{\text{Pth}(\text{Bi}\gamma)}, \eta_\bullet(q) \rangle_\Gamma \right] \\ &= \frac{1}{2} \left[\int_{\frac{1}{2}}^1 \langle \xi'(\theta), \eta(\theta) \rangle_\Gamma d\theta + \int_0^{\frac{1}{2}} \langle \xi'(\theta), \eta(\theta) \rangle_\Gamma d\theta + \langle \Delta_\gamma, \eta(0) \rangle_\Gamma \right]. \end{aligned}$$

This is exactly the formula for $S_H(\xi, \eta)$ in (2.7), which together with (3.23) proves the claim (3.22).

⁷Here, Δ_γ refers to the winding element homomorphism for LH as defined in Section 2.1.

3.2.2 The inclusion of $\tilde{L}H$

We return to the general setup assumed at the beginning of Section 3.2. Recall from Section 3.1.4 that there is a canonical inclusion of abelian groups $\text{Bi}: LH \hookrightarrow L(T_\circ, H, T_\bullet)$ given by $\gamma \mapsto (\gamma_\circ, \gamma|_{\{p,q\}}, \gamma_\bullet)$, where $\gamma_\circ := U(1)\pi_\circ \circ \gamma|_\zeta$ and $\gamma_\bullet := U(1)\pi_\bullet \circ \gamma|_\gamma$. Define a central extension $\tilde{L}H$ of LH via the construction in Section 2.2 using the following input data:

- the even lattice $(\Gamma, \langle \cdot, \cdot \rangle_\Gamma)$,
- the restriction $\tilde{\Gamma}$ of the chosen $U(1)$ -central extension $(\Lambda_\circ - \Lambda_\bullet)^\sim$ of $\Lambda_\circ - \Lambda_\bullet$ to Γ ,
- the restriction ε_Γ to Γ of the chosen 2-cocycle ε on $\Lambda_\circ - \Lambda_\bullet$ for $(\Lambda_\circ - \Lambda_\bullet)^\sim$,
- the point q as a choice of privileged point on S^1 .

The use of $\tilde{\Gamma}$ is permitted, meaning that it indeed has commutator map $(\mu, \mu') \mapsto (-1)^{\langle \mu, \mu' \rangle_\Gamma}$. The function b_0 in (3.15), when restricted to Γ , namely becomes $\frac{1}{2}\langle \mu', \mu \rangle_\Gamma$ modulo \mathbb{Z} since the restriction of the form $\langle \cdot, \cdot \rangle_\Gamma$ on $\Lambda_\circ - \Lambda_\bullet$ to Γ is integral, and so for $\mu, \mu' \in \Gamma$ we have

$$b(\mu, \mu') = e^{2\pi i b_0(\mu, \mu')} = (-1)^{\langle \mu, \mu' \rangle_\Gamma}.$$

We claim that the obvious lift to the underlying sets

$$\tilde{\text{Bi}}: \tilde{L}H \hookrightarrow \tilde{L}(T_\circ, H, T_\bullet), \quad (\gamma, z) \mapsto (\text{Bi}(\gamma), z) \quad (3.24)$$

of Bi is a homomorphism of (non-abelian) groups, meaning that for all $\gamma, \rho \in LH$,

$$c(\text{Bi}(\gamma), \text{Bi}(\rho)) = c_H(\gamma, \rho). \quad (3.25)$$

(We will use the same notations c_H and S_H as we did in Section 3.2.1.)

Indeed, if $\xi: [0, 1] \rightarrow \mathfrak{h}$ is a lift of γ as in (2.1), made by cutting S^1 at q , then we may choose as a glued lift $(\xi_\circ, \xi_m, \xi_\bullet)$ of $\text{Bi}(\gamma)$ the triple of maps

$$\begin{aligned} \xi_\circ &:= \mathbb{R}\pi_\circ \circ \xi|_{[1/2, 1]}: \zeta \rightarrow \mathfrak{t}_\circ \\ \xi_m(p) &:= \xi(1/2), \quad \xi_m(q) := \xi(0) \\ \xi_\bullet &:= \mathbb{R}\pi_\bullet \circ \xi|_{[0, 1/2]}: \gamma \rightarrow \mathfrak{t}_\bullet. \end{aligned}$$

Knowing this, we see that^{8,9}

$$\Delta'_{\text{pth}(\text{Bi}\gamma)} = (\mathbb{R}\pi_\circ)^{-1}(\xi_\circ(q)) - (\mathbb{R}\pi_\bullet)^{-1}(\xi_\bullet(q)) = \xi(1) - \xi(0) = \Delta_\gamma \in \Gamma.$$

⁸See Footnote 7.

⁹This equality of winding elements also follows from the commutative triangle (3.8).

One chooses a glued lift $(\eta_\circ, \eta_m, \eta_\bullet)$ for $\text{Bi}(\rho)$ similarly using a lift $\eta: [0, 1] \rightarrow \mathfrak{h}$ of ρ . This gives us

$$\varepsilon(\Delta'_{\text{Pth}(\text{Bi}\gamma)}, \Delta'_{\text{Pth}(\text{Bi}\rho)}) = \varepsilon_\Gamma(\Delta_\gamma, \Delta_\rho). \quad (3.26)$$

Next, using again that $(\mathbb{R}\pi_\bullet)^{-1}(\eta_\bullet(q)) = \eta(0)$, the second line in (3.20) reads

$$\begin{aligned} & S((\xi_\circ, \xi_m, \xi_\bullet), (\eta_\circ, \eta_m, \eta_\bullet)) \\ &= \frac{1}{2} \left[\int_{\mathfrak{C}} \langle d\xi_\circ, \eta_\circ \rangle_\circ + \int_{\mathfrak{D}} \langle d\xi_\bullet, \eta_\bullet \rangle_\bullet + \langle \Delta'_{\text{Pth}(\text{Bi}\gamma)}, (\mathbb{R}\pi_\bullet)^{-1}(\eta_\bullet(q)) \rangle_\Gamma \right] \\ &= \frac{1}{2} \left[\int_{\frac{1}{2}}^1 \langle \xi'(\theta), \eta(\theta) \rangle_\Gamma d\theta + \int_0^{\frac{1}{2}} \langle \xi'(\theta), \eta(\theta) \rangle_\Gamma d\theta + \langle \Delta_\gamma, \eta(0) \rangle_\Gamma \right]. \end{aligned}$$

This equals the formula for $S_H(\xi, \eta)$ in (2.7), which together with (3.26) proves (3.25).

Remark 3.2.8. The existence of the homomorphisms (3.21) and (3.24) of central extensions can be understood in a more conceptual, but less explicit way, namely via the pullback diagram (3.19).

For the lifting (3.21) of the isomorphism Bi we can observe that in the situation of Section 3.2.1 we have $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma) \cong LH$ and $\text{Pth} = (\text{Bi})^{-1}$, if we identify \mathfrak{C} and \mathfrak{D} with $[1/2, 1]$ and $[0, 1/2]$ respectively. So because Pth is an isomorphism, the same holds for its lift $\widetilde{\text{Pth}}$. That the cocycle c' for $\widetilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ is identical to the one defining \widetilde{LH} then shows that $\widetilde{\text{Pth}}^{-1}$ is the desired lift of Bi .

For the lifting (3.24) of the inclusion Bi , we can think of the cocycle c' as generalising the one defining \widetilde{LH} to a bigger group if we consider LH as a subgroup of $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ through (3.7). Together with the commutativity of the triangle (3.8) we therefore get a commutative diagram

$$\begin{array}{ccc} \widetilde{LH} & \xrightarrow{\quad \tilde{i} \quad} & \widetilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma) \\ \downarrow & & \downarrow \\ LH & \xrightarrow[\text{Bi}]{} L(T_\circ, H, T_\bullet) \xrightarrow[\text{Pth}]{} & P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma). \end{array}$$

The universal property of the pullback (3.19) now gives us the lift of Bi .

3.2.3 Isotony with respect to unicoloured central extensions

Recall the inclusions (3.6) of the non-centrally extended groups $L_{\mathfrak{C}}T_\circ$ and $L_{\mathfrak{D}}T_\bullet$ of unicoloured loops into $L(T_\circ, H, T_\bullet)$ given by $\gamma_\circ \mapsto (\gamma_\circ, 0_H, 0_{T_\bullet})$ and

$\gamma_\bullet \mapsto (0_{T_\circ}, 0_H, \gamma_\bullet)$ respectively. Define central extensions $\tilde{L}T_{\circ/\bullet}$ of $LT_{\circ/\bullet}$ via the construction in Section 2.2 using the following input data:

- the even lattice $(\Lambda_{\circ/\bullet}, \langle \cdot, \cdot \rangle_{\circ/\bullet})$ (or, more precisely, its pre-image in \mathfrak{h} under the isometry $\mathbb{R}\pi_{\circ/\bullet}$),
- the restriction $\tilde{\Lambda}_{\circ/\bullet}$ of the chosen $U(1)$ -central extension $(\Lambda_\circ - \Lambda_\bullet)^\sim$ of $\Lambda_\circ - \Lambda_\bullet$ to $(\mathbb{R}\pi_{\circ/\bullet})^{-1}(\Lambda_{\circ/\bullet})$,
- the restriction $\varepsilon_{\circ/\bullet}$ to $(\mathbb{R}\pi_{\circ/\bullet})^{-1}(\Lambda_{\circ/\bullet})$ of the chosen 2-cocycle ε on $\Lambda_\circ - \Lambda_\bullet$ for $(\Lambda_\circ - \Lambda_\bullet)^\sim$,
- the point q as a choice of privileged point on S^1 .

Indeed, the commutator map of $\tilde{\Lambda}_{\circ/\bullet}$ is $(\lambda_{\circ/\bullet}, \mu_{\circ/\bullet}) \mapsto (-1)^{\langle \lambda_{\circ/\bullet}, \mu_{\circ/\bullet} \rangle_{\circ/\bullet}}$. The function b_0 when restricted to $\Lambda_{\circ/\bullet}$ namely becomes $\frac{1}{2} \langle \mu_{\circ/\bullet}, \lambda_{\circ/\bullet} \rangle_{\circ/\bullet}$ modulo \mathbb{Z} since the restriction $\langle \cdot, \cdot \rangle_{\circ/\bullet}$ of the form $\langle \cdot, \cdot \rangle_\Gamma$ on $\Lambda_\circ - \Lambda_\bullet$ to $(\mathbb{R}\pi_{\circ/\bullet})^{-1}(\Lambda_{\circ/\bullet})$ is integral.

Write $\tilde{L}_\zeta T_\circ$ for the restriction of $\tilde{L}T_\circ$ to $L_\zeta T_\circ$ and define $\tilde{L}_\gamma T_\bullet$ similarly. We claim that the obvious lifts to the underlying sets

$$\tilde{L}_\zeta T_\circ \hookrightarrow \tilde{L}(T_\circ, H, T_\bullet) \hookrightarrow \tilde{L}_\gamma T_\bullet$$

of the inclusions of $L_\zeta T_\circ$ and $L_\gamma T_\bullet$, namely the ones that are the identity on the central subgroups $U(1)$, are group homomorphisms. That is, for all $\gamma_\circ, \rho_\circ \in L_\zeta T_\circ$,

$$c((\gamma_\circ, 0_H, 0_{T_\bullet}), (\rho_\circ, 0_H, 0_{T_\bullet})) = c_\circ(\gamma_\circ, \rho_\circ), \quad (3.27)$$

where c_\circ is the 2-cocycle defining $\tilde{L}_\zeta T_\circ$. We will also denote by S_\circ the map in the definition of c_\circ that takes two \mathfrak{t}_\circ -valued maps as input. Similarly, we claim that

$$c((0_{T_\circ}, 0_H, \gamma_\bullet), (0_{T_\circ}, 0_H, \rho_\bullet)) = c_\bullet(\gamma_\bullet, \rho_\bullet) \quad (3.28)$$

for all $\gamma_\bullet, \rho_\bullet \in L_\gamma T_\bullet$, where c_\bullet is the 2-cocycle defining $\tilde{L}_\gamma T_\bullet$.

We will prove (3.27)—the proof of (3.28) is similar. If $\xi: [0, 1] \rightarrow \mathfrak{t}_\circ$ is a lift of γ_\circ as in (2.1), made by cutting S^1 at q , then we may choose the triple of maps

$$(\xi_\circ, \xi_m, \xi_\bullet) := (\xi|_{[1/2, 1]} - \xi(1/2), 0_{\mathfrak{h}}, 0_{\mathfrak{t}_\bullet})$$

as a glued lift of $(\gamma_\circ, 0, 0)$. Indeed, $\xi(1/2) \in \Lambda_\circ$. What is more, ξ is constant on $[0, 1/2]$. Therefore,

$$\begin{aligned} \Delta'_{\text{Pth}(\gamma_\circ, 0, 0)} &= (\mathbb{R}\pi_\circ)^{-1}(\xi_\circ(q)) - (\mathbb{R}\pi_\bullet)^{-1}(\xi_\bullet(q)) \\ &= (\mathbb{R}\pi_\circ)^{-1}(\xi(1) - \xi(1/2)) - 0_{\mathfrak{h}} \\ &= (\mathbb{R}\pi_\circ)^{-1}(\xi(1) - \xi(0)) = (\mathbb{R}\pi_\circ)^{-1}(\Delta_{\gamma_\circ}). \end{aligned}$$

One chooses a glued lift $(\eta_\circ, \eta_m, \eta_\bullet)$ for $(\rho_\circ, 0, 0)$ similarly using a lift $\eta: [0, 1] \rightarrow \mathfrak{t}_\circ$ of ρ_\circ . The result is

$$\varepsilon(\Delta'_{\text{Pth}(\gamma_\circ, 0, 0)}, \Delta'_{\text{Pth}(\rho_\circ, 0, 0)}) = \varepsilon_\circ((\mathbb{R}\pi_\circ)^{-1}(\Delta_{\gamma_\circ}), (\mathbb{R}\pi_\circ)^{-1}(\Delta_{\rho_\circ})). \quad (3.29)$$

Filling in these glued lifts in the second line in the definition (3.20) of the bicoloured cocycle results in

$$\begin{aligned} & \mathcal{S}((\xi_\circ, \xi_m, \xi_\bullet), (\eta_\circ, \eta_m, \eta_\bullet)) \\ &= \frac{1}{2} \left[\int_{\curvearrowright} \langle d\xi_\circ, \eta_\circ \rangle_\circ + \int_{\curvearrowleft} \langle d\xi_\bullet, \eta_\bullet \rangle_\bullet + \left\langle \Delta'_{\text{Pth}(\gamma_\circ, 0, 0)}, (\mathbb{R}\pi_\bullet)^{-1}(\eta_\bullet(q)) \right\rangle_\Gamma \right] \\ &= \frac{1}{2} \int_{\curvearrowright} \langle d\xi_\circ, \eta_\circ \rangle_\circ \\ &= \frac{1}{2} \int_{\frac{1}{2}}^1 \langle \xi'(\theta), \eta(\theta) - \eta(1/2) \rangle_\circ d\theta \\ &= \frac{1}{2} \int_{\frac{1}{2}}^1 \langle \xi'(\theta), \eta(\theta) \rangle_\circ d\theta - \frac{1}{2} \langle \Delta_{\gamma_\circ}, \eta(0) \rangle_\circ \\ &= \frac{1}{2} \int_{\frac{1}{2}}^1 \langle \xi'(\theta), \eta(\theta) \rangle_\circ d\theta + \frac{1}{2} \langle \Delta_{\gamma_\circ}, \eta(0) \rangle_\circ - \langle \Delta_{\gamma_\circ}, \eta(0) \rangle_\circ. \end{aligned}$$

Because $\eta(0) \in \Lambda_\circ$ and so $\langle \Delta_{\gamma_\circ}, \eta(0) \rangle \in \mathbb{Z}$, this is modulo \mathbb{Z} the same formula as for $S_\circ(\xi, \eta)$ in (2.7). Together with (3.29) this proves the claim (3.27).

3.2.4 Disjoint-commutativity of central extensions

We will prove a disjoint-commutativity property for $\tilde{L}(T_\circ, H, T_\bullet)$ similar to the one in Theorem 2.2.3 for unicoloured torus loop groups. For this we will need a more restrictive notion of *interval* than we used in the unicoloured case:

Definition 3.2.9. (After [BDH15, Section 1.A].¹⁰) A *bicoloured interval* on S^1 is an interval on S^1 which does not contain both points p and q , and if it does contain one of them it does so in its interior.

A bicoloured interval is therefore either contained in the interior of \curvearrowright or \curvearrowleft , or it is split into two (non-singleton) subintervals along the point from $\{p, q\}$ it contains.

The precise statement we will show is as follows:

¹⁰Our definition does not agree exactly with that of [BDH15] because those authors additionally require the datum of a local coordinate around the colour-changing point.

Theorem 3.2.10. *Let (γ, z) and (ρ, w) be two elements of the central extension $\tilde{L}(T_\circ, H, T_\bullet)$ such that the supports of γ and ρ are contained in two disjoint bicoloured intervals on S^1 , respectively. Then (γ, z) and (ρ, w) commute.*

Remember that here we are using the notion of support of Definition 3.1.2.

As preparation for the proof of Theorem 3.2.10 we will first simplify the commutator map of $\tilde{L}(T_\circ, H, T_\bullet)$ without assuming anything about supports of loops:

Proposition 3.2.11. *Let $\gamma = (\gamma_\circ, \gamma_m, \gamma_\bullet)$ and $\rho = (\rho_\circ, \rho_m, \rho_\bullet)$ be in $L(T_\circ, H, T_\bullet)$ and let the 2-cocycle c and the function S be as in (3.20) defined in terms of glued lifts $\xi = (\xi_\circ, \xi_m, \xi_\bullet)$ and $\eta = (\eta_\circ, \eta_m, \eta_\bullet)$ of γ and ρ , respectively. Then*

$$c(\gamma, \rho)c(\rho, \gamma)^{-1} = b(\Delta'_{\text{Pth}\gamma}, \Delta'_{\text{Pth}\rho})e^{2\pi i(S(\xi, \eta) - S(\eta, \xi))}, \tag{3.30}$$

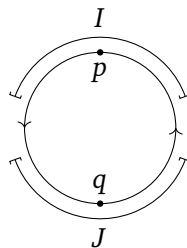
where

$$S(\xi, \eta) - S(\eta, \xi) = \int_{\xi} \langle d\xi_\circ, \eta_\circ \rangle_\circ + \int_{\eta} \langle d\xi_\bullet, \eta_\bullet \rangle_\bullet - \frac{1}{2} \langle \Delta'_{\text{Pth}\rho}, \Delta'_{\text{Pth}\gamma} \rangle_\Gamma - \langle \Delta'_{\text{Pth}\rho}, (\mathbb{R}\pi_\bullet)^{-1}(\xi_\bullet(q)) \rangle_\Gamma. \tag{3.31}$$

The above Proposition can be proven via the definition (3.18) by taking the result (2.9) about unicoloured central extensions and rewriting it in a bicoloured fashion using Lemma 3.1.7(iii). A different method is to imitate the steps in the proof of Theorem 2.2.3 for the bicoloured expression (3.20) instead, using Lemma 3.1.7(i)(b) once.

With this calculation in hand we are ready for

Proof of Theorem 3.2.10. We must show that the commutator (3.30) vanishes. Write I for the bicoloured interval containing the support of γ and J for the one corresponding to ρ . After possibly enlarging them we may assume that either I contains p and J contains q , or the other way around. Suppose we are in the first situation, as for example in the picture



Consider the first term in (3.31). Since ξ_\circ is constant with value $\xi_\circ(q) \in \Lambda_\circ$ outside of $I \cap \mathcal{C}$, this integral is actually only taken over $I \cap \mathcal{C}$. Since I and J are disjoint, η_\circ is constant with value $\eta_\circ(p) \in \Lambda_\circ$ on $I \cap \mathcal{C}$. We may therefore write

$$\int_{\mathcal{C}} \langle d\xi_\circ, \eta_\circ \rangle_\circ = \langle \xi_\circ(q) - \xi_\circ(p), \eta_\circ(p) \rangle_\circ. \quad (3.32)$$

Similarly, the second term in (3.31) is

$$\int_{\mathcal{C}} \langle d\xi_\bullet, \eta_\bullet \rangle_\bullet = \langle \xi_\bullet(p) - \xi_\bullet(q), \eta_\bullet(p) \rangle_\bullet, \quad (3.33)$$

where $\eta_\bullet(p) \in \Lambda_\bullet$. Next, apply $(\mathbb{R}\pi_\circ)^{-1}$ to all entries in the right hand side of (3.32) and $(\mathbb{R}\pi_\bullet)^{-1}$ to all entries in the right hand side of (3.33). We then recall from Lemma 3.1.7(i)(b) that $(\mathbb{R}\pi_\circ)^{-1}(\eta_\circ(p))$ and $(\mathbb{R}\pi_\bullet)^{-1}(\eta_\bullet(p))$ are equal, and moreover, that they are equal to an element $\nu \in \Lambda_\circ \cap \Lambda_\bullet$ since $(\mathbb{R}\pi_{\circ/\bullet})^{-1}(\eta_{\circ/\bullet}(p)) \in (\mathbb{R}\pi_{\circ/\bullet})^{-1}(\Lambda_{\circ/\bullet})$. So we can now write the sum of the first two terms of (3.31) as

$$\langle (\mathbb{R}\pi_\circ)^{-1}(\xi_\circ(q)) - (\mathbb{R}\pi_\bullet)^{-1}(\xi_\bullet(q)), \nu \rangle_\Gamma = \langle \Delta'_{\text{Pth}\gamma}, \nu \rangle_\Gamma.$$

This lies in \mathbb{Z} since $\Delta'_{\text{Pth}\gamma} \in \Lambda_\circ - \Lambda_\bullet$ and $\nu \in \Lambda_\circ \cap \Lambda_\bullet$. We may therefore ignore these first two terms of (3.31).

We will now focus on the last two terms of (3.31). Because of our assumption on the support of γ , if we want to write $\Delta'_{\text{Pth}\gamma} = \lambda_\circ - \lambda_\bullet$ with $\lambda_{\circ/\bullet} \in (\mathbb{R}\pi_{\circ/\bullet})^{-1}(\Lambda_{\circ/\bullet})$, we may choose $\lambda_{\circ/\bullet} := (\mathbb{R}\pi_{\circ/\bullet})^{-1}(\xi_{\circ/\bullet}(q))$. Hence we have

$$b_0(\Delta'_{\text{Pth}\gamma}, \Delta'_{\text{Pth}\rho}) = \frac{1}{2} \langle \Delta'_{\text{Pth}\rho}, \Delta'_{\text{Pth}\gamma} \rangle_\Gamma + \langle \Delta'_{\text{Pth}\rho}, (\mathbb{R}\pi_\bullet)^{-1}(\xi_\bullet(q)) \rangle_\Gamma.$$

We therefore see that $b(\Delta'_{\text{Pth}\gamma}, \Delta'_{\text{Pth}\rho})$ cancels out the contribution of the exponential of the last two terms of (3.31). We conclude that in this case indeed $c(\gamma, \rho)c(\rho, \gamma)^{-1} = 1$ and that (γ, z) and (ρ, w) commute.

Suppose that instead the interval I contains q and J contains p . A similar reasoning as in the first situation shows that we can then write the sum of the first two terms of (3.31) as

$$\begin{aligned} & \langle (\mathbb{R}\pi_\circ)^{-1}(\xi_\circ(q)) - (\mathbb{R}\pi_\circ)^{-1}(\xi_\circ(p)), (\mathbb{R}\pi_\circ)^{-1}(\eta_\circ(q)) \rangle_\Gamma + \\ & \langle (\mathbb{R}\pi_\bullet)^{-1}(\xi_\bullet(p)) - (\mathbb{R}\pi_\bullet)^{-1}(\xi_\bullet(q)), (\mathbb{R}\pi_\bullet)^{-1}(\eta_\bullet(q)) \rangle_\Gamma. \end{aligned}$$

If we substitute $(\mathbb{R}\pi_\circ)^{-1}(\eta_\circ(q))$ by $\Delta'_{\text{Pth}\rho} + (\mathbb{R}\pi_\bullet)^{-1}(\eta_\bullet(q))$ in the above and use that $(\mathbb{R}\pi_\circ)^{-1}(\xi_\circ(p)) = (\mathbb{R}\pi_\bullet)^{-1}(\xi_\bullet(p))$, this becomes

$$\begin{aligned} & \left\langle (\mathbb{R}\pi_\circ)^{-1}(\xi_\circ(q)) - (\mathbb{R}\pi_\circ)^{-1}(\xi_\circ(p)), \Delta'_{\text{Pth}\rho} \right\rangle_\Gamma + \\ & \left\langle (\mathbb{R}\pi_\circ)^{-1}(\xi_\circ(q)) - (\mathbb{R}\pi_\bullet)^{-1}(\xi_\bullet(q)), (\mathbb{R}\pi_\bullet)^{-1}(\eta_\bullet(q)) \right\rangle_\Gamma. \end{aligned}$$

Note that what is in the first slot in this second term is $\Delta'_{\text{Pth}\gamma}$.

Having rewritten these first two terms of (3.31), we fill it back in there so as to get

$$\begin{aligned} S(\xi, \eta) - S(\eta, \xi) &= \frac{1}{2} \langle \Delta'_{\text{Pth}\rho}, \Delta'_{\text{Pth}\gamma} \rangle_\Gamma - \left\langle (\mathbb{R}\pi_\circ)^{-1}(\xi_\circ(p)), \Delta'_{\text{Pth}\rho} \right\rangle_\Gamma + \\ & \left\langle \Delta'_{\text{Pth}\gamma}, (\mathbb{R}\pi_\bullet)^{-1}(\eta_\bullet(q)) \right\rangle_\Gamma. \end{aligned}$$

Because $(\mathbb{R}\pi_\circ)^{-1}(\xi_\circ(p))$ and $(\mathbb{R}\pi_\bullet)^{-1}(\xi_\bullet(p))$ are equal, and moreover, they are equal to an element $\nu \in \Lambda_\circ \cap \Lambda_\bullet$, we see that the second term in the above lies in \mathbb{Z} . We may therefore ignore it from now on.

It follows from the definition of b_0 and the integrality of the lattices $\Lambda_{\circ/\bullet}$ that

$$\begin{aligned} b_0(\Delta'_{\text{Pth}\gamma}, \Delta'_{\text{Pth}\rho}) &= -\frac{1}{2} \langle \Delta'_{\text{Pth}\rho}, \Delta'_{\text{Pth}\gamma} \rangle_\Gamma + \langle \Delta'_{\text{Pth}\rho}, \lambda_\circ \rangle_\Gamma \\ &\equiv -\frac{1}{2} \langle \Delta'_{\text{Pth}\rho}, \Delta'_{\text{Pth}\gamma} \rangle_\Gamma - \langle \Delta'_{\text{Pth}\gamma}, \mu_\bullet \rangle_\Gamma \pmod{\mathbb{Z}}, \end{aligned}$$

where we have written $\Delta'_{\text{Pth}\gamma} = \lambda_\circ - \lambda_\bullet$ and $\Delta'_{\text{Pth}\rho} = \mu_\circ - \mu_\bullet$ for some $\lambda_{\circ/\bullet}, \mu_{\circ/\bullet} \in (\mathbb{R}\pi_{\circ/\bullet})^{-1}(\Lambda_{\circ/\bullet})$. Because of our assumptions on the support of J , we may choose $\mu_{\circ/\bullet} := (\mathbb{R}\pi_{\circ/\bullet})^{-1}(\eta_{\circ/\bullet}(q))$. Therefore, $b(\Delta'_{\text{Pth}\gamma}, \Delta'_{\text{Pth}\rho})$ cancels against

$$e^{2\pi i(S(\xi, \eta) - S(\eta, \xi))}$$

in (3.30). We conclude that also in this case $c(\gamma, \rho)c(\rho, \gamma)^{-1} = 1$. \square

3.3 Actions of covers of $\text{Diff}_+(S^1)$ on central extensions

We assume the setup of Section 3.2. That is, we take the input data from Ingredients 3.2.7 as a given and use the notations $T_{\circ/\bullet}$, H , $\mathfrak{t}_{\circ/\bullet}$ and \mathfrak{h} from Section 3.1 for the tori and Lie algebras associated to the lattices $\Lambda_{\circ/\bullet}$ and

Γ . We constructed from this data central extensions $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ and $\tilde{L}(T_\circ, H, T_\bullet)$ of the group of paths $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ and the bicoloured torus loop group $L(T_\circ, H, T_\bullet)$, respectively.

Recall from Proposition 3.1.3 that $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ and $L(T_\circ, H, T_\bullet)$ both carry actions of the group $\text{Diff}_+^{(n)}(S^1)$, where n is the smallest positive integer such that $n(\Lambda_\circ - \Lambda_\bullet) \subseteq \Gamma$. In the proof of that Proposition the action on $L(T_\circ, H, T_\bullet)$ was built by first constructing it on $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ and then lifting it along the homomorphism Pth (see Section 3.1.1 for the definition of Pth). In this section we will similarly first construct a $\text{Diff}_+^{(n)}(S^1)$ -action on the central extension $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$. This will be done in almost exactly the same way as for the unicoloured case in Section 2.3. Next, we will lift this action to $\tilde{L}(T_\circ, H, T_\bullet)$.

3.3.1 The action of $\text{Diff}_+^{(n)}(S^1)$ on $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$

The proof of Proposition 2.3.1 also calculates the failure for $\text{Diff}_+^{(n)}(S^1)$ to preserve the cocycle c' defining $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$. The result is as follows.

Proposition 3.3.1. *Let $\gamma, \rho \in P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ and $\Xi, H: \mathbb{R} \rightarrow \mathfrak{h}$ be the quasi-periodic extensions to \mathbb{R} of choices of lifts $\xi, \eta: [0, 1] \rightarrow \mathfrak{h}$ of γ and ρ respectively. Take $[\Phi] \in \text{Diff}_+^{(n)}(S^1)$, where $\Phi \in \text{Diff}_+^{(\infty)}(S^1)$ is a choice of representative. Then*

$$c'([\Phi]^*\gamma, [\Phi]^*\rho) = \varepsilon(\Delta'_\gamma, \Delta'_\rho) e^{2\pi i S'(\Phi^*\Xi, \Phi^*H)}$$

and

$$\begin{aligned} S'(\Phi^*\Xi, \Phi^*H) &= S'(\Xi, H) + \frac{1}{2} \left\langle \Delta'_\gamma, H(\Phi^{-1}(0)) - H(0) \right\rangle_\Gamma + \\ &\quad \frac{1}{2} \left\langle \Xi(\Phi^{-1}(0)) - \Xi(0), \Delta'_\rho \right\rangle_\Gamma. \end{aligned}$$

With this precise expression for the failure in hand we can now define the action of $\text{Diff}_+^{(n)}(S^1)$ on $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$. Let $[\Phi] \in \text{Diff}_+^{(n)}(S^1)$ and $(\gamma, z) \in \tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$. Then we set

$$[\Phi] \cdot (\gamma, z) := \left([\Phi]^*\gamma, d'([\Phi], \gamma) \cdot z \right), \quad (3.34)$$

where $[\Phi]^*\gamma$ refers to the action of $\text{Diff}_+^{(n)}(S^1)$ on the non-centrally extended group $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ constructed in the proof of Proposition 3.1.3 and

$$d'([\Phi], \gamma) := e^{\pi i \left\langle \Xi(\Phi^{-1}(0)) - \Xi(0), \Delta'_\gamma \right\rangle_\Gamma} \in U(1). \quad (3.35)$$

The value of d' does not depend on the choice of lift Ξ . We show that it neither depends on the representative Φ of the equivalence class $[\Phi]$ in a similar, but slightly more subtle way compared to the unicoloured case. A different choice would namely be of the form $\Phi + nk$ for some $k \in \mathbb{Z}$, and seeing whether $d'([\Phi + nk], \gamma) = d'([\Phi], \gamma)$ comes down to proving that $nk \langle \Delta'_\gamma, \Delta'_\gamma \rangle_\Gamma \in 2\mathbb{Z}$. We may assume that $k = 1$. Because $\Delta'_\gamma \in \Lambda_\circ - \Lambda_\bullet$ we can make a choice of decomposition $\Delta'_\gamma = \lambda_\circ - \lambda_\bullet$ for some $\lambda_{\circ/\bullet} \in (\mathbb{R}\pi_{\circ/\bullet})^{-1}(\Lambda_{\circ/\bullet})$. Then expand as follows:

$$n \langle \Delta'_\gamma, \Delta'_\gamma \rangle_\Gamma = n \langle \lambda_\circ, \lambda_\circ \rangle_\Gamma - 2n \langle \lambda_\circ, \lambda_\bullet \rangle_\Gamma + n \langle \lambda_\bullet, \lambda_\bullet \rangle_\Gamma.$$

Given that the lattices $\Lambda_{\circ/\bullet}$ are even, the two outer terms on the right hand side are obviously in $2\mathbb{Z}$. But so is the middle term since $n\lambda_\circ \in \Gamma$ which gives $\langle n\lambda_\circ, \lambda_\bullet \rangle_\Gamma \in \mathbb{Z}$. Hence d' is well-defined.

The result of Proposition 3.3.1 shows that d' satisfies an equation similar to (2.15), with $[\Phi]$ and c' in place of φ and c respectively. That is,

$$d'([\Phi], \cdot): P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma) \rightarrow U(1)$$

is a 1-cochain exhibiting the 2-cocycle

$$(\gamma, \rho) \mapsto c'([\Phi]^* \gamma, [\Phi]^* \rho) c'(\gamma, \rho)^{-1}$$

as a 2-coboundary. Therefore (3.34) defines an automorphism of $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$. That d' is compatible with the composition in $\text{Diff}_+^{(n)}(S^1)$ is proven in literally the same way as in the unicoloured situation. We conclude that (3.34) is a well-defined $\text{Diff}_+^{(n)}(S^1)$ -action on $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$.

3.3.2 The action of $\text{Diff}_+^{(n)}(S^1)$ on $\tilde{L}(T_\circ, H, T_\bullet)$

We are now ready to lift the $\text{Diff}_+^{(n)}(S^1)$ -action from $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ to $\tilde{L}(T_\circ, H, T_\bullet)$. Let $[\Phi] \in \text{Diff}_+^{(n)}(S^1)$ and $(\gamma, z) \in \tilde{L}(T_\circ, H, T_\bullet)$. Then define

$$[\Phi] \cdot (\gamma, z) := \left([\Phi] \cdot \gamma, d'([\Phi], \text{Pth } \gamma) \cdot z \right), \quad (3.36)$$

where $[\Phi] \cdot \gamma$ refers to the action of $\text{Diff}_+^{(n)}(S^1)$ on the non-centrally extended group $L(T_\circ, H, T_\bullet)$ constructed in the proof of Proposition 3.1.3.

It is easily checked that, using that we already confirmed it for the action (3.34) on $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$, also (3.36) defines an action of $\text{Diff}_+^{(n)}(S^1)$ on $\tilde{L}(T_\circ, H, T_\bullet)$. Alternatively, we can use the fact that $\tilde{L}(T_\circ, H, T_\bullet)$ is the pull-back of $\tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ along the $\text{Diff}_+^{(n)}(S^1)$ -equivariant homomorphism

Pth, and then note that such a pullback operation is functorial. By design of the $\text{Diff}_+^{(n)}(S^1)$ -actions, also the homomorphism $\widetilde{\text{Pth}}$ that we defined in Construction 3.2.6 is $\text{Diff}_+^{(n)}(S^1)$ -equivariant.

Proposition 3.3.2 ($\text{Diff}_+^{(n)}(S^1)$ -equivariance with respect to \widetilde{LH}). *The inclusion homomorphism $\widetilde{\text{Bi}}$ in (3.24) is equivariant with respect to the $\text{Diff}_+(S^1)$ -action on \widetilde{LH} defined in Section 2.3 and the $\text{Diff}_+^{(n)}(S^1)$ -action on $\widetilde{L}(T_\circ, H, T_\bullet)$ defined by (3.36).*

Proof. We need to prove that for all $\gamma \in LH$, $z \in U(1)$ and $\Phi \in \text{Diff}_+^{(\infty)}(S^1)$ there holds

$$[\Phi] \cdot (\text{Bi}(\gamma), z) = (\text{Bi}([\Phi]^* \gamma), d([\Phi], \gamma) \cdot z),$$

where d is the 1-cochain on LH defined by (2.19). The left hand side is by definition

$$([\Phi] \cdot \text{Bi}(\gamma), d'([\Phi], \text{Pth}(\text{Bi} \gamma)) \cdot z),$$

and we already noted in Remark 3.1.5 the equivariance of Bi . Hence, what is left to show is that

$$d'([\Phi], \text{Pth}(\text{Bi} \gamma)) = d([\Phi], \gamma).$$

We see that this holds if we compare the expressions (3.35) and (2.19). Indeed, a quasi-periodic lift $\Xi: \mathbb{R} \rightarrow \mathfrak{h}$ of $\text{Pth}(\text{Bi} \gamma)$ is also a lift of γ thanks to the commutativity of the triangle (3.8), and we furthermore already learned that $\Delta'_{\text{Pth}(\text{Bi} \gamma)} = \Delta_\gamma$ in Section 3.2.2. \square

3.4 Irreducible, positive energy representations

Assume the setup of Section 3.2 which allowed us to construct a central extension $\widetilde{L}(T_\circ, H, T_\bullet)$ of the bicoloured torus loop group $L(T_\circ, H, T_\bullet)$. In this section we will construct and classify the irreducible, positive energy representations of $\widetilde{L}(T_\circ, H, T_\bullet)$. This will be done in a way entirely similar to our work for unicoloured central extensions \widetilde{LT} . The role of the normal subgroup $(\widetilde{LT})_0$ will in the bicoloured situation be played by a certain normal subgroup denoted by $\widetilde{\ker}(\Delta' \circ \text{Pth})$. Let us therefore define this group and try to understand its structure.

We begin by studying the situation before taking central extensions. The short exact sequence (3.5) involving the groups $L(T_\circ, H, T_\bullet)$ and $P(H, (\Lambda_\circ - \Lambda_\bullet)/H)$ does not split in general. However, its subsequence

$$0 \rightarrow \frac{\Lambda_\circ \cap \Lambda_\bullet}{\Gamma} \rightarrow (\text{Pth})^{-1}(\iota(LH)) \xrightarrow{\text{Pth}} \iota(LH) \rightarrow 0 \quad (3.37)$$

does split. The subgroup $(\text{Pth})^{-1}(\iota(LH))$ namely consists of those bicoloured loops $(\gamma_\circ, \gamma_m, \gamma_\bullet)$ for which the unique lifts $\hat{\gamma}_{\circ/\bullet}$ to H of $\gamma_{\circ/\bullet}$ (see Section 3.1.1 for their definitions) that match at p also match with each other at q , but need not necessarily match with $\gamma_m(q)$. (It might be helpful to compare this with the characterisation of the image of Bi in Section 3.1.4.) That is, there holds $\hat{\gamma}_\circ(p) = \gamma_m(p) = \hat{\gamma}_\bullet(p)$ and $\hat{\gamma}_\circ(q) = \hat{\gamma}_\bullet(q)$, but in general $\gamma_m(q)$ is of the form

$$\hat{\gamma}_\circ(q) + [\nu] = \hat{\gamma}_\bullet(q) + [\nu]$$

for some $[\nu] \in (\Lambda_\circ \cap \Lambda_\bullet)/\Gamma \subseteq H$. Therefore, if we recall the definition of the first arrow in (3.37) from Section 3.1.1, we see that the homomorphism $(\gamma_\circ, \gamma_m, \gamma_\bullet) \mapsto [\nu]$ is a left splitting of (3.37). Equivalently, a right splitting can be given by making the proof in Section 3.1.1 of the surjectivity of the homomorphism Pth more precise: we send a path $\gamma \in \iota(LH)$ to the bicoloured loop $(\gamma_\circ, \gamma_m, \gamma_\bullet)$, where

$$\begin{aligned} \gamma_\circ &:= \text{U}(1)\pi_\circ \circ \gamma|_{[1/2, 1]}: \mathbb{C} \rightarrow T_\circ \\ \gamma_m(p) &:= \gamma(1/2) \in H \\ \gamma_m(q) &:= \gamma(0) = \gamma(1) \in H \\ \gamma_\bullet &:= \text{U}(1)\pi_\bullet \circ \gamma|_{[0, 1/2]}: \mathbb{C} \rightarrow T_\bullet. \end{aligned}$$

These splittings now allow us to understand the structure of $(\text{Pth})^{-1}(\iota(LH))$. There is an isomorphism of abelian groups

$$(\text{Pth})^{-1}(\iota(LH)) \xrightarrow{\sim} \frac{\Lambda_\circ \cap \Lambda_\bullet}{\Gamma} \oplus \iota(LH) \quad (3.38)$$

sending a bicoloured loop $\gamma = (\gamma_\circ, \gamma_m, \gamma_\bullet)$ to the pair $([\nu], \text{Pth } \gamma)$. The $\text{Diff}_+^{(n)}(S^1)$ -action on the full group $L(T_\circ, H, T_\bullet)$ descends to $\text{Diff}_+(S^1)$ when restricted to the subgroup $(\text{Pth})^{-1}(\iota(LH))$. This is because the shift diffeomorphism $\theta \mapsto \theta + 1$ of \mathbb{R} , acting on $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$, acts trivially when restricted to $\iota(LH)$. The isomorphism (3.38) is $\text{Diff}_+(S^1)$ -equivariant if we let $\text{Diff}_+(S^1)$ act on the right hand side by only affecting the $\iota(LH)$ -summand thanks to the equivariance of the homomorphism Pth .

Now define a central extension $\tilde{L}H$ of LH using the input data listed in Section 3.2.2. As explained in Remark 3.2.8, the homomorphism ι of abelian groups then lifts to a homomorphism

$$\tilde{\iota}: \tilde{L}H \hookrightarrow \tilde{P}(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$$

of non-abelian groups which is the identity on the central subgroups $\text{U}(1)$. The isomorphism (3.38) then obviously lifts to a $\text{Diff}_+(S^1)$ -equivariant isomorphism

from the restriction of $\widetilde{L}(T_\bullet, H, T_\bullet)$ to $(\text{Pth})^{-1}(\iota(LH))$ towards

$$\frac{\Lambda_\bullet \cap \Lambda_\bullet}{\Gamma} \oplus \widetilde{\iota}(\widetilde{LH}). \quad (3.39)$$

It sends an element (γ, z) to $([\nu], (\text{Pth}\gamma, z))$.

The subgroup $(\text{Pth})^{-1}(\iota(LH)_0)$ of $(\text{Pth})^{-1}(\iota(LH))$ deserves special attention for the study of the representation theory of $\widetilde{L}(T_\bullet, H, T_\bullet)$ we are about to commence. It is the inverse image under Pth of the identity component $\iota(LH)_0$ of $P(H, (\Lambda_\bullet - \Lambda_\bullet)/\Gamma)$, and contains the identity component $\ker \Delta$ of $L(T_\bullet, H, T_\bullet)$ (strictly, unless $\Gamma = \Lambda_\bullet \cap \Lambda_\bullet$ so that Pth is an isomorphism). An equivalent characterisation of $(\text{Pth})^{-1}(\iota(LH)_0)$ is that it is the kernel of the composite homomorphism

$$L(T_\bullet, H, T_\bullet) \xrightarrow{P} P(H, (\Lambda_\bullet - \Lambda_\bullet)/\Gamma) \xrightarrow{\Delta'} \Lambda_\bullet - \Lambda_\bullet.$$

We will henceforth use the notation $\ker(\Delta' \circ \text{Pth})$ instead.

We write $\widetilde{\ker}(\Delta' \circ \text{Pth})$ for the restriction of $\widetilde{L}(T_\bullet, H, T_\bullet)$ to $\ker(\Delta' \circ \text{Pth})$ and observe that the restriction of the aforementioned isomorphism having codomain (3.39) takes the form

$$\widetilde{\ker}(\Delta' \circ \text{Pth}) \xrightarrow{\sim} \frac{\Lambda_\bullet \cap \Lambda_\bullet}{\Gamma} \times \widetilde{\iota}(\widetilde{LH})_0. \quad (3.40)$$

3.4.1 Irreducible representations of $\widetilde{\ker}(\Delta' \circ \text{Pth})$

Recall from Sections 2.5.1 and 2.5.2 that we are able to equip the abelian group $(LH)_0$ and its central extension $(\widetilde{LH})_0$ with structures of topological groups using the bi-additive form on Γ . Via the isomorphisms (3.38) and (3.40) the groups $\ker(\Delta' \circ \text{Pth})$ and $\widetilde{\ker}(\Delta' \circ \text{Pth})$ then acquire topological group structures as well if we give the finite abelian group $(\Lambda_\bullet \cap \Lambda_\bullet)/\Gamma$ the discrete topology.

Now define for every pair of characters χ and l of the finite abelian group $(\Lambda_\bullet \cap \Lambda_\bullet)/\Gamma$ and the torus H , respectively, a representation $W_{\chi, l}$ of $\widetilde{\ker}(\Delta' \circ \text{Pth})$ on the Hilbert space tensor product

$$\mathcal{S}_{\chi, l} := \mathbb{C}_\chi \otimes_{\mathbb{C}} \mathcal{S}_l := \mathbb{C}_\chi \otimes_{\mathbb{C}} \mathbb{C}_l \otimes_{\mathbb{C}} \mathcal{S},$$

where \mathbb{C}_χ and \mathbb{C}_l denote copies of \mathbb{C} , as follows. Let $(\gamma, z) \in \widetilde{\ker}(\Delta' \circ \text{Pth})$ and consider its image

$$([\nu], (\text{Pth}\gamma, z)) \in \frac{\Lambda_\bullet \cap \Lambda_\bullet}{\Gamma} \times \widetilde{\iota}(\widetilde{LH})_0$$

under the isomorphism (3.40). Then make $[\nu]$ (which can alternatively be written as $\gamma_m(q) - \text{Pth}(\gamma)(1)$) act on \mathbb{C}_χ via χ , and let $(\text{Pth } \gamma, z)$ act on \mathcal{S}_l via the representation W_l of \widetilde{LH} defined in Section 2.5.2. That is, $W_{\chi,l}$ is the tensor product representation of χ and W_l . It is irreducible because χ and W_l are.

Remark 3.4.1. To clarify how $W_{\chi,l}$ depends on l , we note that

$$W_{\chi,l}(\gamma, z) = e^{2\pi i \langle l, \text{avg } \hat{\xi} \rangle_\Gamma} \cdot W_{\chi,0}(\gamma, z),$$

where $\hat{\xi} : [0, 1] \rightarrow \mathfrak{h}$ is any lift of $\text{Pth } \gamma$ as in (3.9) and

$$\text{avg } \hat{\xi} := \int_0^1 \hat{\xi}(\theta) d\theta.$$

Let m be, as in Section 2.5.2, the smallest positive integer such that $m\langle l, l \rangle \in 2\mathbb{Z}$. Now define a representation $R_{\chi,l}$ of $\text{Rot}^{(m)}(S^1)$ on $\mathcal{S}_{\chi,l}$ by acting as the identity on the tensor factor \mathbb{C}_χ and as R_l on \mathcal{S}_l . So $R_{\chi,l}$ can be said to be equal to R_l and the subscript χ is actually irrelevant—it serves, just like in the notation $\mathcal{S}_{\chi,l}$, as a reminder that $R_{\chi,l}$ is associated to the representation $W_{\chi,l}$ and the latter *does* depend on χ .

Proposition 3.4.2. *The $\text{Rot}^{(m)}(S^1)$ -action $R_{\chi,l}$ on the Hilbert space $\mathcal{S}_{\chi,l}$ intertwines in the manner (A.8) with the representation $W_{\chi,l}$ of $\widetilde{\ker}(\Delta' \circ \text{Pth})$.*

Proof. Let $[\Phi_\theta] \in \text{Rot}^{(m)}(S^1)$, $(\gamma, z) \in \widetilde{\ker}(\Delta' \circ \text{Pth})$ and $1 \otimes v \in \mathcal{S}_{\chi,l}$ a vector with $v \in \mathcal{S}_l$. Then there holds on the one hand

$$\begin{aligned} R_{\chi,l}[\Phi_\theta]W_{\chi,l}(\gamma, z)R_{\chi,l}[\Phi_\theta]^*(1 \otimes v) &= R_{\chi,l}[\Phi_\theta]W_{\chi,l}(\gamma, z)(1 \otimes R_l[\Phi_\theta]^*(v)) \\ &= R_{\chi,l}[\Phi_\theta] \left(\chi(\gamma_m(q) - \text{Pth}(\gamma)(1)) \otimes W_l(\text{Pth } \gamma, z)R_l[\Phi_\theta]^*(v) \right) \\ &= \chi(\gamma_m(q) - \text{Pth}(\gamma)(1)) \otimes R_l[\Phi_\theta]W_l(\text{Pth } \gamma, z)R_l[\Phi_\theta]^*(v), \end{aligned}$$

and because of the way that R_l intertwines with W_l we can write

$$R_l[\Phi_\theta]W_l(\text{Pth } \gamma, z)R_l[\Phi_\theta]^* = W_l([\Phi_\theta] \cdot (\text{Pth } \gamma, z)) = W_l([\Phi_\theta]^* \text{Pth}(\gamma), z).$$

On the other hand, since also $[\Phi_\theta] \cdot (\gamma, z) = ([\Phi_\theta]^* \gamma, z)$,

$$\begin{aligned} W_{\chi,l}([\Phi_\theta] \cdot (\gamma, z))(1 \otimes v) &= \chi \left(([\Phi_\theta]^* \gamma)_m(q) - \text{Pth}([\Phi_\theta]^* \gamma)(1) \right) \otimes W_l(\text{Pth}[\Phi_\theta]^* \gamma, z). \end{aligned}$$

We have by the definition of the action of $\text{Diff}_+^{(n)}(S^1)$ on $L(T_\circ, H, T_\bullet)$ in the proof of Proposition 3.1.3 that $\text{Pth}([\Phi_\theta]^*\gamma) = [\Phi_\theta] \cdot (\text{Pth } \gamma)$ and that

$$\begin{aligned} &([\Phi_\theta]^*\gamma)_m(q) - \text{Pth}([\Phi_\theta]^*\gamma)(1) \\ &= \gamma_m(q) - (\text{Pth } \gamma)(0) + ([\Phi_\theta]^* \text{Pth}(\gamma))(0) - ([\Phi_\theta]^* \text{Pth}(\gamma))(1). \end{aligned}$$

Because both $\text{Pth } \gamma$ and $[\Phi_\theta]^* \text{Pth}(\gamma)$ lie in $\iota(LH)$ we have $(\text{Pth } \gamma)(0) = (\text{Pth } \gamma)(1)$ and

$$([\Phi_\theta]^* \text{Pth}(\gamma))(0) = ([\Phi_\theta]^* \text{Pth}(\gamma))(1).$$

We conclude that indeed

$$R_{\chi,l}[\Phi_\theta]W_{\chi,l}(\gamma, z)R_{\chi,l}[\Phi_\theta]^* = W_{\chi,l}([\Phi_\theta] \cdot (\gamma, z)). \quad \square$$

If $a \in (1/m)\mathbb{Z}$ is an energy level, then the a -th energy eigenspace $\mathcal{S}_{\chi,l}(a)$ for $R_{\chi,l}$ is simply $\mathbb{C}_\chi \otimes \mathcal{S}_l(a)$. Since we already concluded in Proposition 2.5.4 that $\mathcal{S}_l(a)$ is zero for $a < 0$ it follows that

Proposition 3.4.3. *The intertwining $\text{Rot}^{(m)}(S^1)$ -action $R_{\chi,l}$ on the representation $W_{\chi,l}$ of $\widetilde{\ker}(\Delta' \circ \text{Pth})$ is of positive energy.*

The aforementioned observation implies more precisely that the character of $R_{\chi,l}$ equals that of R_l :

$$\text{ch}_{R_{\chi,l}}(q) = \text{ch}_{R_l}(q) = q^{\dim H/24} q^{(l,l)/2} \eta(q)^{-\dim H}. \quad (3.41)$$

We define the corresponding normalised character to be

$$Z_{W_{\chi,l}}(q) := q^{-\dim H/24} \text{ch}_{R_{\chi,l}}(q) = q^{(l,l)/2} \eta(q)^{-\dim H}.$$

We have thus constructed a 2-parameter family of mutually non-isomorphic, irreducible, positive energy representations $W_{\chi,l}$ of $\widetilde{\ker}(\Delta' \circ \text{Pth})$. It exhausts the class of such representations:

Theorem 3.4.4. *Every irreducible, positive energy representation of $\widetilde{\ker}(\Delta' \circ \text{Pth})$ such that the central subgroup $U(1)$ acts as $z \mapsto z$ is isomorphic to $W_{\chi,l}$ for some characters χ and l of $(\Lambda_\circ \cap \Lambda_\bullet)/\Gamma$ and H , respectively.*

Proof. The proof is identical to that of Theorem 2.5.5. We first use the $\text{Rot}(S^1)$ -equivariant isomorphism (3.40) to see an arbitrary such representation Q as an irreducible, positive energy representation of $\widetilde{V}\mathfrak{h}$. Here, $\widetilde{V}\mathfrak{h}$ is the Heisenberg group sitting inside $(\widetilde{LH})_0$ as described in Section 2.2.1. An appeal to the unicity result Theorem 2.5.2 for $\widetilde{V}\mathfrak{h}$ then concludes the argument. \square

3.4.2 Irreducible representations of $\widetilde{L}(T_\circ, H, T_\bullet)$

By Lemma A.2.4 there exists a unique topological group structure on $L(T_\circ, H, T_\bullet)$ that is naturally induced by that of its subgroup $\ker(\Delta' \circ \text{Pth})$ under the requirement that this subgroup is open in $L(T_\circ, H, T_\bullet)$. Because for the homomorphism

$$\Delta': P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma) \rightarrow \Lambda_\circ - \Lambda_\bullet$$

there holds $\ker \Delta' = \iota(LH)_0$, we similarly have a unique structure of a topological group on $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ induced by that of $(LH)_0$ such that $\iota(LH)_0$ is open in the former group.

Proposition 3.4.5. *There exists a unique structure of a topological group on the central extension $\widetilde{L}(T_\circ, H, T_\bullet)$ such that $\ker(\Delta' \circ \text{Pth})$ is open in $\widetilde{L}(T_\circ, H, T_\bullet)$.*

Proof. We show this along the same lines as Proposition 2.5.7. That is, we need to check whether for every fixed bicoloured loop $\gamma \in L(T_\circ, H, T_\bullet)$ the map $\ker(\Delta' \circ \text{Pth}) \rightarrow U(1)$ given by

$$\rho \mapsto c(\gamma, \rho)c(\gamma + \rho, -\gamma) = c'(\text{Pth } \gamma, \text{Pth } \rho)c'(\text{Pth } \gamma + \text{Pth } \rho, -\text{Pth } \gamma)$$

is continuous. Note that the homomorphism Δ' admits a splitting. It is given by defining for $\lambda \in \Lambda_\circ - \Lambda_\bullet$ a path $\gamma_\lambda \in P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ as the projection on H of the Lie algebra-valued path $[0, 1] \rightarrow \mathfrak{h}$, $\theta \mapsto \theta\lambda$. Therefore, there exists a decomposition

$$P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma) \xrightarrow{\sim} \iota(LH)_0 \oplus (\Lambda_\circ - \Lambda_\bullet) \xrightarrow{\sim} H \oplus V\mathfrak{h} \oplus (\Lambda_\circ - \Lambda_\bullet)$$

generalising the isomorphism (2.5) in the unicoloured situation. The remainder of the argument now proceeds as in the proof of Proposition 2.5.7. \square

The knowledge of the irreducible, positive energy representations $W_{\chi, l}$ of the subgroup $\widetilde{\ker}(\Delta' \circ \text{Pth})$ now allows us to construct and classify the same class of representations of the full group $\widetilde{L}(T_\circ, H, T_\bullet)$, starting as follows. Let us take such a $W_{\chi, l}$ for characters χ and l of $(\Lambda_\circ \cap \Lambda_\bullet)/\Gamma$ and H , respectively, and consider the induced representation

$$\text{Ind}_{\widetilde{\ker}(\Delta' \circ \text{Pth})}^{\widetilde{L}(T_\circ, H, T_\bullet)} W_{\chi, l}$$

of $\widetilde{L}(T_\circ, H, T_\bullet)$. We will shorten it and its underlying Hilbert space to $\text{Ind } W_{\chi, l}$ and $\text{Ind } \mathcal{S}_{\chi, l}$, respectively. We refer to Appendix A.3.1 and the analogous construction for unicoloured torus loop groups in Section 2.5.3 for details on how this Hilbert space and its action of $\widetilde{L}(T_\circ, H, T_\bullet)$ are defined.

Now take m to be the smallest positive integer such that both $m \geq n$ and $m(l, l) \in 2\mathbb{Z}$. The first condition implies according to Section 3.3.2 that $\text{Rot}^{(m)}(S^1)$ acts on $\widetilde{L}(T_\circ, H, T_\bullet)$, while the second one means by Section 3.4.1 that $\mathcal{S}_{\chi, l}$ carries a positive energy representation $R_{\chi, l}$ of $\text{Rot}^{(m)}(S^1)$ which intertwines with $W_{\chi, l}$. Then define a representation $\text{Ind} R_{\chi, l}$ on $\text{Ind} \mathcal{S}_{\chi, l}$ in terms of $R_{\chi, l}$ in exactly the same way as done in Section 2.5.3. It satisfies the intertwining property (A.8) with respect to $\text{Ind} W_{\chi, l}$ because $R_{\chi, l}$ does so with respect to $W_{\chi, l}$.

To study $\text{Ind} W_{\chi, l}$ we first calculate the representations of $\widetilde{\ker}(\Delta' \circ \text{Pth})$ that are conjugate to $W_{\chi, l}$:

Lemma 3.4.6. (Compare with Lemma 2.5.8.) *Let (γ, z) be an element of the group $\widetilde{L}(T_\circ, H, T_\bullet)$ that is not contained in the (normal) subgroup $\widetilde{\ker}(\Delta' \circ \text{Pth})$ and consider the representation $W_{\chi, l}^{(\gamma, z)}$ of $\widetilde{\ker}(\Delta' \circ \text{Pth})$ conjugate to $W_{\chi, l}$, defined by*

$$W_{\chi, l}^{(\gamma, z)}(\rho, w) := W_{\chi, l}((\gamma, z)^{-1}(\rho, w)(\gamma, z))$$

for $(\rho, w) \in \widetilde{\ker}(\Delta' \circ \text{Pth})$. Then $W_{\chi, l}^{(\gamma, z)}$ is the tensor product representation of $W_{\chi, l}$ and the character

$$\widetilde{\ker}(\Delta' \circ \text{Pth}) \twoheadrightarrow \ker(\Delta' \circ \text{Pth}) \rightarrow \text{U}(1), \quad (\rho, w) \mapsto \rho \mapsto c(\rho, \gamma)c(\gamma, \rho)^{-1}, \quad (3.42)$$

where $\text{U}(1)$ acts on \mathbb{C} as $z \mapsto z$. In turn, for any glued lifts $\xi = (\xi_\circ, \xi_m, \xi_\bullet)$ and $\eta = (\eta_\circ, \eta_m, \eta_\bullet)$ of γ and ρ , respectively, there holds

$$c(\rho, \gamma)c(\gamma, \rho)^{-1} = e^{2\pi i(S(\eta, \xi) - S(\xi, \eta))}, \quad (3.43)$$

where

$$S(\eta, \xi) - S(\xi, \eta) = - \int_{\zeta} \langle d\xi_\circ, \eta_\circ \rangle_\circ - \int_{\lambda} \langle d\xi_\bullet, \eta_\bullet \rangle_\bullet. \quad (3.44)$$

The equations (3.43) and (3.44) follow from the expression for the commutator map associated to c we found in Proposition 3.2.11.

Proposition 3.4.7. *The induced representation $\text{Ind} W_{\chi, l}$ of $\widetilde{L}(T_\circ, H, T_\bullet)$ is irreducible.*

Proof. It is sufficient to show that all the conjugate representations $W_{\chi, l}^{(\gamma, z)}$ as in Lemma 3.4.6 are not isomorphic to $W_{\chi, l}$. To do this, we will examine the restriction of $W_{\chi, l}^{(\gamma, z)}$ to the subgroup of $\widetilde{\ker}(\Delta' \circ \text{Pth})$ consisting of the elements of the form $(\rho, 1)$ where ρ is a constant bicoloured loop, that is, $\rho \in \text{Bi}(H)$.

Because this subgroup is canonically isomorphic to $\text{Bi}(H)$, we will denote it as such and write its elements simply as ρ .

Let $\rho \in \text{Bi}(H)$, meaning that there is an $\alpha \in \mathfrak{h}$ such that the maps $\rho_{\circ\bullet}$ are constant with values $\exp_{\circ\bullet}(\mathbb{R}\pi_{\circ\bullet})(\alpha)$ and $\rho_{\text{m}}(p) = \rho_{\text{m}}(q) = \exp_H \alpha$. Then

$$\eta := \left((\mathbb{R}\pi_{\circ})(\alpha), \alpha, (\mathbb{R}\pi_{\bullet})(\alpha) \right)$$

is a glued lift of ρ . Plugging this into (3.44) gives

$$\begin{aligned} S(\eta, \xi) - S(\xi, \eta) &= -\langle \xi_{\circ}(q) - \xi_{\circ}(p), (\mathbb{R}\pi_{\circ})(\alpha) \rangle_{\circ} - \langle \xi_{\bullet}(p) - \xi_{\bullet}(q), (\mathbb{R}\pi_{\bullet})(\alpha) \rangle_{\bullet} \\ &= -\langle \Delta'_{\text{Pth}\gamma}, \alpha \rangle_{\Gamma}. \end{aligned}$$

So (3.42) has $\iota(H)$ acting by the character $-\Delta'_{\text{Pth}\gamma} \in \Lambda_{\circ} - \Lambda_{\bullet} \hookrightarrow \Gamma^{\vee}$, which implies that $W_{\chi,l}^{(\gamma,z)}$ is letting $\text{Bi}(H)$ act by $l - \Delta'_{\text{Pth}\gamma}$. Because $\Delta'_{\text{Pth}\gamma} \neq 0$, we have $l - \Delta'_{\text{Pth}\gamma} \neq l$ and therefore $W_{\chi,l}^{(\gamma,z)}$ and $W_{\chi,l}$ are not isomorphic. \square

The reason we are inducing up representations not from the identity component of $\widetilde{L}(T_{\circ}, H, T_{\bullet})$, but from the larger subgroup $\widetilde{\ker}(\Delta' \circ \text{Pth})$, is precisely to make the step in the above proof true where we note that $\Delta'_{\text{Pth}\gamma} \neq 0$.

Let us examine when these representations $\text{Ind } W_{\chi,l}$ are isomorphic or not. For the next result we will make use of the bicoloured loops $\gamma_{[\lambda_{\circ}, \lambda_{\bullet}]}$ defined in Section 3.1.9 associated to elements $[\lambda_{\circ}, \lambda_{\bullet}] \in (\Lambda_{\circ} \oplus \Lambda_{\bullet})/\Gamma$. Note that if

$$\lambda = (\mathbb{R}\pi_{\circ})^{-1}(\lambda_{\circ}) - (\mathbb{R}\pi_{\bullet})^{-1}(\lambda_{\bullet}) \in \Lambda_{\circ} - \Lambda_{\bullet}$$

and σ is the (left) coset of $\widetilde{\ker}(\Delta' \circ \text{Pth})$ in $\widetilde{L}(T_{\circ}, H, T_{\bullet})$ consisting of all elements (γ, z) such that $\Delta'_{\text{Pth}\gamma} = \lambda$, then the element $(\gamma_{[\lambda_{\circ}, \lambda_{\bullet}]}, 1)$ is a representative of σ .

We furthermore observe that because $\widetilde{\ker}(\Delta' \circ \text{Pth})$ is a normal subgroup of $\widetilde{L}(T_{\circ}, H, T_{\bullet})$, the restriction of $\text{Ind } W_{\chi,l}$ to it restricts to each subspace $\mathcal{S}_{\chi,l}^{\sigma}$ for all cosets σ .

Theorem 3.4.8 (Restriction of $\text{Ind } W_{\chi,l}$ from $\widetilde{L}(T_{\circ}, H, T_{\bullet})$ to $\widetilde{\ker}(\Delta' \circ \text{Pth})$). *Fix a character l of H , an element $\lambda \in \Lambda_{\circ} - \Lambda_{\bullet}$ and let σ be the (left) coset of $\widetilde{\ker}(\Delta' \circ \text{Pth})$ in $\widetilde{L}(T_{\circ}, H, T_{\bullet})$ corresponding to λ . Pick any pre-image $[\lambda_{\circ}, \lambda_{\bullet}]$ of λ under the homomorphism*

$$(\mathbb{R}\pi_{\circ})^{-1} - (\mathbb{R}\pi_{\bullet})^{-1}: \frac{\Lambda_{\circ} \oplus \Lambda_{\bullet}}{\Gamma} \rightarrow \Lambda_{\circ} - \Lambda_{\bullet}.$$

Then the composite unitary map

$$f_{\chi,l}^{\sigma}: \mathcal{S}_{\chi,l}^{\sigma} \xrightarrow{\sim} \mathcal{S}_{\chi,l} \xrightarrow{\sim} \mathcal{S}_{\chi,l-\lambda}, \quad [(\gamma_{[\lambda_{\circ}, \lambda_{\bullet}]}, 1), v] \mapsto v \mapsto v$$

intertwines the representations $(\text{Res Ind } W_{\chi,l})|_{\mathcal{S}_{\chi,l}^\sigma}$ and $W_{\chi,l-\lambda}$ of $\widetilde{\ker}(\Delta' \circ \text{Pth})$ and the representations $\text{Ind } R_{\chi,l}|_{\mathcal{S}_{\chi,l}^\sigma}$ and $R_{\chi,l-\lambda}$ of $\text{Rot}^{(m)}(S^1)$.

Proof. The first map in the composition $f_{\chi,l}^\sigma$ is an isomorphism from the restriction of $\text{Res Ind } W_{\chi,l}$ to $\mathcal{S}_{\chi,l}^\sigma$ to the conjugate representation $W_{\chi,l}^{(\gamma_{[\lambda_\circ, \lambda_\bullet]}, 1)}$. The latter was calculated partially in Lemma 3.4.6. For a more precise result we substitute a specific glued lift $(\xi_\circ, \xi_m, \xi_\bullet)$ of $\gamma_{[\lambda_\circ, \lambda_\bullet]}$ into (3.44), namely the one defined in Section 3.1.9. This gives¹¹

$$\begin{aligned} S(\eta, \xi) - S(\xi, \eta) &= -\left\langle \lambda_\circ - (\mathbb{R}\pi_\circ) \circ (\mathbb{R}\pi_\bullet)^{-1}(\lambda_\bullet), \int_{\mathcal{C}} \eta_\circ(\theta) d\theta \right\rangle_\circ - \\ &\quad \left\langle (\mathbb{R}\pi_\bullet) \circ (\mathbb{R}\pi_\circ)^{-1}(\lambda_\circ) - \lambda_\bullet, \int_{\mathcal{A}} \eta_\bullet(\theta) d\theta \right\rangle_\bullet \\ &= -\left\langle (\mathbb{R}\pi_\circ)^{-1}(\lambda_\circ) - (\mathbb{R}\pi_\bullet)^{-1}(\lambda_\bullet), \right. \\ &\quad \left. (\mathbb{R}\pi_\circ)^{-1} \left(\int_{\mathcal{C}} \eta_\circ(\theta) d\theta \right) + (\mathbb{R}\pi_\bullet)^{-1} \left(\int_{\mathcal{A}} \eta_\bullet(\theta) d\theta \right) \right\rangle_\Gamma \\ &= -\langle \lambda, \text{avg } \hat{\eta} \rangle_\Gamma \end{aligned}$$

and therefore,

$$c(\rho, \gamma_{[\lambda_\circ, \lambda_\bullet]})c(\gamma_{[\lambda_\circ, \lambda_\bullet]}, \rho)^{-1} = e^{-2\pi i \langle \lambda, \text{avg } \hat{\eta} \rangle_\Gamma}.$$

By Remark 3.4.1 we have

$$e^{2\pi i \langle -\lambda, \text{avg } \hat{\eta} \rangle_\Gamma} \cdot W_{\chi,l}(\rho, w)(v) = W_{\chi,l-\lambda}(\rho, w)(v)$$

for all v . We conclude that the second map in the composition $f_{\chi,l}^\sigma$ is an isomorphism from $W_{\chi,l}^{(\gamma_{[\lambda_\circ, \lambda_\bullet]}, 1)}$ to $W_{\chi,l-\lambda}$.

For the second claim of the Theorem, we first assert that m , which we defined to be smallest positive integer such that both $m \geq n$ and $m\langle l, l \rangle \in 2\mathbb{Z}$, is also the smallest positive integer m such that both $m \geq n$ and $m\langle l-\lambda, l-\lambda \rangle \in 2\mathbb{Z}$. We namely have

$$m\langle l-\lambda, l-\lambda \rangle = m\langle l, l \rangle - 2m\langle l, \lambda \rangle + m\langle \lambda, \lambda \rangle.$$

Because $m\lambda \in \Gamma$, there holds $2m\langle l, \lambda \rangle \in 2\mathbb{Z}$. To show that $m\langle \lambda, \lambda \rangle \in 2\mathbb{Z}$ we refer to the proof of the well-definedness of d' in Section 3.3.1. We conclude

¹¹See Remark 3.4.1 for the definition of $\text{avg } \hat{\eta}$.

that $\text{Rot}^{(m)}(S^1)$ acts on both $\mathcal{S}_{\chi,l}$ and $\mathcal{S}_{\chi,l-\lambda}$, although the restrictions of these actions to their respective tensor factors \mathbb{C}_l and $\mathbb{C}_{l-\lambda}$ are different.

In order to now prove that $f_{\chi,l}^\sigma$ intertwines the representations $\text{Ind} R_{\chi,l}|_{\mathcal{S}_{\chi,l}^\sigma}$ and $R_{\chi,l-\lambda}$ by imitating the proof of the analogous statement in Theorem 2.5.10 we require the following observations and calculations. Given the definition of $\gamma_{[\lambda_\circ, \lambda_\bullet]}$, it is clear that

$$\text{Pth}(\gamma_{[\lambda_\circ, \lambda_\bullet]}) = \exp_H \circ \hat{\xi}.$$

Evaluating Δ' on $\text{Pth}(\gamma_{[\lambda_\circ, \lambda_\bullet]})$ therefore gives

$$\hat{\xi}(1) - \hat{\xi}(0) = (\mathbb{R}\pi_\circ)^{-1}(\lambda_\circ) - (\mathbb{R}\pi_\bullet)^{-1}(\lambda_\bullet) = \lambda.$$

(A different proof of this last fact uses that we already learned in Section 3.1.9 that $\Delta_{\gamma_{[\lambda_\circ, \lambda_\bullet]}} = [\lambda_\circ, \lambda_\bullet]$, together with the commutativity of the diagram (3.11).) It then follows from the formula for d' in (3.35) that for $[\Phi_\theta] \in \text{Rot}^{(m)}(S^1)$

$$d'([\Phi_\theta], \text{Pth}(\gamma_{[\lambda_\circ, \lambda_\bullet]})) = e^{-\pi i \langle \lambda, \lambda \rangle_{\Gamma} \theta}.$$

There furthermore holds, by the construction of the $\text{Diff}_+^{(m)}(S^1)$ -action on the groups $P(H, (\Lambda_\circ - \Lambda_\bullet)/\Gamma)$ and $L(T_\circ, H, T_\bullet)$ described in the proof of Proposition 3.1.3, that

$$[\Phi_\theta]^* \text{Pth}(\gamma_{[\lambda_\circ, \lambda_\bullet]}) = \text{Pth}(\gamma_{[\lambda_\circ, \lambda_\bullet]}) + \exp_H(-\lambda\theta)$$

and

$$([\Phi_\theta] \cdot \gamma_{[\lambda_\circ, \lambda_\bullet]})_{\text{m}}(q) = \exp_H(-\lambda\theta),$$

which implies

$$[\Phi_\theta] \cdot \gamma_{[\lambda_\circ, \lambda_\bullet]} = \gamma_{[\lambda_\circ, \lambda_\bullet]} + \text{Bi}(\exp_H(-\lambda\theta)).$$

Collecting the above calculations, we see from the definition of the lifting of the $\text{Diff}_+^{(m)}(S^1)$ -action to $\tilde{L}(T_\circ, H, T_\bullet)$ described in Section 3.3.2 that

$$\begin{aligned} [\Phi_\theta] \cdot (\gamma_{[\lambda_\circ, \lambda_\bullet]}, 1) &= ([\Phi_\theta] \cdot \gamma_{[\lambda_\circ, \lambda_\bullet]}, d'([\Phi_\theta], \text{Pth}(\gamma_{[\lambda_\circ, \lambda_\bullet]}))) \\ &= (\gamma_{[\lambda_\circ, \lambda_\bullet]} + \text{Bi}(\exp_H(-\lambda\theta)), e^{-\pi i \langle \lambda, \lambda \rangle_{\Gamma} \theta}). \end{aligned}$$

Next, we observe that by (3.18),

$$c(\gamma_{[\lambda_\circ, \lambda_\bullet]}, \text{Bi}(\exp_H(-\lambda\theta))) = c'(\exp_H \circ \hat{\xi}, \exp_H(-\lambda\theta))$$

and the latter 2-cocycle value is easily computed to be $e^{-2\pi i \langle \lambda, \lambda \rangle_{\Gamma} \theta}$. An imitation of the proof of Theorem 2.5.10 now goes through without a hitch. \square

Summarising, Theorem 3.4.8, together with the isomorphism

$$\widetilde{L}(T_\circ, H, T_\bullet) / \widetilde{\ker}(\Delta' \circ \text{Pth}) \xrightarrow{\sim} \Lambda_\circ - \Lambda_\bullet$$

induced by the homomorphisms $\widetilde{\text{Pth}}$ and Δ' , describe how $\text{Ind } W_{\chi, l}$ combined with the intertwining $\text{Rot}^{(m)}(S^1)$ -action breaks up into irreducible subrepresentations after restriction to $\widetilde{\ker}(\Delta' \circ \text{Pth}) \rtimes \text{Rot}^{(m)}(S^1)$. We namely have a unitary isomorphism

$$\bigoplus_{\lambda \in \Lambda_\circ - \Lambda_\bullet} f_{\chi, l}^{\sigma_\lambda} : \text{Res}_{\widetilde{L}(T_\circ, H, T_\bullet)}^{\widetilde{\ker}(\Delta' \circ \text{Pth})} \text{Ind}_{\widetilde{\ker}(\Delta' \circ \text{Pth})}^{\widetilde{L}(T_\circ, H, T_\bullet)} W_{\chi, l} \xrightarrow{\sim} \bigoplus_{\lambda \in \Lambda_\circ - \Lambda_\bullet} W_{\chi, l - \lambda}$$

of representations of $\widetilde{\ker}(\Delta' \circ \text{Pth}) \rtimes \text{Rot}^{(m)}(S^1)$, where σ_λ is the coset associated to λ as in the statement of Theorem 3.4.8.

One can now copy the proof of Theorem 2.5.11 to show the following result. The applications of Theorem 2.5.10 and Proposition 2.5.9 in that proof should be replaced by Theorem 3.4.8 and Proposition 3.4.7, respectively, and the role of the winding element homomorphism on a unicoloured torus loop group is taken over by the composition $\Delta' \circ \text{Pth}$.

Theorem 3.4.9. *Two representations $\text{Ind } W_{\chi, l}$ and $\text{Ind } W_{\chi', l'}$ of $\widetilde{L}(T_\circ, H, T_\bullet)$, where χ and χ' are characters of $(\Lambda_\circ \cap \Lambda_\bullet) / \Gamma$ and l and l' are characters of H , are (unitarily) isomorphic if and only if both $\chi' = \chi$ and $l' = l - \lambda$ for some $\lambda \in \Lambda_\circ - \Lambda_\bullet$.*

We can calculate the character of $\text{Ind } W_{\chi, l}$ in the same way as we did in the unicoloured situation. Let us fix an element $\lambda \in \Lambda_\circ - \Lambda_\bullet$ and its corresponding coset $\sigma \subseteq \widetilde{L}(T_\circ, H, T_\bullet)$ as in the statement of Theorem 3.4.8. Then we find via (3.41) the character of the subspace $\mathcal{S}_{\chi, l}^\sigma$ to be

$$\text{ch}_{\mathcal{S}_{\chi, l}^\sigma}(q) = \text{ch}_{R_{\chi, l - \lambda}}(q) = q^{\text{rank } \Gamma / 24} q^{(l - \lambda, l - \lambda)_\Gamma / 2} \eta(q)^{-\text{rank } \Gamma},$$

since $\text{rank } \Gamma = \dim H$. The character of $\text{Ind } R_{\chi, l}$ is then obtained by summing over all the subspaces $\mathcal{S}_{\chi, l}^\sigma$:

$$\begin{aligned} \text{ch}_{\text{Ind } R_{\chi, l}}(q) &= \sum_{\lambda \in \Lambda_\circ - \Lambda_\bullet} \text{ch}_{\mathcal{S}_{\chi, l}^\sigma}(q) \\ &= q^{\text{rank } \Gamma / 24} \sum_{\lambda \in \Lambda_\circ - \Lambda_\bullet} q^{(l - \lambda, l - \lambda)_\Gamma / 2} \cdot \eta(q)^{-\text{rank } \Gamma} \\ &= q^{\text{rank } \Gamma / 24} \theta_{l + (\Lambda_\circ - \Lambda_\bullet)}(q) \cdot \eta(q)^{-\text{rank } \Gamma}. \end{aligned}$$

Note that, even if $l = 0$, the character will in general be a series in non-integral powers of q , in contrast to the unicoloured situation. We learn in particular that

Proposition 3.4.10. *The intertwining $\text{Rot}^{(m)}(S^1)$ -action $\text{Ind}R_{\chi,l}$ on the representation $\text{Ind}W_{\chi,l}$ of $\widetilde{L}(T_{\circ}, H, T_{\bullet})$ is of positive energy.*

We do not know whether $\text{ch}_{\text{Ind}R_{\chi,l}}(q)$ satisfies a type of modular behaviour, but we expect this not to hold in any case unless we apply an energy correction and define

$$Z_{\text{Ind}W_{\chi,l}}(q) := q^{-\text{rank}\Gamma/24} \text{ch}_{\text{Ind}R_{\chi,l}}(q) = \theta_{l+(\Lambda_{\circ}-\Lambda_{\bullet})}(q) \cdot \eta(q)^{-\text{rank}\Gamma}.$$

Theorem 3.4.11. *Every irreducible, positive energy representation of $\widetilde{L}(T_{\circ}, H, T_{\bullet})$ such that the central subgroup $U(1)$ acts as $z \mapsto z$ is (unitarily) isomorphic to $\text{Ind}W_{\chi,l}$ for some characters χ and l of $(\Lambda_{\circ} \cap \Lambda_{\bullet})/\Gamma$ and H , respectively. The isomorphism classes of such representations are therefore labelled by two parameters: one is an element of the dual group of the finite abelian group $(\Lambda_{\circ} \cap \Lambda_{\bullet})/\Gamma$ and the other is an element of the finite abelian group $\Gamma^{\vee}/(\Lambda_{\circ} - \Lambda_{\bullet})$.*

Proof. This is shown along the lines of the proof of Theorem 2.5.13. The role of the subgroup $(\widetilde{L}T)_0$ of $\widetilde{L}T$ and the knowledge of the representation theory of the former afforded by Theorem 2.5.5 is taken over by that of the subgroup $\widetilde{\ker}(\Delta' \circ \text{Pth})$ and Theorem 3.4.4. We saw in Proposition 3.4.7 that $\text{Ind}W_{\chi,l}$ is irreducible and it is seen through Theorem 3.4.8 that for different cosets σ the subspaces $\mathcal{S}_{\chi,l}^{\sigma}$ of $\text{Ind}\mathcal{S}_{\chi,l}$ are mutually non-isomorphic representations of $\widetilde{\ker}(\Delta' \circ \text{Pth})$. \square

We conclude from Theorems 3.4.9 and 3.4.11 that, up to the ambiguity of the character by which $U(1)$ acts, $\widetilde{L}(T_{\circ}, H, T_{\bullet})$ has only finitely many isomorphism classes of irreducible, positive energy representations. There exists precisely one isomorphism class, represented by $\text{Ind}W_{0,0}$, if and only if both inclusions $\Gamma \subseteq \Lambda_{\circ} \cap \Lambda_{\bullet}$ and $\Lambda_{\circ} - \Lambda_{\bullet} \hookrightarrow (\Lambda_{\circ} \cap \Lambda_{\bullet})^{\vee}$ are equalities. Since $(\Lambda_{\circ} \cap \Lambda_{\bullet})^{\vee} = \Lambda_{\circ}^{\vee} - \Lambda_{\bullet}^{\vee}$, the latter condition is fulfilled when Λ_{\circ} and Λ_{\bullet} are both unimodular.

Chapter 4

Outlook

In this Chapter we offer some speculative thoughts on further directions in which to continue the study of bicoloured torus loop groups started in this thesis.

4.1 Defects between lattice conformal nets

Recall from Chapter 1 that our motivation for introducing and studying bicoloured torus loop groups is to find new examples of defects in the sense of Definition 1.3.4 between lattice conformal nets. That is, if $\Lambda_{\circ/\bullet}$ are two even lattices that give rise to conformal nets $\mathcal{A}_{\circ/\bullet}$, as explained in Example 1.2.9, then we would like to construct (some) \mathcal{A}_{\circ} - \mathcal{A}_{\bullet} -defects.

The results achieved in Chapter 3 suggest the following approach, which imitates the construction in Example 1.2.9. If

$$\Lambda_{\circ} \xleftrightarrow{\pi_{\circ}} \Gamma \xleftrightarrow{\pi_{\bullet}} \Lambda_{\bullet}$$

is a span of even lattices, then belonging to it (and some minor extra data) there is a centrally extended bicoloured torus loop group $\tilde{L}(T_{\circ}, H, T_{\bullet})$. Here, $T_{\circ/\bullet}$ and H are the tori associated to the above three lattices. When these lattices are positive definite this group has, up to isomorphism and the character by which the central subgroup $U(1)$ acts, finitely many irreducible, positive energy representations denoted by $\text{Ind } W_{\chi, l}$. We single out one of them, $\text{Ind } W_{0,0}$, and dub it the *vacuum representation* of $\tilde{L}(T_{\circ}, H, T_{\bullet})$. If $I \subseteq S^1$ is a bicoloured interval, where we consider \frown as being white and \smile as black, and $\tilde{L}_I(T_{\circ}, H, T_{\bullet})$ is the subgroup of those elements (γ, z) of $\tilde{L}(T_{\circ}, H, T_{\bullet})$ for which $\text{supp } \gamma \subseteq I$, then we may form the von Neumann algebra

$$D_H(I) := \text{vN}\left(\left(\text{Ind } W_{0,0}\right)\left(\tilde{L}_I(T_{\circ}, H, T_{\bullet})\right)\right) \quad (4.1)$$

acting on the underlying Hilbert space $\text{Ind } \mathcal{S}_{0,0}$ of $\text{Ind } W_{0,0}$. Let $\tilde{L}T_{\circ,\bullet}$ be the centrally extended unicoloured torus loop groups described in Section 3.2.3 and assume that the lattice nets $\mathcal{A}_{\circ,\bullet}$ are constructed from these. We then conjecture that there exists an $\mathcal{A}_{\circ}-\mathcal{A}_{\bullet}$ -defect D_H of which the algebras attached to those bicoloured intervals that are embedded in S^1 are defined as in (4.1).

This (candidate) family of von Neumann algebras obviously satisfies the functoriality, isotony and locality axioms of a defect. That also the requirement holds stating that D_H restricts to \mathcal{A}_{\circ} and \mathcal{A}_{\bullet} on ζ and \succ , respectively, follows from the fact that the vacuum representations $\text{Ind } W_0$ of $\tilde{L}T_{\circ,\bullet}$ inject as direct summands into $\text{Ind } W_{0,0}$ in a way that is equivariant with respect to the group homomorphisms

$$\tilde{L}_{\zeta}T_{\circ} \hookrightarrow \tilde{L}(T_{\circ}, H, T_{\bullet}) \hookleftarrow \tilde{L}_{\succ}T_{\bullet}.$$

With the belief that the formalism of bicoloured torus loop groups produces lattice net defects comes the question whether such a defect can be alternatively constructed through other, more traditional methods. Recall namely that, even though the central extension $\tilde{L}(T_{\circ}, H, T_{\bullet})$ only contains the two unicoloured ‘halves’ $\tilde{L}_{\zeta}T_{\circ}$ and $\tilde{L}_{\succ}T_{\bullet}$ of $\tilde{L}T_{\circ}$ and $\tilde{L}T_{\bullet}$, respectively, there exists a $\text{Diff}_+^{(n)}(S^1)$ -equivariant inclusion of the full group $\tilde{L}H$ into it as well (see Section 3.2.2 and Proposition 3.3.2). Let \mathcal{A}_{Γ} denote the lattice conformal net constructed from $\tilde{L}H$. One can then speculate that there is an inclusion $\mathcal{A}_{\Gamma}(I) \hookrightarrow D_H(I)$ of the associated von Neumann algebras which is of so called *finite index*. Even stronger: the defect D_H might be a relatively local, finite index extension of \mathcal{A}_{Γ} . As we already mentioned in Example 1.3.9, such extensions are known to be classified by data internal to the net \mathcal{A}_{Γ} , namely via Q -systems. This would open a different angle to constructing these defects—one that follows operator-algebraic means.

4.2 Generalising to different tori at the two defect points

Adding to our conjecture that a centrally extended bicoloured torus loop group $\tilde{L}(T_{\circ}, H, T_{\bullet})$ gives rise to a defect D_H through the construction (4.1), we predict that any positive energy representation Q of this group can be equipped with the structure of a D_H - D_H -sector in the sense of [BDH15, Definition 2.2]:

$$\begin{array}{ccc} & \xrightarrow{D_H} & \\ \mathcal{A}_{\circ} & \Downarrow Q & \mathcal{A}_{\bullet} \\ & \xleftarrow{D_H} & \end{array} .$$

properties. The positive energy representations of such a central extension could then possibly give rise to sectors between the two defects D_p and D_q constructed from $\tilde{L}(T_\circ, H_p, T_\bullet)$ and $\tilde{L}(T_\circ, H_q, T_\bullet)$.

Evidence that such central extensions might sometimes exist is given by the observation that in the special case when $\Lambda_\circ, \Lambda_\bullet, \Gamma_p$ and Γ_q are all equal to the same lattice Γ , and the homomorphisms $\pi_{p,\circ/\bullet}$ and $\pi_{q,\circ}$ are chosen to be the identity but $\pi_{q,\bullet}$ is a lattice automorphism of Γ , the group $L(T_\circ, H_p, H_q, T_\bullet)$ is isomorphic to the so called *twisted* unicoloured torus loop group

$$L_{(g)}H := \left\{ \gamma \in C^\infty([0, 1], H) \mid \gamma(1) = g_H(\gamma(0)) \right\},$$

where g_H is the automorphism of the torus $H := \Gamma \otimes_{\mathbb{Z}} \mathbb{U}(1)$ induced by g . (In this definition of $L_{(g)}H$ we again omit conditions on the derivatives at 0 and 1 for γ .) Such twisted loop groups and their central extensions have been studied before (see [PS86, Section 3.7, 175]) in the case that g has finite order—an assumption that is indeed fulfilled when Γ is positive definite, as explained in Appendix A.1.

4.3 Generalising from tori to non-abelian Lie groups

Except for selected parts of the Introduction, we spent our attention in this thesis exclusively on torus loop groups and lattice conformal nets. One may wonder whether the notion of a bicoloured torus loop group $L(T_\circ, H, T_\bullet)$ can be adapted to the situation when the two tori $T_{\circ/\bullet}$ are instead compact, connected, simple Lie groups $G_{\circ/\bullet}$. Of course, the definition of this group itself does make sense upon replacing H with an arbitrary Lie group with homomorphisms towards $T_{\circ/\bullet}$, but the difficulty lies again in finding central extensions with desirable properties.

Motivated by the discussion in [KS11a, Section 5] we suggest that the search for appropriate matching conditions for bicoloured loops in this non-abelian situation could be guided slightly better by the following observation. The authors of [Kaw+07] classify the finite index, relatively local extensions of the affine Kac–Moody nets $\mathcal{A}_{\text{SU}(2),k}$ for all levels $k \geq 1$. By the comments made in Example 1.3.9 we hence obtain a family of defects from $\mathcal{A}_{\text{SU}(2),k}$ to itself. It then seems fruitful to ask whether there exists a family of matching conditions from $\text{SU}(2)$ to itself, depending on a level k , with which one is able to reproduce these defects. We leave this investigation for future research.

Appendix A

Background material

In this Appendix we supply background material on various notions used throughout the thesis.

A.1 Lattices

This section is a brief introduction to lattices. We present some basic examples and we attempt to give the reader a feeling for their richness by discussing their automorphism groups and techniques on how to construct new examples of lattices from given ones. Their role in this thesis is that a lattice is the major ingredient for building the centrally extended unicoloured torus loop groups studied in Chapter 2 and, similarly, a span of even lattices is the most important input datum for the central extensions of bicoloured torus loop groups that we construct in Chapter 3. We finish by defining theta series of lattices for their appearance when calculating the (graded) characters of the representations of these loop groups in Section 2.5 and Section 3.4.

References. The material in this section is largely taken from [Ebe13], [CS99] and [Nik80, §1]. More specific references will be given in the text.

By a *lattice* we mean a pair of a free \mathbb{Z} -module Λ of finite rank, together with a non-degenerate, symmetric, bi-additive form $\langle \cdot, \cdot \rangle : \Lambda \times \Lambda \rightarrow \mathbb{Z}$. We will often omit the reference to the form and denote the lattice simply by Λ . We call Λ *even* if $\langle \lambda, \lambda \rangle \in 2\mathbb{Z}$ for all $\lambda \in \Lambda$, and *odd* otherwise. The property of a lattice being *positive definite*, *negative definite* or *indefinite* is defined in the obvious way. *Morphisms* of lattices are injective \mathbb{Z} -module homomorphisms which respect the forms.

The automorphism group of Λ is denoted by $\text{Aut}(\Lambda; \langle \cdot, \cdot \rangle)$. It is a finite group if Λ is definite since it is a discrete group on the one hand, while on the

other hand, it is a subgroup of the compact orthogonal group $\text{Aut}(\Lambda \otimes_{\mathbb{Z}} \mathbb{R}, \langle \cdot, \cdot \rangle)$, where we extended $\langle \cdot, \cdot \rangle$ bilinearly to $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and used the same notation for it. If Λ is indefinite, then $\text{Aut}(\Lambda; \langle \cdot, \cdot \rangle)$ is typically infinite as shown in [Kne02, Satz (30.4)].

An element $\mu \in \Lambda$ is called a (*long*) *root* if $\langle \mu, \mu \rangle = 2$. In general, a lattice might not contain any roots. We say that Λ is a *root lattice* if it is generated by its roots. Root lattices are automatically even. They have many automorphisms, since for every root μ the reflection

$$\lambda \mapsto \lambda - 2 \frac{\langle \lambda, \mu \rangle}{\langle \mu, \mu \rangle} \mu = \lambda - \langle \lambda, \mu \rangle \mu, \quad \lambda \in \Lambda,$$

in the hyperplane orthogonal to μ is a lattice automorphism.

A.1.1 The dual lattice and the discriminant group

For a lattice Λ , define the \mathbb{Z} -module $\Lambda^{\vee} := \text{Hom}_{\text{Ab}}(\Lambda, \mathbb{Z})$. Then Λ^{\vee} is also free and finitely generated, of the same rank as Λ , and there is an injective homomorphism of \mathbb{Z} -modules $\Lambda \hookrightarrow \Lambda^{\vee}$ given by $\lambda \mapsto \langle \lambda, \cdot \rangle$. We claim that Λ^{\vee} carries a non-degenerate, symmetric, bi-additive form $\Lambda^{\vee} \times \Lambda^{\vee} \rightarrow \mathbb{Q}$, denoted by $\langle \cdot, \cdot \rangle$ as well, which extends the form on Λ . One way to define it is to first extend the form on Λ bilinearly to $\mathbb{Q}\Lambda$. Next, one notices that the inclusion $\Lambda \hookrightarrow \Lambda^{\vee}$ induces an isomorphism of vector spaces $\mathbb{Q}\Lambda \xrightarrow{\sim} \mathbb{Q}(\Lambda^{\vee})$. This can be used to transport the form to $\mathbb{Q}(\Lambda^{\vee})$ and one finally restricts it to Λ^{\vee} . We call Λ^{\vee} together with this form the *dual lattice* of Λ .

We prove that $\langle \lambda, l \rangle := \langle \langle \lambda, \cdot \rangle, l \rangle = l(\lambda) \in \mathbb{Z}$ for all $\lambda \in \Lambda$ and $l \in \Lambda^{\vee}$. Since both sides of this equality are additive in the arguments λ and l , it is sufficient to show that $\langle \lambda_i, \lambda_j^{\vee} \rangle = \delta_{ij}$ for all i, j , where $\{\lambda_i\}_i$ is a basis for Λ and $\{\lambda_j^{\vee}\}_j$ is the associated dual basis for Λ^{\vee} . There is a unique $\lambda^j \in \mathbb{Q}\Lambda$ such that $\lambda_j^{\vee} = \langle \lambda^j, \cdot \rangle$, so $\langle \lambda_i, \lambda_j^{\vee} \rangle = \langle \lambda_i, \lambda^j \rangle = \lambda_j^{\vee}(\lambda_i)$, which shows what we wanted.

The action of the group $\text{Aut}(\Lambda; \langle \cdot, \cdot \rangle)$ on Λ extends to a (left) action on Λ^{\vee} which preserves the \mathbb{Q} -valued form, namely by $g \cdot l := l \circ g^{-1}$ for $g \in \text{Aut}(\Lambda; \langle \cdot, \cdot \rangle)$ and $l \in \Lambda^{\vee}$.

If the injection $\Lambda \hookrightarrow \Lambda^{\vee}$ is surjective, we say that Λ is *unimodular*.

Concrete examples of positive definite lattices are often constructed as full rank submodules Λ of some linear subspace E of \mathbb{R}^n for some n and the bi-additive form is then defined to be the restriction of the Euclidean inner product $\langle \cdot, \cdot \rangle_E$ on E . The dual lattice can then be identified with $\{\alpha \in E \mid \langle \alpha, \lambda \rangle_E \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}$. Indeed, this module has an obvious homomorphism $\alpha \mapsto \langle \alpha, \cdot \rangle_E|_{\Lambda}$ to Λ^{\vee} . The inverse is given by first equating for $l \in \Lambda^{\vee}$ its \mathbb{R} -linear

extension $\mathbb{R}l: \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \mathbb{R}$ with an \mathbb{R} -linear map $E \rightarrow \mathbb{R}$, and next using the isomorphism of the \mathbb{R} -linear dual E^{\vee} with E that is induced by $\langle \cdot, \cdot \rangle_E$.

Since Λ and its dual have the same rank, the quotient group Λ^{\vee}/Λ is a finite abelian group which we denote by D_{Λ} . Its order is written as $\text{disc } \Lambda$ and named the *discriminant* of Λ . This group comes with the extra structure of the symmetric bi-additive form

$$b_{\Lambda}: D_{\Lambda} \times D_{\Lambda} \rightarrow \mathbb{Q}/\mathbb{Z}, \quad (l + \Lambda, l' + \Lambda) \mapsto \langle l, l' \rangle + \mathbb{Z},$$

called the *discriminant bi-additive form* of Λ . To see that b is non-degenerate we show that if $l \in \Lambda^{\vee}$ is such that $\langle l, l' \rangle \in \mathbb{Z}$ for all $l' \in \Lambda^{\vee}$, then $l \in \Lambda$. Such an l defines an element $\langle l, \cdot \rangle$ in $(\Lambda^{\vee})^{\vee} := \text{Hom}_{\text{Ab}}(\Lambda^{\vee}, \mathbb{Z})$. By the evaluation isomorphism $\Lambda \xrightarrow{\sim} (\Lambda^{\vee})^{\vee}$ of \mathbb{Z} -modules there is a unique $\lambda \in \Lambda$ such that $\langle l, l' \rangle = l'(\lambda)$ for all $l' \in \Lambda^{\vee}$. In turn, $l'(\lambda) = \langle \langle \lambda, \cdot \rangle, l' \rangle$, so by the non-degeneracy of the form on Λ^{\vee} we must have $l = \langle \lambda, \cdot \rangle$.

If Λ is even, b can be refined to the quadratic form

$$q_{\Lambda}: D_{\Lambda} \rightarrow \mathbb{Q}/2\mathbb{Z}, \quad l + \Lambda \mapsto \langle l, l \rangle + 2\mathbb{Z},$$

called the *discriminant quadratic form* of Λ . By this, we mean the following:

Definition A.1.1. A *quadratic form* on an abelian group D with values in another abelian group A is a function $q: D \rightarrow A$ such that

- (i) $q(-d) = q(d)$ for all $d \in D$, and
- (ii) the symmetric form $b(d, d') := q(d + d') - q(d) - q(d')$ on D is bi-additive.

We call b the *bi-additive form* of q and say that q is *non-degenerate* when b is.

The composition of the bi-additive form b of q_{Λ} with the projection homomorphism $\mathbb{Q}/2\mathbb{Z} \rightarrow \mathbb{Q}/\mathbb{Z}$ equals 2 times b_{Λ} . Since we learned that b_{Λ} is non-degenerate, so is b and therefore by definition also q_{Λ} . Together with the form b_{Λ} if Λ is odd, or q_{Λ} if Λ is even, D_{Λ} is named the *discriminant group* of Λ (after [Nik80, §1.3]).

A.1.2 Gluing of lattices

An obvious method of constructing new lattices from given ones is as follows. The *direct sum* of two lattices $(\Lambda_1, \langle \cdot, \cdot \rangle)$ and $(\Lambda_2, \langle \cdot, \cdot \rangle)$ has as its underlying \mathbb{Z} -module the direct sum of the underlying \mathbb{Z} -modules of the Λ_i , and its bi-additive form is set to be

$$\langle (\lambda_1, \lambda_2), (\lambda'_1, \lambda'_2) \rangle := \langle \lambda_1, \lambda'_1 \rangle_1 + \langle \lambda_2, \lambda'_2 \rangle_2$$

for $\lambda_i, \lambda'_i \in \Lambda_i$. So the form is defined in such a way that lattice elements in different summands are orthogonal to each other. Of course, this construction generalises to an arbitrary finite number of lattices.

It is interesting and desirable to have a method of constructing a new lattice from a given one which does not increase the rank, as well. We discuss one such technique known as *self-gluing*. It can be thought of as sliding additional copies of the given lattice in between the points of this same lattice at certain prescribed positions. This is explained in [CS99, Chapter 4, Section 3] for example, but we will follow the exposition of [Nik80, §1.4] in terms of the discriminant group instead.

Let $\Gamma \hookrightarrow \Lambda$ be a morphism of lattices. Then there are homomorphisms

$$\Gamma \hookrightarrow \Lambda \hookrightarrow \Lambda^\vee \rightarrow \Gamma^\vee,$$

where the last map is precomposition with $\Gamma \hookrightarrow \Lambda$ and which also respects the (\mathbb{Q} -valued) bi-additive forms. It is injective if Γ and Λ have the same rank. Suppose namely that $l \in \Lambda^\vee$ and that its image in Γ^\vee is zero. Then it factors as a homomorphism $\Lambda/\Gamma \rightarrow \mathbb{Z}$, but Λ/Γ is a finite abelian group by our assumption. Since \mathbb{Z} has no torsion, $\Lambda/\Gamma \rightarrow \mathbb{Z}$ must vanish and so the same holds for l .

For a given lattice Γ , we will use the term *overlattice* sometimes to refer to a lattice Λ of the same rank as Γ together with a specified morphism of lattices $\Gamma \hookrightarrow \Lambda$, and at other times to such a lattice Λ alone when the context implies a canonical inclusion.

Lemma A.1.2. (See [Gri11, Theorem 2.3.3].) *For an overlattice $\Gamma \hookrightarrow \Lambda$, the discriminants are related by*

$$\text{disc}(\Gamma) = \text{disc}(\Lambda) \cdot [\Lambda : \Gamma]^2.$$

As we will see in a moment, overlattices give rise to subgroups of the discriminant group of a special kind:

Definition A.1.3. A subgroup U of the discriminant group D_Γ of a lattice Γ is called *b-isotropic* if $b_\Gamma(u, u') = 0 \in \mathbb{Q}/\mathbb{Z}$ for all $u, u' \in U$. When Γ is even U is *q-isotropic* if $q_\Gamma(u) = 0 \in \mathbb{Q}/2\mathbb{Z}$ for all $u \in U$.

A subgroup that is *q-isotropic* is *b-isotropic* as well by the polarisation identity, but the converse does not necessarily hold.

Theorem A.1.4 (Classification of overlattices). (Taken from [Nik80, Proposition 1.4.1].) *For a given lattice Γ , the map $\Lambda \mapsto U_\Lambda := \Lambda/\Gamma$ is an isomorphism of posets between the overlattices Λ of Γ contained in Γ^\vee , and the b-isotropic subgroups of*

D_Γ . If Γ is even, even such overlattices correspond to the q -isotropic subgroups. Unimodular such overlattices Λ correspond to the isotropic subgroups U_Λ for which $|U_\Lambda|^2 = \text{disc } \Gamma$.

There is a canonical isomorphism of abelian groups $U_\Lambda^\perp/U_\Lambda \cong D_\Lambda$, where the orthogonal complement is taken inside D_Γ with respect to b_Γ . It respects the forms $b_\Gamma|_{U_\Lambda^\perp}$ and b_Λ and, if Λ is even, the forms $q_\Gamma|_{U_\Lambda^\perp}$ and q_Λ .

Since the overlattices of a given lattice Γ have canonical inclusions into Γ^\vee , the above Theorem shows that they can be classified and constructed through the study of the isotropic subgroups of D_Γ . When an overlattice Λ of a direct sum $\Gamma_1 \oplus \cdots \oplus \Gamma_n$ has been constructed in this way from an isotropic subgroup U of the discriminant group of $\Gamma_1 \oplus \cdots \oplus \Gamma_n$ we say that the Γ_i are *components* of Λ which have been ‘glued together’ along representatives of U in the dual lattices Γ_i^\vee .

All even, unimodular, positive definite lattices of rank 24 have been classified in [Nie73] (there are 24 up to isomorphism) and one use of the technique of gluing lattices is the construction of all but one of them (the *Leech lattice*) from certain simpler lattices of lower or equal rank (see [Ebe13, Chapter 3] and [CS99, Chapter 16] for expositions).

A.1.3 Examples of lattices

It is high time we present some basic examples of lattices.

Example A.1.5 (The A_n -series). One series of lattices is the so called A_n -series, defined for $n \geq 1$. The underlying \mathbb{Z} -module of A_n is the following submodule of \mathbb{Z}^{n+1} :

$$A_n := \left\{ (x_1, \dots, x_{n+1}) \in \mathbb{Z}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0 \right\}$$

and it inherits a positive definite form from the Euclidean inner product on \mathbb{R}^{n+1} . One proves that A_n is even by noting that for $(x_1, \dots, x_{n+1}) \in A_n$,

$$\sum_{i=1}^{n+1} x_i^2 + \sum_{i \neq j} 2x_i x_j = \left(\sum_{i=1}^{n+1} x_i \right)^2 = 0.$$

Its rank is n because if $\varepsilon_1, \dots, \varepsilon_{n+1}$ denotes the standard basis of \mathbb{R}^{n+1} , then one possible basis for A_n is

$$\{\varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_2, \dots, \varepsilon_{n+1} - \varepsilon_n\}. \quad (\text{A.1})$$

It is a root lattice of which the $n(n+1)$ roots are $\varepsilon_i - \varepsilon_j$, where $1 \leq i, j \leq n+1$ and $i \neq j$.

For $n \geq 2$, the automorphism group of A_n is isomorphic to a product $\{\pm 1\} \times S_{n+1}$, where -1 is the negation of elements and the symmetric group S_{n+1} on $n+1$ letters acts by permuting the $n+1$ coordinates. Equivalently, the transposition that switches the i -th and j -th coordinates can be seen as a reflection in the hyperplane orthogonal to the root $\varepsilon_i - \varepsilon_j$. If $n = 1$, negation coincides with flipping the two coordinates, so $\text{Aut}(A_1) \cong S_2$.

The dual lattice A_n^\vee is generated by A_n together with the single element

$$l := \left(\frac{n}{n+1}, -\frac{1}{n+1}, \dots, -\frac{1}{n+1} \right) \in \mathbb{Q}^{n+1}.$$

Therefore, D_{A_n} is isomorphic to the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$. Each overlattice, respectively even overlattice, of A_n is generated by A_n together with an element $a \cdot l$, where $0 \leq a \leq n$ is an integer such that $a^2 \langle l, l \rangle$ lies in \mathbb{Z} , respectively in $2\mathbb{Z}$.

Example A.1.6 (The D_n -series). For $n \geq 3$, another series of positive definite lattices can be defined as

$$D_n := \left\{ (x_1, \dots, x_n) \in \mathbb{Z}^n \mid \sum_{i=1}^n x_i \in 2\mathbb{Z} \right\}.$$

It is clear that D_n contains A_{n-1} and that it is generated by it together with the element $\varepsilon_1 + \varepsilon_2 \in \mathbb{Z}^n$. This observation shows that D_n has rank n , and gives via (A.1) a basis for D_n . One proves that D_n is even in the same way as one does for A_n . It is a root lattice of which the $2n(n-1)$ roots are $\pm(\varepsilon_i + \varepsilon_j)$ and $\varepsilon_i - \varepsilon_j$, where $1 \leq i, j \leq n$ and $i \neq j$.

The dual lattice D_n^\vee is generated by \mathbb{Z}^n together with the single element $l_1 := \frac{1}{2}(1, 1, \dots, 1) \in \mathbb{Q}^n$. If n is even the discriminant group D_{D_n} is a direct sum of the two groups of order 2 generated by the equivalence classes $[l_1]$ and $[l_2]$ of l_1 and $l_2 := (1, 0, \dots, 0) \in \mathbb{Z}^n$, while in the odd case $[l_2] = 2[l_1]$, so D_{D_n} is generated by $[l_1]$ alone, which then has order 4. Summarising, D_{D_n} has the following structure:

$$D_{D_n} \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ is even,} \\ \mathbb{Z}/4\mathbb{Z} & \text{if } n \text{ is odd.} \end{cases}$$

If n is odd, then D_{D_n} has no non-zero b -isotropic subgroups since

$$b_{D_n}([l_1], [l_1]) = n/4 \pmod{\mathbb{Z}}$$

and so D_n has no non-trivial overlattices. If n is even, then D_{D_n} always has at least one b -isotropic subgroup that is not q -isotropic, namely the one generated by $[l_2]$ because $q_{D_n}([l_2]) = 1 \pmod{2\mathbb{Z}}$. If $n \in 4\mathbb{Z}$, then also the two subgroups generated by $[l_1]$ and $[l_1] + [l_2]$ are b -isotropic. If moreover $n \in 8\mathbb{Z}$, then both are even q -isotropic.

Example A.1.7 (The E_6 , E_7 and E_8 lattices). If $n \in 8\mathbb{Z}$, the even, positive definite overlattice of D_n corresponding to the q -isotropic subgroup of the discriminant group D_{D_n} which is generated by $[l_1]$ is denoted by D_n^+ . In simpler words, $D_n^+ := D_n + \mathbb{Z} \cdot l_1$. Since $[l_1]$ has order 2 in D_{D_n} and $\text{disc } D_n = 4$, the discriminant group of D_n^+ is trivial by Lemma A.1.2. That is, D_n^+ is a unimodular lattice. If $n = 8$, it is again a root lattice because D_8 is and l_1 is in this case a root. We denote D_8^+ as E_8 . It is known to be the unique even, unimodular, positive definite lattice of rank 8 up to isomorphism (see [Ebe13, Proposition 2.5]).

Take now any root $\mu \in E_8$ and consider the sublattice $\{\lambda \in E_8 \mid \langle \lambda, \mu \rangle = 0\}$ that is the orthogonal complement of μ in E_8 . It turns out that all choices of μ give in this way an isomorphic sublattice. We denote it by E_7 . It is a root lattice as well, but it is no longer unimodular. Its discriminant group has order 2 with generator $[l_1]$ (assuming that μ was chosen orthogonal to l_1).

The lattice E_8 contains copies of A_2 . The orthogonal complements in E_8 of all of these are isomorphic and denoted by E_6 . This is again a root lattice and its discriminant group has order 3.

Example A.1.8 (The hyperbolic plane). The underlying \mathbb{Z} -module of the *hyperbolic plane lattice* is $\mathbb{Z} \oplus \mathbb{Z}$ and its form is given by

$$\langle (\lambda_1, \lambda_2), (\mu_1, \mu_2) \rangle := \lambda_1 \mu_2 + \lambda_2 \mu_1.$$

It is even, unimodular and is an example of an indefinite lattice. One namely has both $\langle (1, 1), (1, 1) \rangle = 2$ and $\langle (1, -1), (1, -1) \rangle = -2$.

Example A.1.9 (Leech lattices). Any even, unimodular, positive definite lattice of rank 24 without roots is called a *Leech lattice*. There exists only one Leech lattice up to isomorphism and many different constructions of it have been devised. This characterisation was proven independently in [CS99, Chapter 12] and [Nie73].

Even though the Leech lattice has no roots, which means that it does not have corresponding reflection automorphisms, its automorphism group $\text{Aut}(\Lambda_{24}, \langle \cdot, \cdot \rangle)$ is large and rich in structure as shown in [Con68]. It is known as the *zeroth Conway group* and denoted by Co_0 . It has order

$$2^{22} 3^9 5^4 7^2 11 \cdot 13 \cdot 23 = 8\,315\,553\,613\,086\,720\,000$$

and is known not to be a simple group. Its quotient by the central subgroup $\{\pm \text{id}\}$ is simple, though, and is called the *first Conway group* Co_1 . It is one of the 26 sporadic finite simple groups. Two other groups in that list, Co_2 and Co_3 , can be obtained as subgroups of Co_0 that fix certain elements in Λ_{24} .

A.1.4 The theta series of a lattice

For a lattice Λ , the series

$$\theta_\Lambda(q) := \sum_{\lambda \in \Lambda} q^{\langle \lambda, \lambda \rangle / 2}$$

in the formal variable $q^{1/2}$ is called the *theta series* of Λ . Clearly, if Λ is even, $\theta_\Lambda(q)$ contains only integral powers of q . If Λ is definite it can be seen as a generating function for the lengths of the elements of Λ because we can rewrite it as

$$\theta_\Lambda(q) = \sum_{k \in \mathbb{Z}} |\Lambda_k| q^{k/2}, \quad \Lambda_k := \{\lambda \in \Lambda \mid \langle \lambda, \lambda \rangle = k\},$$

and Λ_k is indeed a finite set since the closed ball of radius \sqrt{k} in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ is compact.

In this thesis we will also have use for *rational lattices*. A rational lattice is defined identically as a lattice, except that the bi-additive form may be \mathbb{Q} -valued instead of \mathbb{Z} -valued. We will sometimes distinguish lattices from rational ones by calling the former *integral*. Examples of rational lattices are the dual Λ^\vee of an integral lattice Λ and the sum of two integral lattices which contain a common integral sublattice. Of course one can define the theta series of a rational lattice also, which will then be a series in fractional powers of q .

A.2 Central extensions of groups

In this thesis central extensions of certain abelian groups are constructed. This section is devoted to defining central extensions and explaining how the ones we are interested in can be defined by a particular type of maps, called (*group*) *2-cocycles*. Group cocycles of a group G are in general defined for an abelian group A and an action of G on A , but for our purposes we merely need the case that G acts trivially on A . That is, we will omit this action. Even though only the situation when G is abelian matters to us, we will begin by discussing the non-abelian case as well because doing so does not require any extra effort.

References. The material in this section can be found in [FLM88, Section 5.1–5.2].

We start our discussion by assuming that G and A are abstract groups, with A abelian. We will write the addition in A multiplicatively.

Definition A.2.1. A *central extension* of G by A is a group \tilde{G} together with two homomorphisms $A \hookrightarrow \tilde{G}$ and $\tilde{G} \twoheadrightarrow G$ that make it fit into a short exact sequence

$$1 \rightarrow A \rightarrow \tilde{G} \rightarrow G \rightarrow 1,$$

such that the image of A is a subgroup of the centre of \tilde{G} . We will often denote the central extension simply by \tilde{G} , thereby suppressing the data of the other two homomorphisms in our notation. A *morphism* between two central extensions \tilde{G} and \tilde{G}' is a homomorphism $\tilde{G} \rightarrow \tilde{G}'$ making the following diagram commute:

$$\begin{array}{ccccc} & & \tilde{G} & & \\ & \nearrow & \downarrow & \searrow & \\ 1 & \rightarrow & A & & G \rightarrow 1. \\ & \searrow & \downarrow & \nearrow & \\ & & \tilde{G}' & & \end{array}$$

It is not hard to show that a morphism of central extensions is necessarily an isomorphism. In other words, the category of central extensions of G by A is a groupoid.

The algebraic study of central extensions starts with the following observation. Pick a set-theoretic section $s: G \rightarrow \tilde{G}$ of the homomorphism $\tilde{G} \rightarrow G$. Then s is usually not a homomorphism as well. Instead, $s(g)s(g')s(gg')^{-1}$ lies in (the image of) A for all $g, g' \in G$. The function

$$c: G \times G \rightarrow A, \quad (g, g') \mapsto s(g)s(g')s(gg')^{-1}$$

can be seen as a measurement for the failure of the multiplicativity of s . By writing out both sides of the equation $s((g_1g_2)g_3) = s(g_1(g_2g_3))$ in two different ways it can be checked that c satisfies

$$c(g_1, g_2)c(g_1g_2, g_3) = c(g_1, g_2g_3)c(g_2, g_3) \tag{A.2}$$

for all g_1, g_2 and g_3 in G . Moreover, if $s(1_G) = 1_{\tilde{G}}$ then

$$c(g, 1_G) = c(1_G, g) = 1_A \tag{A.3}$$

for all $g \in G$.

We analyse the dependence of c on s by assuming that $f: \tilde{G} \xrightarrow{\sim} \tilde{G}'$ is an isomorphism to another central extension. (We allow the case when $\tilde{G}' = \tilde{G}$,

the two homomorphisms to G are the same and f is the identity.) If s' is a section for \tilde{G}' we can associate a function c' to it in the same way as we did above. Note that $f \circ s$ is a section of \tilde{G}' also. Now define an auxiliary function $d: G \rightarrow A$ as $g \mapsto s'(g)(f \circ s)(g)^{-1}$ for all $g \in G$. If $s'(1_G) = (f \circ s)(1_G)$ then $d(1_G) = 1_A$. Finally setting

$$\delta d: G \times G \rightarrow A, \quad (g, g') \mapsto d(g)d(g')d(gg')^{-1} \quad (\text{A.4})$$

for all $g, g' \in G$, one can prove that $c' = (\delta d) \cdot c$. We thus see that c is not associated canonically to \tilde{G} . Nevertheless, we were able to express its ill-definedness in a precise manner.

We formalise these observations by

Definition A.2.2. A 1-cochain (for the pair (G, A)) is a function $d: G \rightarrow A$. We say that a 1-cochain d is *normalised* if $d(1_G) = 1_A$. A 1-cochain is called a 1-cocycle if it is a group homomorphism. The trivial 1-cocycle is called the 1-coboundary. Given a 1-cochain d , we define a function δd as (A.4) for all $g, g' \in G$.

A 2-cochain is a function $c: G \times G \rightarrow A$. It is called a 2-cocycle if it satisfies (A.2) for all g_1, g_2 and g_3 in G . We say that a 2-cocycle c is *normalised* if (A.3) holds for all $g \in G$. A 2-cochain c is called a 2-coboundary if $c = \delta d$ for some 1-cochain d . Two 2-cocycles are *cohomologous* if they differ by a 2-coboundary.

It is easily checked that a 2-coboundary is a 2-cocycle. A 2-coboundary is normalised if and only if it comes from a normalised 1-cochain.

Remark A.2.3. If a 2-cocycle c is normalised, then this implies by the cocycle relation that $c(g, g^{-1}) = c(g^{-1}, g)$ for all $g \in G$. Any 2-cocycle c is cohomologous to a normalised one. It namely follows from the cocycle relation that $c(g, 1) = c(1, g) = c(1, 1)$ for all $g \in G$. So let d be the constant 1-cochain $d(g) := c(1, 1)$ for all $g \in G$. Then $(\delta d)^{-1} \cdot c$ is normalised.

This procedure of associating (cohomology classes of) 2-cocycles to central extensions can be reversed. Given a normalised 2-cocycle c the set $\tilde{G}_c := G \times A$ namely becomes a group under the multiplication

$$(g, a) \cdot (g', a') := (gg', aa' \cdot c(g, g')).$$

Its unit element is $(1_G, 1_A)$ and the inverse of (g, a) is given by

$$(g^{-1}, a^{-1}c(g, g^{-1})^{-1}).$$

It is a central extension of G by A when equipped with the obvious inclusion of A and projection to G .

This construction is inverse to the one we studied earlier in the sense that if c comes from a section s of a central extension \tilde{G} such that $s(1_G) = 1_{\tilde{G}}$, then there is a canonical isomorphism of central extensions $\tilde{G}_c \xrightarrow{\sim} \tilde{G}$ given by $(g, a) \mapsto s(g)a$. If a normalised 2-cocycle c' is cohomologous to c , say, $c' = (\delta d) \cdot c$ for some (normalised) 1-cochain d , then $(g, a) \mapsto (g, d(g)^{-1} \cdot a)$ is an isomorphism $\tilde{G}_c \xrightarrow{\sim} \tilde{G}_{c'}$ of the corresponding central extensions.

Let us revisit our study by assuming that G and A are topological groups. The definition of a *topological* central extension \tilde{G} in Definition A.2.1 then additionally demands that the two homomorphisms are continuous. The difference with the situation for abstract groups is that there might not exist any globally continuous section $s: G \rightarrow \tilde{G}$. In general, s is continuous only on a neighbourhood around 1_G . A discontinuous section can then result in a discontinuous cocycle c and so the central extension \tilde{G}_c is not a topological group when we give the underlying set $G \times A$ of \tilde{G}_c the product topology.

However, it turns out that under certain conditions \tilde{G}_c is a topological group anyway when s is only locally continuous (see [Nee02, Proposition 2.2 and Remark 2.3]). We state a preparatory Lemma and then give a particular version of such a result that is sufficient for the purposes in this thesis.

Lemma A.2.4. *Let G be an abstract group, G_0 a normal topological subgroup and suppose that for every element of G the associated conjugation map on G_0 is continuous. Then there exists a unique structure of a topological group on G such that G_0 is open in G . The topology is given by declaring a subset $U \subseteq G$ to be open when $gU \cap G_0$ is open in G_0 for all $g \in G$.*

Corollary A.2.5. *Let G and A be topological groups with A abelian, G_0 an open normal topological subgroup of G and c a normalised 2-cocycle on G such that*

- (i) *c is continuous when restricted to G_0 , and*
- (ii) *for every $g \in G$ the map $G_0 \rightarrow A$ given by $g' \mapsto c(g, g')c(gg', g^{-1})$ is continuous.*

Then there exists a unique structure of a topological group on \tilde{G}_c such that $(\tilde{G}_0)_c$ is open in \tilde{G}_c . Here, $(\tilde{G}_0)_c$ is the topological group of which its underlying set $G_0 \times A$ carries the product topology.

Proof. Note first that $(\tilde{G}_0)_c$ is normal in \tilde{G}_c . Next, let $g \in G$ and $(g', a) \in (\tilde{G}_0)_c$. Then we calculate that

$$\begin{aligned} (g, 1_A)(g', a)(g, 1_A)^{-1} &= (gg', a \cdot c(g, g'))(g^{-1}, c(g, g^{-1})^{-1}) \\ &= (gg'g^{-1}, a \cdot c(g, g')c(g, g^{-1})^{-1}c(gg', g^{-1})). \end{aligned}$$

Hence, conjugation by $(g, 1_A)$ is a continuous map on $(\tilde{G}_0)_c$. The result now follows from Lemma A.2.4. \square

A.2.1 Central extensions of abelian groups

If Λ is an abelian group, then every central extension $\tilde{\Lambda}$ of Λ by A has a canonically associated *commutator map* $b: \Lambda \times \Lambda \rightarrow A$. It is defined by picking a section $s: \Lambda \rightarrow \tilde{\Lambda}$ and setting $b(\lambda, \mu) := s(\lambda)s(\mu)s(\lambda)^{-1}s(\mu)^{-1}$ for all $\lambda, \mu \in \Lambda$, which is independent of the choice of s . It is bi-additive and satisfies $b(\lambda, \lambda) = 0$ for all $\lambda \in \Lambda$. (These two properties together imply that b is skew-symmetric.) There furthermore holds

$$b(\lambda, \mu) = \varepsilon(\lambda, \mu)\varepsilon(\mu, \lambda)^{-1}$$

for any choice of 2-cocycle ε for $\tilde{\Lambda}$, so this can be used as an alternative definition of b .

If Λ is a free \mathbb{Z} -module of finite rank, then also conversely, every bi-additive map $b: \Lambda \times \Lambda \rightarrow A$ which satisfies $b(\lambda, \lambda) = 0$ arises in this way. We can namely pick an ordered basis $\{\lambda_1, \dots, \lambda_n\}$ of Λ and define a function $\varepsilon: \Lambda \times \Lambda \rightarrow A$ by first setting

$$\varepsilon(\lambda_i, \lambda_j) := \begin{cases} b(\lambda_i, \lambda_j) & \text{if } i < j, \\ 1 & \text{if } i \geq j, \end{cases}$$

and extending this bi-additively next. Since ε is bi-additive, it is a 2-cocycle and therefore determines a central extension $\tilde{\Lambda}$. It satisfies

$$\varepsilon(\lambda, \mu)\varepsilon(\mu, \lambda)^{-1} = b(\lambda, \mu)$$

for all $\lambda, \mu \in \Lambda$, which can be checked by hand for the basis elements λ_i and must then hold for arbitrary elements of Λ since both sides of this equation are bi-additive. This shows that b is the commutator map of $\tilde{\Lambda}$. With the assumption about Λ still in place one can furthermore show that b determines $\tilde{\Lambda}$ up to non-unique isomorphism.

A.3 Group representations

In Sections 2.5 and 3.4 representations are constructed and classified of central extensions of uni- and bicoloured torus loop groups, respectively, and of related groups. These representations are of the following kind:

Definition A.3.1. A unitary representation $Q: G \rightarrow U(\mathcal{H})$ of a topological group G on a Hilbert space \mathcal{H} is called *strongly continuous* if Q is a continuous map with respect to the strong operator topology on $U(\mathcal{H})$ or, equivalently, if for all vectors $v \in \mathcal{H}$ the map $G \rightarrow \mathcal{H}$, $g \mapsto Q(g)(v)$ is continuous at $g = 1$.

A *morphism*, or *intertwiner*, between two strongly continuous, unitary representations of G on Hilbert spaces \mathcal{H} and \mathcal{K} is a bounded linear map $\mathcal{H} \rightarrow \mathcal{K}$ which intertwines the respective actions of G . Two such representations are called *unitarily isomorphic* if there exists a unitary morphism between them.

The category of strongly continuous, unitary representations of G is denoted by $\text{Rep } G$ and if Q and Q' are in $\text{Rep } G$ we write $\text{Hom}_G(Q, Q')$ for the complex vector space $\text{Hom}_{\text{Rep } G}(Q, Q')$.

In the rest of this section *groups* will always be topological and *representations* will always be meant to be strongly continuous and unitary, unless stated otherwise.

Using the fact that the operator-theoretic adjoint of a morphism is again a morphism one can show

Proposition A.3.2. (See [Kir04, p. 361] or [Sug90, Proposition I.2.6].) *Two representations of a group are isomorphic if and only if they are unitarily isomorphic.*

Definition A.3.3. Let Q be a representation of a group on a Hilbert space \mathcal{H} . A *subrepresentation* of Q is a closed linear subspace \mathcal{K} of \mathcal{H} that is invariant under Q , equipped with the restriction of Q to \mathcal{K} . We say that Q is *irreducible* if \mathcal{H} is non-zero and its only subrepresentations are $\{0\}$ and \mathcal{H} itself.

The following equivalent characterisation of the invariance of a closed subspace is often useful:

Lemma A.3.4. (See [Fol95, Proposition 3.4].) *Let Q be a representation of a group on a Hilbert space \mathcal{H} . Then a closed linear subspace \mathcal{K} of \mathcal{H} is invariant under Q if and only if the orthogonal projection of \mathcal{H} onto \mathcal{K} commutes with Q .*

A basic fact is that if $\mathcal{K} \subseteq \mathcal{H}$ is a subrepresentation then the same holds for its orthogonal complement \mathcal{K}^\perp .

Morphisms between irreducible representations are rigid in the following sense, as is well known from the theory of finite-dimensional representations:

Lemma A.3.5 (Schur's lemma). (See [Fol95, Lemma 3.5].) *A non-zero representation Q of a group is irreducible if and only if every endomorphism of Q is a scalar multiple of the identity. A morphism between two irreducible representations is zero when the representations are non-isomorphic and a multiple of a unitary isomorphism otherwise.*

We denote the external or internal direct sum of a family of representations Q_i as $\bigoplus_i Q_i$. If \mathcal{H}_i is the underlying Hilbert space of Q_i , then that of $\bigoplus_i Q_i$ is the Hilbert space completion

$$\overline{\bigoplus_i \mathcal{H}_i}$$

of the algebraic direct sum of the \mathcal{H}_i .

A finite-dimensional representation of a group always contains an irreducible subrepresentation because one can pick a non-zero subrepresentation of minimal dimension. This argument cannot be applied to infinite-dimensional representations. Even worse, the left regular representation of the additive group \mathbb{R} on the Hilbert space $L^2(\mathbb{R})$ is an example which has plenty of subrepresentations but no irreducible ones (as explained in for example [Fol95, p. 72]), showing that this property is not satisfied in general. When we do have this knowledge at our disposal a general argument can be used for

Proposition A.3.6. *A non-zero representation of a group such that every non-zero subrepresentation contains an irreducible subrepresentation is the (internal) direct sum of a set of mutually orthogonal, irreducible subrepresentations.*

Proof. (As for example in [Kow14, Proof of Corollary 5.4.2].) Denote the representation and its underlying Hilbert space by Q and \mathcal{H} , respectively. Consider the set Σ of which each element is a set of mutually orthogonal, irreducible subrepresentations of Q . Such sets can be ordered by inclusion, making Σ a poset. Our assumption says that Q contains at least one irreducible subrepresentation, so the singleton set that it forms belongs to Σ and therefore Σ is non-empty. Every non-empty chain in Σ has an upper bound in Σ , namely the union of the sets belonging to that chain. Zorn's lemma now tells us that Σ contains a maximal element, say, a set $\{\mathcal{K}_\alpha\}_{\alpha \in A}$ of subrepresentations for some index set A . Define \mathcal{K} to be the (internal, Hilbert space) direct sum of the \mathcal{K}_α . If $\mathcal{K} = \mathcal{H}$, we are done. If not, then \mathcal{K}^\perp is a non-zero subrepresentation of Q because Q is unitary. Applying the assumption again to exhibit an irreducible subrepresentation of \mathcal{K}^\perp and adding this to the set $\{\mathcal{K}_\alpha\}_{\alpha \in A}$ contradicts its maximality. We conclude that $\mathcal{K} = \mathcal{H}$ after all. \square

Of course, the decomposition that we obtain in this way is in no way unique. Moreover, this result and its proof do not tell you anything about how to concretely decompose a given representation.

A.3.1 Induction of representations in the case of a discrete coset space

Throughout this section G will be a group, H a subgroup such that the coset space G/H is discrete and countable and Q a representation¹ of H on a Hilbert space \mathcal{K} . (This implies that H is open, and hence closed in G .) We explain a method of *inducing* Q up to a representation of G and we prove some of its basic properties.

The conditions we impose on G/H are sufficient for the purposes of this thesis because for the pairs (G, H) considered in Sections 2.5.3 and 3.4.2 the coset space is isomorphic to the underlying abelian group of a lattice and of a rational lattice, respectively. Furthermore, for those pairs H is normal in G . Therefore we will often include that assumption in our results as well, although this is not essential.

We start by constructing from the data G , H and Q a new Hilbert space $\text{Ind}_H^G \mathcal{K}$ that is larger than \mathcal{K} as the following (external) Hilbert space direct sum indexed over the left cosets σ of H in G :

$$\text{Ind}_H^G \mathcal{K} := \overline{\bigoplus_{\sigma \in G/H} \mathcal{K}^\sigma}.$$

Here, \mathcal{K}^σ stands for the Hilbert space $\sigma \times_H \mathcal{K}$. Its vectors are equivalence classes of pairs $(x, v) \in \sigma \times \mathcal{K}$ for the relation $(xh, v) \sim (x, Q(h)v)$. We denote such a class as $[x, v]$. Addition of vectors is defined as $[x, v] + [x', v'] := [x, v + Q(x^{-1}x')v']$. The inverse of $[x, v]$ is $[x, -v]$ and the zero vector is $[x, 0]$. Scalar multiplication is defined as $\alpha \cdot [x, v] := [x, \alpha v]$, and, lastly, the inner product is $\langle [x, v], [x', v'] \rangle := \langle v, Q(x^{-1}x')v' \rangle$. Any choice of representative $x \in \sigma$ induces a unitary map $\mathcal{K}^\sigma \xrightarrow{\sim} \mathcal{K}$ given by $[x, v] \mapsto v$. Hence \mathcal{K}^σ can be seen as a copy of \mathcal{K} associated to σ which is constructed without making any choices.

We will often abbreviate $\text{Ind}_H^G \mathcal{K}$ as $\text{Ind } \mathcal{K}$ when it is clear which groups are under discussion. A general vector of $\text{Ind } \mathcal{K}$ is a tuple of vectors

$$([x^\sigma, v^\sigma])_{\sigma \in G/H}, \quad [x^\sigma, v^\sigma] \in \mathcal{K}^\sigma,$$

¹We inherit the conventions on terminology from Appendix A.3. So groups are meant to be topological and representations are strongly continuous and unitary.

such that

$$\sum_{\sigma \in G/H} \|v^\sigma\|_{\mathcal{K}}^2 < \infty.$$

Next, we define an action $\text{Ind}_H^G Q$ of G on $\text{Ind } \mathcal{K}$ by setting for $g \in G$

$$(\text{Ind}_H^G Q)(g) \cdot ([x^\sigma, v^\sigma])_\sigma := ([gx^{g^{-1}\sigma}, v^{g^{-1}\sigma})]_\sigma.$$

It is easily checked that $(\text{Ind}_H^G Q)(g)$ is well-defined, linear and unitary, and we will often abbreviate this action as $\text{Ind } Q$. Informally, and more geometrically speaking, it can be understood by seeing $G \times_H \mathcal{K}$ as the total space of a complex vector bundle over the space of cosets G/H . The action $\text{Ind } Q$ on $\text{Ind } \mathcal{K}$ is then the natural one when the latter is considered as a space of sections of this bundle that are square-integrable with respect to the counting measure on G/H .

Proposition A.3.7. *The action $\text{Ind } Q$ of G on $\text{Ind } \mathcal{K}$ is strongly continuous.*

Proof. Let $([x^\sigma, v^\sigma])_\sigma$ be a vector in $\text{Ind } \mathcal{K}$ and $(g_n)_n$ a sequence of elements in G converging to 1. We then wish to show that the sequence of vectors

$$\left((\text{Ind } Q)(g_n) \cdot ([x^\sigma, v^\sigma])_\sigma \right)_n$$

converges to $([x^\sigma, v^\sigma])_\sigma$. For every n there holds

$$\begin{aligned} ([x^\sigma, v^\sigma])_\sigma - (\text{Ind } Q)(g_n) \cdot ([x^\sigma, v^\sigma])_\sigma &= ([x^\sigma, v^\sigma])_\sigma - ([g_n x^{g_n^{-1}\sigma}, v^{g_n^{-1}\sigma})]_\sigma \\ &= \left([x^\sigma, v^\sigma - Q((x^\sigma)^{-1} g_n x^{g_n^{-1}\sigma})(v^{g_n^{-1}\sigma})] \right)_\sigma, \end{aligned}$$

and hence

$$\begin{aligned} \left\| ([x^\sigma, v^\sigma])_\sigma - (\text{Ind } Q)(g_n) \cdot ([x^\sigma, v^\sigma])_\sigma \right\|_{\text{Ind } \mathcal{K}}^2 &= \sum_{\sigma \in G/H} \left\| v^\sigma - Q((x^\sigma)^{-1} g_n x^{g_n^{-1}\sigma})(v^{g_n^{-1}\sigma}) \right\|_{\mathcal{K}}^2. \end{aligned}$$

Now note that, since G/H is discrete, $(g_n)_n$ converging to 1 implies that there exists some $N \geq 1$ such that for $n \geq N$ we have $g_n^{-1}\sigma = \sigma$ for all cosets σ simultaneously. An appeal to the strong continuity of Q then finishes the argument. \square

We call $\text{Ind } Q$ the *representation of G induced from Q* .

Proposition A.3.8. *Let Q_1 and Q_2 be two representations of H with underlying Hilbert space \mathcal{K}_1 and \mathcal{K}_2 , respectively. If $f : Q_1 \rightarrow Q_2$ is an H -intertwiner, then the function $\text{Ind } f : \text{Ind } \mathcal{K}_1 \rightarrow \text{Ind } \mathcal{K}_2$ defined by*

$$(\text{Ind } f)([x^\sigma, v^\sigma])_\sigma := \left([x^\sigma, f(v^\sigma)] \right)_\sigma \quad (\text{A.5})$$

for all vectors $([x^\sigma, v^\sigma])_\sigma \in \text{Ind } \mathcal{K}_1$ is a G -intertwiner. It is isometric when f is.

Proof. That $\text{Ind } f$ is linear and compatible with the actions of G is easily checked. Because

$$\sum_\sigma \|f(v^\sigma)\|_{\mathcal{K}_2}^2 \leq \|f\| \cdot \sum_\sigma \|v^\sigma\|_{\mathcal{K}_1}^2 < \infty,$$

the tuple on the right hand side of (A.5) is indeed a vector of $\text{Ind } \mathcal{K}_2$. This inequality also shows that $\|\text{Ind } f\| = \|f\|$. \square

Corollary A.3.9. *The assignment $Q \mapsto \text{Ind } Q$ defines a functor $\text{Ind}_H^G : \text{Rep } H \rightarrow \text{Rep } G$.*

Unfortunately, this induction functor produces in general representations of G that are too large or too small for the functor to be a left or a right adjoint, respectively, to the restriction functor $\text{Res}_G^H : \text{Rep } G \rightarrow \text{Rep } H$. Regarding the first of these two claims: if Q' is a representation of G there always exists the injective complex linear map

$$\text{Hom}_G(\text{Ind } Q, Q') \hookrightarrow \text{Hom}_H(Q, \text{Res } Q') \quad (\text{A.6})$$

given by pre-composition with the H -intertwining inclusion $\mathcal{K} \hookrightarrow \text{Ind } \mathcal{K}$, $v \mapsto [1, v]$. To show that (A.6) is surjective means that for an H -intertwiner $f : Q \rightarrow \text{Res } Q'$ we should produce a G -intertwiner $\hat{f} : \text{Ind } Q \rightarrow Q'$ such that $\hat{f}[1, v] = f(v)$ for all $v \in \mathcal{K}$. Algebraic considerations force us to define

$$\hat{f}[x^\sigma, v^\sigma] := Q'(x^\sigma)f(v^\sigma)$$

for every coset $\sigma \in G/H$.

That \hat{f} can have problems of convergence is illustrated by taking $G := \mathbb{Z}$, $H = \{0\}$ and letting Q and Q' be the trivial representations of H and G , respectively. In this case the underlying Hilbert space $\text{Ind } \mathcal{K}$ of $\text{Ind } Q$ consists of all square-integrable functions $\mathbb{Z} \rightarrow \mathcal{K} := \mathbb{C}$. It contains in particular the function $k \mapsto 1/k$. The domain of the H -intertwiner $Q \rightarrow \text{Res } Q'$ given by $1_{\mathbb{C}} \mapsto 1_{\mathbb{C}}$ then cannot be enlarged to $\text{Ind } Q$ because the series $\sum_{k \in \mathbb{Z}} 1/k$ does not converge in $\mathcal{K}' := \mathbb{C}$.

To show that Ind is neither right adjoint to Res we observe that there always exists the injective complex linear map

$$\text{Hom}_G(Q', \text{Ind } Q) \hookrightarrow \text{Hom}_H(\text{Res } Q', Q) \quad (\text{A.7})$$

given by post-composition with the H -intertwining orthogonal projection P of $\text{Ind } \mathcal{K}$ to \mathcal{K} . Showing that (A.7) is surjective means that for an H -intertwiner $f : \text{Res } Q' \rightarrow Q$ we should produce an G -intertwiner $\hat{f} : Q' \rightarrow \text{Ind } Q$ such that $P \circ \hat{f} = f$. We have no choice but to define

$$\hat{f}(v) = \sum_{\sigma \in G/H} [x^\sigma, f(Q'(x^\sigma)^{-1}(v))]$$

for every vector $v \in \mathcal{K}'$, where $\{x^\sigma\}_{\sigma \in G/H}$ is any set of representatives of the (left) cosets of H in G .

Now choose again $G := \mathbb{Z}$, $H = \{0\}$ and Q and Q' to be the trivial representations of H and G respectively. Then the codomain of the H -intertwiner $\text{Res } Q' \rightarrow Q$ given by $1_{\mathbb{C}} \mapsto 1_{\mathbb{C}}$ cannot be enlarged to $\text{Ind } Q$ because the vector

$$\sum_{\sigma \in G/H} [x^\sigma, 1_{\mathbb{C}}]$$

does not converge in $\text{Ind } \mathcal{K}$.

Nevertheless, these weak forms (A.6) and (A.7) of *Frobenius reciprocity* are sufficient to show that the irreducibility of a representation that is induced from a normal subgroup can be tested by calculating conjugate representations. Either one will do, and we will choose to use (A.6). We begin with a little preparatory material that is useful for the study of induced representations in general:

Definition A.3.10. Let $g \in G$. The associated representation Q^g of the subgroup $gHg^{-1} \subseteq G$ that is *conjugate* to Q is defined by $Q^g(ghg^{-1}) := Q(h)$ for all $h \in H$ on the Hilbert space \mathcal{K} .

Obviously, the conjugates of Q are either all reducible or all irreducible. The importance of conjugate representations is their appearance in the restriction back to a normal subgroup of an induced representation:

Lemma A.3.11. Assume that the subgroup H of G is normal. Then for every set of representatives $\{x^\sigma\}_{\sigma \in G/H}$ of the (left) cosets of H in G the conjugate representations Q^{x^σ} are representations of H and the unitary maps $\mathcal{K}^\sigma \xrightarrow{\sim} \mathcal{K}$ given by $[x^\sigma, v^\sigma] \mapsto v^\sigma$ induce a unitary isomorphism

$$\text{Res}_G^H \text{Ind}_H^G Q \cong \bigoplus_{\sigma \in G/H} Q^{x^\sigma}$$

of H -representations.

Proof. Since H is normal in G , we have $x^\sigma H(x^\sigma)^{-1} = H$ and so Q^{x^σ} is a representation of H given by $Q^{x^\sigma}(h) = Q((x^\sigma)^{-1}hx^\sigma)$ for all $h \in H$. Furthermore, if $h \in H$, then H being normal implies that $h^{-1}\sigma = \sigma$ for every coset σ . Hence,

$$\begin{aligned} (\text{Res Ind } Q)(h)([x^\sigma, v^\sigma]_\sigma) &= ([hx^\sigma, v^\sigma]_\sigma) \\ &= ([x^\sigma(x^\sigma)^{-1}hx^\sigma, v^\sigma]_\sigma) \\ &= \left([x^\sigma, Q((x^\sigma)^{-1}hx^\sigma)(v^\sigma)]_\sigma \right). \end{aligned}$$

This proves what was asked. \square

Theorem A.3.12 (Mackey's irreducibility criterion in the case of a normal subgroup). *Assume that the subgroup H of G is normal. If Q is irreducible and for some set of representatives $\{x^\sigma\}_{\sigma \in G/H}$ the associated conjugate representations Q^{x^σ} for $\sigma \neq H$ are not isomorphic to Q , then the induced representation $\text{Ind } Q$ is irreducible as well.*

Proof. Let us at first not make any assumptions about the representations at hand. It follows from (A.6) that there is an inclusion

$$\text{Hom}_G(\text{Ind } Q, \text{Ind } Q) \hookrightarrow \text{Hom}_H(Q, \text{Res Ind } Q)$$

of complex vector spaces. Applying Lemma A.3.11 next to some set of coset representatives $\{x^\sigma\}_{\sigma \in G/H}$, we may expand this as:

$$\begin{aligned} \text{Hom}_G(\text{Ind } Q, \text{Ind } Q) &\hookrightarrow \text{Hom}_H\left(Q, \bigoplus_{\sigma \in G/H} Q^{x^\sigma}\right) \\ &\hookrightarrow \prod_{\sigma \in G/H} \text{Hom}_H(Q, Q^{x^\sigma}) \\ &= \text{Hom}_H(Q, Q^{x^H}) \oplus \prod_{\substack{\sigma \in G/H \\ \sigma \neq H}} \text{Hom}_H(Q, Q^{x^\sigma}). \end{aligned}$$

Now suppose that Q is irreducible and that $Q^{x^\sigma} \not\cong Q$ when $\sigma \neq H$. Then also Q^{x^σ} is irreducible for all x^σ , so by Schur's lemma $\text{Hom}_H(Q, Q^{x^\sigma})$ is zero when $\sigma \neq H$ and $\text{Hom}_H(Q, Q^{x^H})$ is 1-dimensional. Therefore, $\text{Hom}_G(\text{Ind } Q, \text{Ind } Q)$ is 1-dimensional and hence $\text{Ind } Q$ is irreducible by Schur's lemma. \square

A.3.2 The positive energy condition

The representations of the various groups studied in Section 2.5 and Section 3.4 satisfy one more property besides the ones listed in Definition A.3.1: they are of positive energy. It is this attribute which makes them amenable to classification. We introduce this notion in this section and explain how it interacts with that of irreducibility.

References. The material in this section is largely taken from [PS86, Section 9.2] and [Was98, Section I.6]. More specific references will be given in the text.

Recall from the Introduction that we denote the topological group of counterclockwise rotations of the manifold S^1 by $\text{Rot}(S^1)$. We will use the following models for the covering groups of $\text{Rot}(S^1)$:

$$\text{Rot}^{(\infty)}(S^1) \cong \{\Phi_\theta: \mathbb{R} \xrightarrow{\sim} \mathbb{R}, \quad \theta' \mapsto \theta' + \theta\}_{\theta \in \mathbb{R}}$$

and for $m \geq 1$

$$\text{Rot}^{(m)}(S^1) \cong \text{Rot}^{(\infty)}(S^1)/m\mathbb{Z},$$

where \mathbb{Z} stands for the subgroup of $\text{Rot}^{(\infty)}(S^1)$ generated by the shift $\theta \mapsto \theta + 1$. We denote elements of $\text{Rot}^{(m)}(S^1)$ as $[\Phi_\theta]$, where $\Phi_\theta \in \text{Rot}^{(\infty)}(S^1)$ and the square brackets stand for its equivalence class in $\text{Rot}^{(m)}(S^1)$. The image $[\Phi_\theta]$ in $\text{Rot}(S^1)$ of $\Phi_\theta \in \text{Rot}^{(\infty)}(S^1)$ is the anti-clockwise rotation by angle θ , so $[\Phi_1] = \text{id}_{S^1}$.

It is a well-known fact that a non-zero representation of a compact group can be written as the (internal) direct sum of a set of mutually orthogonal, irreducible, finite-dimensional subrepresentations. (See for example [Kow14, Corollary 5.4.2].) This applies in particular to the group $\text{Rot}^{(m)}(S^1)$ for some $m \geq 1$. Collecting the irreducible representations together into isotypic components, such a representation R on a Hilbert space \mathcal{H} can then be written as the completion

$$\mathcal{H} = \overline{\bigoplus_{a \in (1/m)\mathbb{Z}} \mathcal{H}(a)},$$

where $\mathcal{H}(a)$ is the isotypic component

$$\mathcal{H}(a) := \{v \in \mathcal{H} \mid R[\Phi_\theta](v) = e^{-2\pi i a \theta} v \text{ for all } \theta \in [0, 1]\}$$

on which R acts by the $(-a)$ -th character of $\text{Rot}^{(m)}(S^1)$.

Definition A.3.13. A representation R of $\text{Rot}^{(m)}(S^1)$ for some $m \geq 1$ on a Hilbert space \mathcal{H} is said to be of *positive energy* if both

- the isotypic components $\mathcal{H}(a)$ are zero for $a < 0$, and
- R is of *finite type*, that is, for each $a \in (1/m)\mathbb{Z}_{\geq 0}$ the dimension of $\mathcal{H}(a)$ is finite.

We then call $\mathcal{H}(a)$ the a -th energy eigenspace of \mathcal{H} .

Let N be a group together with a continuous $\text{Rot}^{(m)}(S^1)$ -action on it. A representation Q of N on a Hilbert space \mathcal{H} is said to be of *positive energy* if there exists an extension of Q to a representation of the semidirect product $N \rtimes \text{Rot}^{(m)}(S^1)$ on \mathcal{H} such that its restriction to $\text{Rot}^{(m)}(S^1)$ is of positive energy. A *morphism* between two positive energy N -representations is defined to be a morphism of N -representations.

Notice that the extension to $N \rtimes \text{Rot}^{(m)}(S^1)$ is not part of the data of a positive energy representation which is why, correspondingly, morphisms are not required to intertwine the rotation actions on the respective Hilbert spaces either.

Remark A.3.14. Write the (left) action of $\text{Rot}^{(m)}(S^1)$ on N as $[\Phi_\theta]^*g := [\Phi_\theta] \cdot g$, where $g \in N$. Recall that $N \rtimes \text{Rot}^{(m)}(S^1)$ has $N \times \text{Rot}^{(m)}(S^1)$ as its underlying topological space and that its multiplication is defined by

$$(g, [\Phi_\theta]) \cdot (g', [\Phi_{\theta'}]) := (g[\Phi_\theta]^*(g'), [\Phi_{\theta+\theta'}])$$

for all $g, g' \in N$ and $[\Phi_\theta], [\Phi_{\theta'}] \in \text{Rot}^{(m)}(S^1)$. It is then easily seen that Q extending to $N \rtimes \text{Rot}^{(m)}(S^1)$ is equivalent to there existing a representation R of $\text{Rot}^{(m)}(S^1)$ on \mathcal{H} satisfying the intertwining property

$$R[\Phi_\theta]Q(g)R[\Phi_\theta]^* = Q([\Phi_\theta]^*g). \quad (\text{A.8})$$

That is, $R[\Phi_\theta]$ is an isomorphism from Q to the ‘twisted’ representation $Q \circ [\Phi_\theta]$.

If χ_a is the character $[\Phi_\theta] \mapsto e^{2\pi ia\theta}$ of $\text{Rot}^{(m)}(S^1)$, then of course $\chi_a \cdot R$ also satisfies (A.8) and $\chi_a \cdot R$ is again of positive energy if $a \leq 0$. This shows that the lift of Q to $N \rtimes \text{Rot}^{(m)}(S^1)$ is never unique. However, this is the only indeterminacy if Q is irreducible. Let R' namely be another representation of $\text{Rot}^{(m)}(S^1)$ on \mathcal{H} intertwining like R with Q . Then $[\Phi_\theta] \mapsto R[\Phi_\theta]^*R'[\Phi_\theta]$ is a representation of $\text{Rot}^{(m)}(S^1)$ on \mathcal{H} which commutes with Q , hence it must be a character by Schur’s lemma.

Remark A.3.15. Of course, any finite-dimensional representation of any group N on a Hilbert space \mathcal{H} is of positive energy by making $\text{Rot}(S^1)$ act as the identity on both N and \mathcal{H} . However, the groups N we study in this thesis already come with natural, non-trivial actions of $\text{Rot}^{(m)}(S^1)$ for some m , hence making the demand of positivity of energy for their representations restrictive and interesting.

Remark A.3.16. One could obviously relax the assumption of positivity of energy to obtain the more general notion of a representation of which the energy is *bounded from below*. However, for such a representation it would again not be appropriate to require the $\text{Rot}^{(m)}(S^1)$ -action R to be part of the data. Therefore we would be free to ‘shift the energy’, that is, to multiply R with a character χ_a , obtaining a positive energy representation for small enough negative a . We thus see that this generalisation is vacuous.

The notions of irreducibility for the groups N and $N \rtimes \text{Rot}^{(m)}(S^1)$ agree when positivity of energy is assumed:

Lemma A.3.17. *Let N be a group together with a continuous $\text{Rot}^{(m)}(S^1)$ -action on it. An irreducible representation of $N \rtimes \text{Rot}^{(m)}(S^1)$ such that the restriction to $\text{Rot}^{(m)}(S^1)$ is of positive energy is also irreducible as a representation of N .*

The proof of this Lemma will actually not use that the representation of $\text{Rot}^{(m)}(S^1)$ is of finite type.

Proof. (Taken from [PS86, Proposition 9.2.3].) Denote the underlying Hilbert space of the representation in the Lemma by \mathcal{H} and the representations of N and $\text{Rot}^{(m)}(S^1)$ on \mathcal{H} by Q and R respectively. Let P be the orthogonal projection of \mathcal{H} onto a subrepresentation of Q . It commutes with Q . We will prove that P also commutes with R , which will then imply that the subrepresentation is invariant under $N \rtimes \text{Rot}^{(m)}(S^1)$ and must therefore be either $\{0\}$ or \mathcal{H} itself.

Consider the m -periodic function $\theta \mapsto R[\Phi_\theta]PR[\Phi_\theta]^*$ on the real line with values in the orthogonal projections on \mathcal{H} . We want to show that it is constant because filling in $\theta = 0$ will then imply that P commutes with R . Using the intertwining relation of R with Q and the fact that P commutes with Q it is easily checked that this family of projections commutes with Q . Define for each $a \in (1/m)\mathbb{Z}$ the a -th Fourier coefficient of this function as the operator

$$P_a := \frac{1}{m} \int_0^m e^{-2\pi ia\theta} R[\Phi_\theta]PR[\Phi_\theta]^* d\theta.$$

We give a precise meaning to this integral as follows. If $v, w \in \mathcal{H}$ are two fixed vectors, then the two functions $\mathbb{R} \rightarrow \mathcal{H}$ given by $\theta \mapsto PR[\Phi_\theta]^*v$ and $\theta \mapsto R[\Phi_\theta]^*w$ are continuous thanks to the strong continuity of R and the continuity of P . The function

$$\theta \mapsto \langle PR[\Phi_\theta]^*v, R[\Phi_\theta]^*w \rangle = \langle R[\Phi_\theta]PR[\Phi_\theta]^*v, w \rangle \quad (\text{A.9})$$

is then continuous as well and so the a -th Fourier coefficient

$$\frac{1}{m} \int_0^m e^{-2\pi ia\theta} \langle R[\Phi_\theta]PR[\Phi_\theta]^*v, w \rangle d\theta$$

of (A.9) exists. Next, we observe that the absolute value of this integral is bounded by

$$\begin{aligned} \frac{1}{m} \int_0^m \left| e^{-2\pi i a \theta} \langle R[\Phi_\theta] P R[\Phi_\theta]^* v, w \rangle \right| d\theta &\leq \frac{1}{m} \int_0^m \|R[\Phi_\theta] P R[\Phi_\theta]^* v\| \cdot \|w\| d\theta \\ &\leq \frac{1}{m} \int_0^m \|v\| \cdot \|w\| d\theta = \|v\| \cdot \|w\|, \end{aligned}$$

where we used that R is unitary and P is bounded. This estimate shows both that for $v \in \mathcal{H}$ there exists a unique vector $P_a(v) \in \mathcal{H}$ satisfying

$$\langle P_a v, w \rangle = \frac{1}{m} \int_0^m e^{-2\pi i a \theta} \langle R[\Phi_\theta] P R[\Phi_\theta]^* v, w \rangle d\theta$$

for all $w \in \mathcal{H}$, and that the operator P_a thus defined is bounded.

Using the fact that $R[\Phi_\theta] P R[\Phi_\theta]^*$ commutes with Q one can show that P_a commutes with Q as well. Furthermore, since $R[\Phi_\theta] P R[\Phi_\theta]^*$ is self-adjoint we have $P_a^* = P_{-a}$:

$$\begin{aligned} \langle P_a v, w \rangle &= \frac{1}{m} \int_0^m e^{-2\pi i a \theta} \langle v, R[\Phi_\theta] P R[\Phi_\theta]^* w \rangle d\theta \\ &= \frac{1}{m} \int_0^m \overline{e^{2\pi i a \theta} \langle R[\Phi_\theta] P R[\Phi_\theta]^* w, v \rangle} d\theta \\ &= \overline{\langle P_{-a} w, v \rangle} = \langle v, P_{-a} w \rangle. \end{aligned}$$

Pick $b \in (1/m)\mathbb{Z}$ and a vector $v \in \mathcal{H}(b)$ in the associated energy eigenspace. Then we calculate the energy of $P_a(v)$ as follows. Let $\theta' \in \mathbb{R}$ and write

$$\begin{aligned} &\langle R[\Phi_{\theta'}] P_a v, w \rangle \\ &= \frac{1}{m} \int_0^m e^{-2\pi i a \theta} \langle R[\Phi_{\theta'+\theta}] P R[\Phi_\theta]^* v, w \rangle d\theta \\ &= e^{2\pi i a \theta'} \frac{1}{m} \int_0^m e^{-2\pi i a (\theta'+\theta)} \langle R[\Phi_{\theta'+\theta}] P R[\Phi_{\theta'+\theta}]^* R[\Phi_{-\theta'}]^* v, w \rangle d\theta \\ &= e^{2\pi i (a-b)\theta'} \frac{1}{m} \int_0^m e^{-2\pi i a (\theta'+\theta)} \langle R[\Phi_{\theta'+\theta}] P R[\Phi_{\theta'+\theta}]^* v, w \rangle d\theta \\ &= e^{2\pi i (a-b)\theta'} \frac{1}{m} \int_{\theta'}^{\theta'+m} e^{-2\pi i a \theta} \langle R[\Phi_\theta] P R[\Phi_\theta]^* v, w \rangle d\theta \\ &= e^{-2\pi i (b-a)\theta'} \langle P_a v, w \rangle. \end{aligned}$$

We conclude from this that P_a maps $\mathcal{H}(b)$ to $\mathcal{H}(b - a)$.

Let $b \in (1/m)\mathbb{Z}$ be the lowest energy level of R . Then P_a annihilates $\mathcal{H}(b)$ for $a > 0$. Because \mathcal{H} is irreducible as a representation of $N \rtimes \text{Rot}^{(m)}(S^1)$ it is generated by $\mathcal{H}(b)$ under the action of $N \rtimes \text{Rot}^{(m)}(S^1)$. In other words, \mathcal{H} is the closure of the span of the set of vectors

$$\{Q(g)R[\Phi_\theta](v) \mid g \in N, \Phi_\theta \in \text{Rot}^{(\infty)}(S^1) \text{ and } v \in \mathcal{H}(b)\}.$$

However, since R leaves $\mathcal{H}(b)$ invariant \mathcal{H} is also generated by $\mathcal{H}(b)$ under the action of Q alone. The continuity of P_a and it commuting with Q then implies that P_a for $a > 0$ annihilates all of \mathcal{H} , that is, $P_a = 0$.

From the relation $P_a^* = P_{-a}$ it now follows that $P_a = 0$ for all $a \neq 0$. Because for each two fixed v and w the scalars $\langle P_a v, w \rangle$ are the Fourier coefficients of the continuous function (A.9), this function is constant by Fejér's theorem. In particular, $\langle R[\Phi_\theta]PR[\Phi_\theta]^*v, w \rangle = \langle P v, w \rangle$ for all $\theta \in \mathbb{R}$. We conclude that $R[\Phi_\theta]PR[\Phi_\theta]^* = P$ for all $\theta \in \mathbb{R}$. \square

An alternative proof of the above result is demonstrated in [Nee14, Theorem 1.5] which uses the Borchers–Arveson theorem from the theory of von Neumann algebras.

The next criterion guarantees existence of an irreducible subrepresentation.

Proposition A.3.18. *Let G be a group containing $\text{Rot}^{(m)}(S^1)$. Then a non-zero representation of G contains an irreducible subrepresentation when at least one non-zero isotypic component of the $\text{Rot}^{(m)}(S^1)$ -action is finite-dimensional.*

Our only use of the above result in this thesis is when G is of the form $N \rtimes \text{Rot}^{(m)}(S^1)$ for some group N carrying an action of $\text{Rot}^{(m)}(S^1)$.

Proof. (Compare with [Was98, Proposition I.6(c)].) Denote the representation and its underlying Hilbert space by Q and \mathcal{H} , respectively. Pick a non-zero, finite-dimensional $\text{Rot}^{(m)}(S^1)$ -isotypic component $\mathcal{H}(a)$, where $a \in (1/m)\mathbb{Z}$. We first show that for any subrepresentation \mathcal{K} of Q there is a decomposition

$$\mathcal{H}(a) = (\mathcal{H}(a) \cap \mathcal{K}) \oplus (\mathcal{H}(a) \cap \mathcal{K}^\perp). \quad (\text{A.10})$$

One inclusion is obvious. For the reverse inclusion, let $v \in \mathcal{H}(a)$ and write it as $v = w + w^\perp$ with $w \in \mathcal{K}$ and $w^\perp \in \mathcal{K}^\perp$. We want to prove that $w, w^\perp \in \mathcal{H}(a)$. Take $[\Phi_\theta] \in \text{Rot}^{(m)}(S^1)$. Then on the one hand

$$Q([\Phi_\theta])(v) = Q([\Phi_\theta])(w) + Q([\Phi_\theta])(w^\perp).$$

On the other hand,

$$Q([\Phi_\theta])(v) = e^{2\pi ia\theta} v = e^{2\pi ia\theta} w + e^{2\pi ia\theta} w^\perp,$$

which proves (A.10) by the unicity of decompositions of vectors along a direct sum.

We may without loss of generality now assume that \mathcal{H} is generated by $\mathcal{H}(a)$ under Q . Otherwise we could consider the subrepresentation of \mathcal{H} which is generated by $\mathcal{H}(a)$ under Q instead because this satisfies the same hypotheses in the Proposition as \mathcal{H} .

Consider the following set

$$\{\mathcal{K} \cap \mathcal{H}(a) \mid \mathcal{K} \text{ is a subrepresentation of } Q\} \setminus \{0\}$$

of (finite-dimensional) $\text{Rot}^{(m)}(S^1)$ -subrepresentations of $\mathcal{H}(a)$. It is non-empty because it contains at least $\mathcal{H}(a) = \mathcal{H} \cap \mathcal{H}(a)$. Let \mathcal{K}_0 be a subrepresentation of Q such that $\mathcal{K}_0 \cap \mathcal{H}(a)$ has the smallest dimension of all the elements in the above set. We claim that \mathcal{K}_0 is irreducible for Q .

Let $\mathcal{K} \subseteq \mathcal{K}_0$ be a subrepresentation of Q . Suppose that $\mathcal{K} \cap \mathcal{H}(a) = \mathcal{K}_0 \cap \mathcal{H}(a)$. Then $(\mathcal{K}^\perp \cap \mathcal{K}_0) \cap \mathcal{H}(a) = \{0\}$ and so by (A.10) there holds

$$\mathcal{H}(a) \subseteq (\mathcal{K}^\perp \cap \mathcal{K}_0)^\perp = \mathcal{K} + \mathcal{K}_0^\perp.$$

Because \mathcal{H} is generated under Q by $\mathcal{H}(a)$ it follows that $\mathcal{K} + \mathcal{K}_0^\perp = \mathcal{H}$ and so $\mathcal{K}^\perp \cap \mathcal{K}_0 = \{0\}$. Hence we have $\mathcal{K} = \mathcal{K}_0$. Suppose on the contrary that $\mathcal{K} \cap \mathcal{H}(a) \subsetneq \mathcal{K}_0 \cap \mathcal{H}(a)$. Then, by the minimality assumption on \mathcal{K}_0 , we have $\mathcal{K} \cap \mathcal{H}(a) = \{0\}$. By (A.10) this implies that $\mathcal{H}(a)$ is entirely contained in \mathcal{K}^\perp . Using again that \mathcal{H} is generated under Q by $\mathcal{H}(a)$ we conclude that $\mathcal{K}^\perp = \mathcal{H}$ and so $\mathcal{K} = \{0\}$. \square

Having proved these results, we show that positive energy representations are similar to finite-dimensional ones in the following regard:

Proposition A.3.19. (See [Was98, Proposition I.6(d)].) *A non-zero, positive energy representation of a group is the (internal) direct sum of a set of mutually orthogonal, irreducible, positive energy subrepresentations.*

Proof. Denote the group, the representation and its underlying Hilbert space by N , Q and \mathcal{H} , respectively. We are given that for some $m \geq 1$ there exists a positive energy representation R of $\text{Rot}^{(m)}(S^1)$ on \mathcal{H} which intertwines with Q .

A non-zero subrepresentation $\mathcal{K} \subseteq \mathcal{H}$ for $N \rtimes \text{Rot}^{(m)}(S^1)$ is again the direct sum of its energy eigenspaces for the restriction of R to \mathcal{K} , each of which is of

course again finite-dimensional. Therefore at least one of these eigenspaces must be non-zero and so \mathcal{H} contains an irreducible subrepresentation for $N \rtimes \text{Rot}^{(m)}(S^1)$ by Proposition A.3.18. The representation of $N \rtimes \text{Rot}^{(m)}(S^1)$ on \mathcal{H} thus satisfies the criterion of Proposition A.3.6. The summands in the resulting orthogonal decomposition into irreducible subrepresentations for $N \rtimes \text{Rot}^{(m)}(S^1)$ are then irreducible for Q as well by Lemma A.3.17. \square

We close this section by introducing notation and a name for the generating function of the dimensions of the energy eigenspaces of a positive energy representation.

Definition A.3.20. (After [FLM88, Section 1.10] and [PS86, Definition 14.1.1].) Let R be a positive energy representation of $\text{Rot}^{(m)}(S^1)$ for some $m \geq 1$ on a Hilbert space \mathcal{H} . The formal Laurent series

$$\text{ch}_R(q) := \sum_{a \in (1/m)\mathbb{Z}_{\geq 0}} \dim(\mathcal{H}(a)) q^a$$

in a formal variable q is called the (*graded*) *character* of R .

A.4 Heisenberg groups

In this section we will define Heisenberg groups and their Weyl representations. They are relevant to this thesis because the central extensions of uni- and bicoloured torus loop groups constructed in Sections 2.2 and 3.2 turn out to contain Heisenberg groups. This fact is the key to the representation theory of the former.

References. The material in this section is taken from [PS86, Section 9.5], [Par92, Chapter II] and [Ism96, Chapter I]. More specific references will be given in the text.

Definition A.4.1. Let V be a topological real vector space carrying a continuous, non-degenerate, bilinear skew form $S: V \times V \rightarrow \mathbb{R}$. The *Heisenberg group* \tilde{V} associated to the pair (V, S) is the $U(1)$ -central extension of the underlying topological abelian group of V defined by the 2-cocycle

$$c: V \times V \rightarrow U(1), \quad c(\xi, \eta) := e^{-2\pi i S(\xi, \eta)}$$

on V .

Spelling this out, \tilde{V} has as underlying topological space $V \times U(1)$ and its continuous multiplication and inverse are given by

$$(\xi, z) \cdot (\eta, w) := (\xi + \eta, zw \cdot c(\xi, \eta)), \quad (\xi, z)^{-1} = (-\xi, z^{-1} \cdot c(\xi, -\xi)^{-1}),$$

respectively, for $\xi, \eta \in V$ and $z, w \in U(1)$.

Note that c is indeed a normalised 2-cocycle because S is bi-additive. The centre of \tilde{V} consists of (the image of) $U(1)$ alone. If namely an element $(\xi, z) \in \tilde{V}$ commutes with all $(\eta, w) \in \tilde{V}$, then $c(\xi, \eta)^2 = 1$ using the skew-symmetry of S and so $2S(\xi, \eta) \in \mathbb{Z}$ for all $\eta \in V$. The bilinearity of S then implies that $S(\xi, \eta) = 0$ and therefore $\xi = 0$ because S is non-degenerate. The non-degeneracy and skew-symmetry of S hence ensure that the centre of $U(1)$ is as small as possible.

Of course, we did not need the structure of a scalar multiplication on V in order to define the Heisenberg group. The vector space structure is needed to get a hold on the representation theory—a topic we turn to now. We will first construct the underlying Hilbert spaces of the representations we will be studying.

A.4.1 Bosonic Fock spaces

Let V be a real vector space carrying a *complex structure* $J : V \rightarrow V$, meaning a linear endomorphism (which is automatically an automorphism) such that $J^2 = -\text{id}_V$. Let furthermore $\langle \cdot, \cdot \rangle_J$ be a Hermitian inner product on the corresponding complex vector space V_J . Then we associate a new, larger complex Hilbert space to the complex pre-Hilbert space $(V_J, \langle \cdot, \cdot \rangle_J)$ as follows.

First, form the *symmetric algebra* $\text{Sym}^*(V_J)$ on V_J . It is an algebraic direct sum

$$\text{Sym}^*(V_J) := \bigoplus_{k=0}^{\infty} \text{Sym}^k(V_J),$$

where the k -th *symmetric power* $\text{Sym}^k(V_J)$ of V_J is the complex linear span of monomials in the vectors of V_J of degree k , which are considered as commuting variables. By definition, $\text{Sym}^0(V_J) := \mathbb{C}$ and the 0-th power of any vector in V_J is $1 \in \text{Sym}^0(V_J)$. We will ignore the algebra structure and consider $\text{Sym}^*(V_J)$ as a complex vector space. A general vector of it is a sum

$$\sum_{k=0}^{\infty} v_k, \tag{A.11}$$

where $v_k \in \text{Sym}^k(V_J)$ and only finitely many summands are non-zero.

This vector space $\text{Sym}^*(V_J)$ inherits a Hermitian inner product from V_J . If $\xi_1 \xi_2 \cdots \xi_k$ and $\xi'_1 \xi'_2 \cdots \xi'_k$ are namely two monomials of the same degree k then we set their inner product to be

$$\langle \xi_1 \xi_2 \cdots \xi_k, \xi'_1 \xi'_2 \cdots \xi'_k \rangle := \sum_{\sigma \in S_k} \langle \xi_1, \xi'_{\sigma(1)} \rangle_J \cdots \langle \xi_k, \xi'_{\sigma(k)} \rangle_J, \quad (\text{A.12})$$

where S_k is the symmetric group on k symbols. Two monomials of different degrees are set to be orthogonal. This inner product on monomials is finally extended to general vectors of $\text{Sym}^*(V_J)$ by (Hermitian) linearity.

The positive definiteness can be seen by picking an orthonormal basis for V_J . The values of the inner product are then determined on the monomials in these basis elements. If in the expression (A.12) the vectors ξ_i and ξ'_i are such basis elements, this value is only non-zero when there is a permutation σ such that $\xi'_{\sigma(i)} = \xi_i$ for all i and in that case it is clearly non-negative because $\langle \cdot, \cdot \rangle_J$ is positive definite.

We conclude that $\text{Sym}^*(V_J)$ in this way becomes a complex pre-Hilbert space.

Definition A.4.2. The *bosonic Fock space* $\mathcal{S}(V_J)$ associated to the triple

$$(V, J, \langle \cdot, \cdot \rangle_J)$$

is the Hilbert space completion of $\text{Sym}^*(V_J)$.

A general vector of $\mathcal{S}(V_J)$ is a series of the form (A.11) where possibly infinitely many summands are non-zero, but

$$\sum_{k=0}^{\infty} \|v_k\|^2 < \infty.$$

The inner product of two such vectors is given summand-wise.

In order to later define unitary operators on $\mathcal{S}(V_J)$ we single out a special class of its vectors.

Definition A.4.3. Let $\xi \in V_J$. The *coherent vector* associated to ξ is the formal power series

$$e^\xi := \sum_{k=0}^{\infty} \frac{\xi^k}{k!}$$

seen as an element of the infinite product $\prod_{k=0}^{\infty} \text{Sym}^k(V_J)$.

Proposition A.4.4. *Coherent vectors satisfy the following properties:*

- (i) $e^\xi \in \mathcal{S}(V_J)$ for all $\xi \in V_J$, that is, the partial sums defining e^ξ converge in $\mathcal{S}(V_J)$,
- (ii) if also $\eta \in V_J$ then $\langle e^\xi, e^\eta \rangle = e^{\langle \xi, \eta \rangle_J}$,
- (iii) every finite set of coherent vectors is linearly independent,
- (iv) the space of finite linear combinations of coherent vectors lies densely in $\mathcal{S}(V_J)$,
- (v) the function $V_J \rightarrow \mathcal{S}(V_J)$ given by $\xi \mapsto e^\xi$ is continuous with respect to the norm topology on V_J induced by $\langle \cdot, \cdot \rangle_J$.

Proof. If $n \geq 0$, then we calculate that, according to the definition of the inner product (A.12),

$$\left\langle \sum_{k=0}^n \frac{\xi^k}{k!}, \sum_{l=0}^n \frac{\eta^l}{l!} \right\rangle = \sum_{k,l=0}^n \frac{\langle \xi^k, \eta^l \rangle}{k! \cdot l!} = \sum_{k=0}^n \frac{k! \cdot \langle \xi, \eta \rangle_J^k}{(k!)^2} = \sum_{k=0}^n \frac{\langle \xi, \eta \rangle_J^k}{k!}.$$

This proves both (i) and (ii).

(iii): Let $e^{\xi_1}, \dots, e^{\xi_n}$ be a finite set of coherent vectors and suppose that $\sum_{k=1}^n \alpha_k e^{\xi_k} = 0$ for some scalars $\alpha_k \in \mathbb{C}$. Then define for each (unordered) pair $\{k, l\} \subseteq \{1, \dots, n\}$ of indices with $k \neq l$ the following subset of vectors of V_J :

$$E_{k,l} := \{ \xi \in V_J \mid \langle \xi, \xi_k \rangle_J \neq \langle \xi, \xi_l \rangle_J \}.$$

One can think of $E_{k,l}$ as the complement of the hyperplane orthogonal to the line $\mathbb{C} \cdot (\xi_k - \xi_l)$. It is open and lies densely in V_J . Since a finite intersection of open and dense subsets is again dense, the intersection

$$\bigcap_{\substack{1 \leq k, l \leq n \\ k \neq l}} E_{k,l}$$

is in particular not empty. In other words: there exists a vector $\xi \in V_J$ such that the n scalars $z_k := \langle \xi, \xi_k \rangle_J$ are all distinct. It is well known that this implies that the n functions $\mathbb{C} \rightarrow \mathbb{C}$ given by $z \mapsto e^{z \cdot z_k}$ are linearly independent. Because for all $z \in \mathbb{C}$ there holds

$$\sum_{k=1}^n \overline{\alpha_k} e^{z \cdot z_k} = \sum_{k=1}^n \overline{\alpha_k} \langle e^{z \cdot \xi}, e^{\xi_k} \rangle = \left\langle e^{z \cdot \xi}, \sum_{k=1}^n \alpha_k e^{\xi_k} \right\rangle = 0$$

by (ii), we conclude that $\alpha_k = 0$ for all k .

(iv): Set $\mathcal{H} \subseteq \mathcal{S}(V_J)$ to be the closure of the span of all the exponential vectors. Let $\xi \in V_J$ and consider the function $\mathbb{R} \rightarrow \mathcal{H}$ given by $\theta \mapsto e^{\theta \cdot \xi}$. It is smooth because the series $e^{\theta \cdot \xi}$ is absolutely convergent. Its n -th derivative is valued in \mathcal{H} also and is given by term-wise differentiation:

$$\theta \mapsto \sum_{k=0}^{\infty} \xi^k \cdot \frac{(\theta \xi)^k}{k!}.$$

In particular, the monomial

$$\xi^k = \left. \frac{d^k}{d\theta^k} e^{\theta \cdot \xi} \right|_{\theta=0} \quad (\text{A.13})$$

lies in \mathcal{H} for all $k \geq 0$. Now note that by taking partial derivatives a more general monomial $\xi_1 \cdots \xi_k$ can be expressed in terms of a power of a single vector:

$$k! \cdot \xi_1 \cdots \xi_k = \left. \frac{\partial^k}{\partial \theta_1 \cdots \partial \theta_k} (\theta_1 \xi_1 + \cdots + \theta_k \xi_k)^k \right|_{\theta_1 = \cdots = \theta_k = 0}. \quad (\text{A.14})$$

Having just learned that $(\sum_{i=1}^k \theta_i \xi_i)^k \in \mathcal{H}$, we see that $\xi_1 \cdots \xi_k \in \mathcal{H}$ as well. Hence, \mathcal{H} equals $\mathcal{S}(V_J)$.

(v): If $\xi, \eta \in V_J$ then, using (ii),

$$\|e^\xi - e^\eta\|^2 = e^{\|\xi\|_J^2} + e^{\|\eta\|_J^2} - 2 \cdot \text{Re} e^{\langle \xi, \eta \rangle_J}.$$

Together with the continuity of the inner product $\langle \cdot, \cdot \rangle_J$ this proves what was asked. \square

The properties (i)–(iv) in this Proposition allow one to uniquely specify a unitary operator on $\mathcal{S}(V_J)$ by prescribing its values on the coherent vectors and checking whether it then preserves inner products.

A.4.2 The Weyl representations of a Heisenberg group

Given a real vector space, we saw in Definition A.4.1 how to associate a Heisenberg group to a skew form, and in Appendix A.4.1 how to forge a bosonic Fock space from a complex structure and a Hermitian inner product. In this subsection we will build a representation of the former group on the latter Hilbert space, but in order to do so the skew form and the complex structure will need to be compatible in a certain sense and, moreover, the inner product should be a specific one derived from these two pieces of structure.

Proposition A.4.5. *Let V be a real vector space carrying a non-degenerate, bilinear skew form S and a complex structure $J: V \rightarrow V$. Write V_J for the associated complex vector space. Then the following compatibility requirements between S and J are equivalent:*

(i) *S is J -invariant, that is, $S(J\xi, J\eta) = S(\xi, \eta)$ for all $\xi, \eta \in V$, and J tames S , meaning that $S(\xi, J\xi) > 0$ for all non-zero $\xi \in V$,*

(ii) *the bilinear form*

$$g_J: V \times V \rightarrow \mathbb{R}, \quad g_J(\xi, \eta) := S(\xi, J\eta)$$

is J -invariant and makes V a real pre-Hilbert space,

(iii) *the form*

$$\langle \cdot, \cdot \rangle_J: V_J \times V_J \rightarrow \mathbb{C}, \quad \langle \xi, \eta \rangle_J := 2\pi(S(\xi, J\eta) - iS(\xi, \eta)),$$

makes V_J a complex pre-Hilbert space, that is, $\langle \cdot, \cdot \rangle_J$ is a Hermitian inner product.^{2,3}

Remark A.4.6. If a pair (S, J) satisfies the equivalent properties of Proposition A.4.5, then so does the pair $(-S, -J)$. One gets a Hermitian inner product on the conjugate complex pre-Hilbert space V_{-J} .

Let V be a real vector space carrying a non-degenerate, bilinear skew form S and a complex structure $J: V \rightarrow V$. Write V_J for the associated complex vector space and suppose that S and J satisfy the equivalent compatibility requirements of Proposition A.4.5. Then S is continuous with respect to the norm topology on V induced by the Hermitian inner product $\langle \cdot, \cdot \rangle_J$ because the latter is continuous and S is up to a scalar its imaginary part:

$$S(\xi, \eta) = \frac{i}{4\pi} (\langle \xi, \eta \rangle_J - \langle \eta, \xi \rangle_J) = -\frac{1}{2\pi} \cdot \text{Im} \langle \xi, \eta \rangle_J.$$

We may therefore construct a Heisenberg group \tilde{V} from S .

Theorem A.4.7. *Let $\mathcal{S}(V_J)$ be the bosonic Fock space associated to the Hermitian inner product $\langle \cdot, \cdot \rangle_J$ in Proposition A.4.5(iii). Define for $(\xi, z) \in \tilde{V}$ and $\kappa \in V_J$ a vector⁴*

$$W_J(\xi, z)(e^\kappa) := z \cdot e^{-\frac{1}{2}\langle \xi, \xi \rangle_J - \langle \kappa, \xi \rangle_J} \cdot e^{\kappa + \xi} \in \mathcal{S}(V_J). \quad (\text{A.15})$$

²Recall our convention that a Hermitian inner product is complex linear in its *first* variable.

³The factor 2π in the definition of $\langle \cdot, \cdot \rangle_J$ is of course irrelevant for the statement of this Proposition on its own. It serves as a normalisation to ensure compatibility with our Definition A.4.1 of a Heisenberg group.

⁴On the right hand side of the following equation we consider ξ as a vector of V_J .

Then

- (i) this is an action of the group \tilde{V} on the space of finite linear combinations of the coherent vectors in $\mathcal{S}(V_J)$,
- (ii) which preserves inner products.

Therefore, (A.15) extends to a unitary representation W_J of \tilde{V} on $\mathcal{S}(V_J)$. It is strongly continuous.

Proof. After filling in the definitions of the multiplication in \tilde{V} and of W_J , checking the equation

$$W_J((\xi, z) \cdot (\eta, w))(e^\kappa) = W_J(\xi, z)(W_J(\eta, w)(e^\kappa))$$

for all $(\eta, w) \in \tilde{V}$ quickly comes down to verifying whether

$$\frac{1}{2}\langle \xi, \eta \rangle_J - \frac{1}{2}\langle \eta, \xi \rangle_J = -2\pi i \cdot S(\xi, \eta).$$

This follows from the definition in Proposition A.4.5(iii) of $\langle \cdot, \cdot \rangle_J$ in terms of S and J :

$$\frac{1}{2}(\langle \xi, \eta \rangle_J - \langle \eta, \xi \rangle_J) = \frac{1}{2}(\langle \xi, \eta \rangle_J - \overline{\langle \xi, \eta \rangle_J}) = i \cdot \text{Im} \langle \xi, \eta \rangle_J = -2\pi i \cdot S(\xi, \eta).$$

Proving that inner products are preserved is a matter of writing out

$$\begin{aligned} \langle W_J(\xi, z)(e^\eta), W_J(\xi, z)(e^\kappa) \rangle &= |z| e^{-\frac{1}{2}\langle \xi, \xi \rangle_J - \langle \eta, \xi \rangle_J} \cdot \overline{e^{-\frac{1}{2}\langle \xi, \xi \rangle_J - \langle \kappa, \xi \rangle_J}} \cdot \langle e^{\eta+\xi}, e^{\kappa+\xi} \rangle \\ &= e^{-\langle \xi, \xi \rangle_J} e^{-\langle \eta, \xi \rangle_J} e^{-\langle \xi, \kappa \rangle_J} \cdot e^{\langle \eta+\xi, \kappa+\xi \rangle_J} \\ &= e^{\langle \eta, \kappa \rangle_J} = \langle e^\eta, e^\kappa \rangle, \end{aligned}$$

where we used Proposition A.4.4(ii) for the last equality.

That the action (A.15) first extends from coherent vectors to finite linear combinations of those is implied by Proposition A.4.4(iii), and that it then extends further to all of $\mathcal{S}(V_J)$ follows from Proposition A.4.4(iv).

To prove strong continuity of this representation we show that for all vectors $v \in \mathcal{S}(V_J)$ the map $\tilde{V} \rightarrow \mathcal{S}(V_J)$ given by $(\xi, z) \mapsto W_J(\xi, z)(v)$ is continuous. It is sufficient to prove this when v is a coherent vector e^κ and in that case this can be seen from the continuity of the inner product $\langle \cdot, \cdot \rangle_J$ together with Proposition A.4.4(v), which tells us that the composite function

$$\tilde{V} \rightarrow V \xrightarrow{\sim} V_J \rightarrow \mathcal{S}(V_J), \quad (\xi, z) \mapsto \xi \mapsto \kappa + \xi \mapsto e^{\kappa+\xi}$$

is continuous. □

The factor $e^{-\frac{1}{2}\langle \xi, \xi \rangle_J - \langle \kappa, \xi \rangle_J}$ in (A.15) can be understood as a correction to make the action unitary. It has a more conceptual meaning also, though, namely that of a certain *Radon-Nikodym derivative*. This is explained in, for example, [Ism96, Chapter 1, §1.3].

Definition A.4.8. The representation (A.15) of the Heisenberg group \tilde{V} on the bosonic Fock space $\mathcal{S}(V_J)$ is called the *Weyl representation* (associated to the complex structure J on V).

Theorem A.4.9. *The Weyl representation W_J is irreducible.*

Proof. (After [Ism96, Theorem 7.1].) Let T be an endomorphism of W_J . We wish to show that T is a scalar multiple of the identity operator because that will imply the irreducibility of W_J by Schur's lemma.

Let us first study the vector $T(1)$, where $1 \in \mathbb{C} =: \text{Sym}^0(V_J) \subseteq \mathcal{S}(V_J)$. Define for a fixed vector $\xi \in V_J$ the function

$$\mathbb{C} \rightarrow \mathbb{C}, \quad z \mapsto \langle T1, e^{z\xi} \rangle. \quad (\text{A.16})$$

By expanding $e^{z\xi}$ into a power series we see that this function is anti-holomorphic. On the other hand, using the unitarity of W_J and that $1 = e^0$, we can rewrite it as

$$\begin{aligned} \langle T1, e^{z\xi} \rangle &= \langle W_J(-z\xi, 1)(T1), W_J(-z\xi, 1)(e^{z\xi}) \rangle \\ &= \langle TW_J(-z\xi, 1)(1), e^{\frac{1}{2}|z|^2\langle \xi, \xi \rangle_J} \cdot 1 \rangle \\ &= \langle T(e^{-\frac{1}{2}|z|^2\langle \xi, \xi \rangle_J} \cdot e^{-z\xi}), e^{\frac{1}{2}|z|^2\langle \xi, \xi \rangle_J} \cdot 1 \rangle \\ &= \langle Te^{-z\xi}, 1 \rangle = \langle e^{-z\xi}, T^*1 \rangle. \end{aligned}$$

Hence (A.16) is also holomorphic. Therefore, it is a constant function. It then follows from the expressions (A.13) and (A.14) respectively that $\langle T1, \xi^k \rangle = 0$ and $\langle T1, \xi_1 \cdots \xi_k \rangle = 0$ for all $k \geq 1$ and monomials $\xi_1 \cdots \xi_k \in \text{Sym}^k(V_J)$. This means that $T(1)$ is a scalar multiple of 1, say, $T(1) = \alpha \in \mathbb{C}$. Then also $T(e^\xi) = \alpha \cdot e^\xi$ holds for all $\xi \in V_J$ since

$$e^\xi = e^{\frac{1}{2}\langle \xi, \xi \rangle_J} W_J(\xi, 1)(1).$$

The density of the coherent vectors in $\mathcal{S}(V_J)$ finishes the argument. In other words, we use that 1 (or, more generally, any other coherent vector) is cyclic for W_J . \square

The following Proposition explains that the construction of a Weyl representation is in a certain sense functorial.

Proposition A.4.10. *Let (V_i, S_i, J_i) for $i = 1, 2$ be two triples satisfying the demands above and let $g: V_1 \xrightarrow{\sim} V_2$ be an \mathbb{R} -linear isomorphism which preserves the skew forms S_i and intertwines the complex structures J_i . Write $g^*\xi_1 \in V_2$ for the image of $\xi_1 \in V_1$. Then*

- (i) *g lifts to a continuous group isomorphism $\tilde{V}_1 \xrightarrow{\sim} \tilde{V}_2$ between the associated Heisenberg groups via $g \cdot (\xi_1, z_1) := (g^*\xi_1, z_1)$, where $(\xi_1, z_1) \in \tilde{V}_1$, and*
- (ii) *there is a unitary operator $U(g): \mathcal{S}((V_1)_{J_1}) \xrightarrow{\sim} \mathcal{S}((V_2)_{J_2})$ which extends the (\mathbb{R} -linear) isomorphism g between the subspaces $(V_i)_{J_i} = \text{Sym}^1((V_i)_{J_i}) \subseteq \mathcal{S}((V_i)_{J_i})$,*

such that $U(g)$ intertwines the Weyl representations W_{J_i} of \tilde{V}_i on $\mathcal{S}((V_i)_{J_i})$, that is,

$$U(g)W_{J_1}(\xi_1, z_1)U(g)^* = W_{J_2}(g \cdot (\xi_1, z_1)).$$

The proof of the claim above is identical to the one in the ‘absolute’ (as opposed to ‘relative’) situation when $(V_1, S_1, J_1) = (V_2, S_2, J_2)$. We will therefore only give a proof in this latter case and we simultaneously include the hypothesis that we have a group of such automorphisms g :

Proposition A.4.11. *Let G be a topological group acting by \mathbb{R} -linear automorphisms on V which preserves S , commutes with J and is strongly continuous with respect to the norm topology on V induced by $\langle \cdot, \cdot \rangle_J$. Write $g^*\xi$ for the translate of $\xi \in V$ by $g \in G$. Then*

- (i) *G acts strongly continuously on \tilde{V} as $g \cdot (\xi, z) := (g^*\xi, z)$, where $(\xi, z) \in \tilde{V}$, and*
- (ii) *there is a representation $U: G \rightarrow \text{U}(\mathcal{S}(V_J))$ which extends the (\mathbb{R} -linear) action on the subspace $V_J = \text{Sym}^1(V_J) \subseteq \mathcal{S}(V_J)$,*

such that the intertwining property

$$U(g)W_J(\xi, z)U(g)^* = W_J(g \cdot (\xi, z)) \tag{A.17}$$

is satisfied.

Proof. Indeed, (i) is true because G preserves S , and so it also does not perturb the cocycle c defining \tilde{V} .

To prove (ii), we first note that since G commutes with J it acts by \mathbb{C} -linear operators on V_J . Moreover, it commuting with J and preserving S implies by the definition of the Hermitian inner product $\langle \cdot, \cdot \rangle_J$ in Proposition A.4.5(iii)

that G preserves $\langle \cdot, \cdot \rangle_J$. That is, G acts by unitary operators on the complex pre-Hilbert space $(V_J, \langle \cdot, \cdot \rangle_J)$.

By applying G factor-wise to monomials in the vectors of V_J , we see from the definition of the inner product (A.12) that we get a unitary G -action on $\text{Sym}^*(V_J)$ and hence also on its completed Hilbert space $\mathcal{S}(V_J)$. We denote the latter action by U . Equivalently, we may define U first argument-wise on the coherent vectors by setting $U(g)(e^\kappa) := e^{g^*\kappa}$. Proposition A.4.4(ii) says that this respects inner products and therefore the action extends to all of $\mathcal{S}(V_J)$.

Proving the strong continuity of U comes down to checking whether, if $\kappa \in V_J$, then $g_n \rightarrow g$ in G implies that $e^{g_n^*\kappa} \rightarrow e^{g^*\kappa}$ in $\mathcal{S}(V_J)$. This follows from the strong continuity of the G -action on V_J and Proposition A.4.4(v).

To show the intertwining property (A.17) it suffices to equate the actions of the operators on both sides on coherent vectors. If $\kappa \in V_J$, then

$$\begin{aligned} U(g)W_J(\xi, z)U(g)^*(e^\kappa) &= U(g)W_J(\xi, z)e^{(g^{-1})^*\kappa} \\ &= z \cdot e^{-\frac{1}{2}\langle \xi, \xi \rangle_J - \langle (g^{-1})^*\kappa, \xi \rangle_J} \cdot U(g)(e^{(g^{-1})^*\kappa + \varepsilon}) \\ &= z \cdot e^{-\frac{1}{2}\langle \xi, \xi \rangle_J - \langle (g^{-1})^*\kappa, \xi \rangle_J} \cdot e^{\kappa + g^*\xi}. \end{aligned}$$

Because g preserves $\langle \cdot, \cdot \rangle_J$ this is equal to

$$W_J(g^*\xi, z)(e^\kappa) = W_J(g \cdot (\xi, z))(e^\kappa). \quad \square$$

Remark A.4.12 (The groupoid of Weyl representations). While the above result easily allows one to exhibit many symmetries of a Weyl representation—sufficiently many for the purposes in this thesis—it is worth noting that these are not the only ones such a representation possesses. It namely turns out that, in order to implement the elements of a group G , they need not necessarily commute with J , but conjugating J should merely ‘not distort J too much’. More precisely, for an element $g \in G$ the commutator $[g, J]$ should be a *Hilbert–Schmidt operator* on V_J . This condition is necessary as well. The result is then not a representation of G itself on $\mathcal{S}(V_J)$, but of a certain $U(1)$ -central extension instead. We refer for these claims to [PS86, Proposition 9.5.9].

We can therefore say that to the space of complex structures on V that satisfy Proposition A.4.5 there is associated a category of which the objects are the corresponding Weyl representations and the morphisms are the \tilde{V} -intertwiners. Schur’s lemma then tells us that this is a groupoid, while the aforementioned claims (or, rather, their generalisations to the ‘relative’ case) declare that it is in general not connected: the connected components are exactly the full subgroupoids associated to each *polarisation class* of complex structures that differ from each other by a Hilbert–Schmidt operator.

With these Weyl representations we have constructed one family of irreducible representations of a Heisenberg group. Having the classical Stone–von Neumann theorem for finite-dimensional Heisenberg groups in mind (see for example [Fol89, Theorem 1.50]), it is reasonable to ask to what extent a similar uniqueness result holds for the possibly infinite-dimensional Heisenberg groups we are considering here. Of course, the action (A.15) can be tweaked a little by letting the central subgroup $U(1)$ act by a non-trivial character instead. It turns out that this is the only freedom we have when we additionally assume the positive energy property:

Theorem A.4.13 (The Stone–von Neumann theorem for positive energy representations). *(See [PS86, Proposition 9.5.10].) Suppose there exists a complex structure J and, for some $m \geq 1$, a $\text{Rot}^{(m)}(S^1)$ -action on V satisfying the demands of Proposition A.4.11 such that the resulting representation of $\text{Rot}^{(m)}(S^1)$ on $\mathcal{S}(V_J)$ is of positive energy. Then W_J is, up to isomorphism, the unique irreducible, positive energy representation of \tilde{V} such that the central subgroup $U(1)$ acts as $z \mapsto z$.*

According to Proposition A.3.19, the positive energy assumption additionally gives us complete knowledge of representations of \tilde{V} that are not necessarily irreducible: they are simply direct sums of the unique irreducible one.

The proof of the above Theorem involves the Lie algebra of \tilde{V} and its representation by densely defined, skew-adjoint operators on any representation of \tilde{V} . These are topics we did not have the chance to treat.

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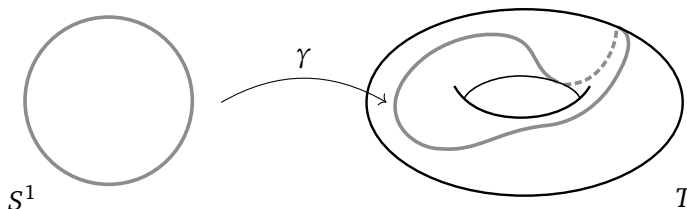
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Samenvatting

Hier zal ik een poging doen de belangrijkste onderwerpen en resultaten uit dit proefschrift op een meer elementaire manier uiteen te zetten. We zullen eerst de unigekleurde toruslusgroepen uit Hoofdstuk 2 behandelen, om vervolgens over te gaan naar de theorie van de bigekleurde toruslusgroepen uit Hoofdstuk 3.

Unigekleurde toruslusgroepen

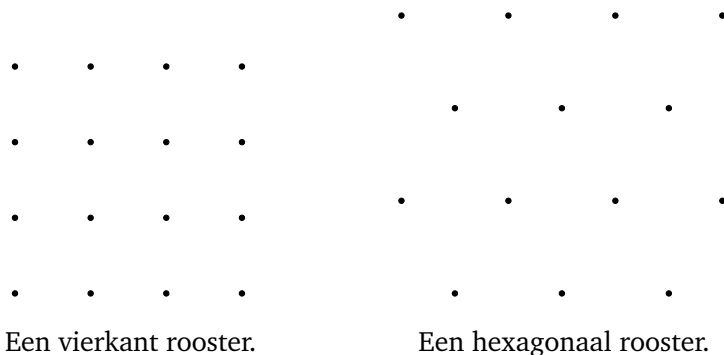
Om uit te leggen wat een toruslusgroep is beginnen we met het kiezen van een willekeurige, maar vaste torus T waar de toruslusgroep van af zal hangen. (De betekenis van het bijvoeglijk naamwoord ‘unigekleurde’ in de titel van deze sectie zal toegelicht worden in de volgende sectie.) Vervolgens beschouwen we de verzameling van alle gesloten lussen die op T liggen. Wij eisen van elke lus slechts dat deze in zekere zin glad is, maar het is hem toegestaan om zichzelf te doorsnijden of meerdere keren om de torus te wikkelen vóór hij sluit. Deze collectie van lussen is dus zeer groot, maar blijkt over een interessante structuur te beschikken. Merk namelijk eerst op dat T als torus niet alleen een meetkundig object is, maar ook een algebraïsche structuur bezit. De punten op T kunnen namelijk bij elkaar worden opgeteld, van elkaar worden afgetrokken en ook is er voor deze optelling een neutraal punt 0 . Dit maakt T een *groep*. Stel een lus γ op T nu voor als een afbeelding van de eenheidscirkel S^1 naar T toe.



(In de illustratie is T als zijnde 2-dimensionaal weergegeven, maar een

hoger-dimensionale torus is in onze discussie zeker ook toegestaan.) Dit wil zeggen dat we de punten op de lus γ schrijven als $\gamma(\theta)$, waar θ een punt op S^1 is. Als ρ dan een andere lus op T is kan deze met γ puntsgewijs worden opgeteld, wat een nieuwe lus $\gamma + \rho$ oplevert. Hiermee zien we dat de verzameling van alle lussen op T zelf weer een groep is, die we noteren met LT en de (*torus*)lusgroep behorende bij T noemen. Zijn neutrale element voor de optelling is de lus die volledig in het punt 0 van T geconcentreerd is.

In dit werk hebben niet zo zeer toruslusgroepen zelf onze interesse, als wel bepaalde van hun *centrale uitbreidingen* die geassocieerd zijn aan *roosters*. Een (positief definitief) rooster van dimensie n is een oneindige verzameling regelmatig verdeelde punten in de n -dimensionale Euclidische ruimte \mathbb{R}^n die 1) de oorsprong 0 bevat, 2) \mathbb{R}^n opspant, en 3) zodanig gepositioneerd is dat het inproduct tussen de twee vectoren vanuit 0 naar iedere twee punten toe een geheel getal is. Onderstaand zijn twee voorbeelden van (delen van) 2-dimensionale roosters weergegeven.



In hogere dimensies zijn exotischere voorbeelden te vinden en het is vaak vruchtbaar om een rooster als een meetkundig object op zich te beschouwen.

In Sectie 2.2 van dit proefschrift wordt verteld hoe, vanuit de toruslusgroep LT , met behulp van ieder rooster Λ van een speciaal type en van dezelfde dimensie als T een bepaalde grotere groep $\tilde{L}T$ gecreëerd kan worden. Een element van $\tilde{L}T$ is niet langer slechts een lus, maar een paar (γ, z) , waar γ een lus op T is en dus behoort tot LT , terwijl z een complex getal van modulus 1 is. Merk op dat zulke complexe getallen zelf ook een groep vormen, welke we noteren met $U(1)$. Voor ieder element van LT bevat $\tilde{L}T$ dus zoveel kopieën als er in $U(1)$ liggen. Tot zover hebben wij $\tilde{L}T$ slechts als verzameling beschreven. De rol van het rooster Λ ligt in het bepalen van de vermenigvuldiging van de elementen van $\tilde{L}T$ met elkaar op zo een manier dat $\tilde{L}T$, net zoals LT , een groep vormt. We zeggen dat $\tilde{L}T$ een *centrale uitbreiding* van LT is (*langs de groep*

$U(1)$).

Een reden om in deze centrale uitbreidingen geïnteresseerd te zijn is het feit dat ze een inzichtelijke *representatietheorie* hebben. Grof gezegd is een *representatie* van een groep een specifieke manier waarop de groep zich kan manifesteren als symmetrieën van een vectorruimte met een inproduct. Onder een representatie wordt een element van een groep dus voorgesteld als een lineaire transformatie die een gegeven inproduct behoudt. Het bestuderen van de representaties van een groep kan leiden tot een beter begrip van de groep zelf. Het is handig om hiermee bij de meest elementaire te beginnen, de zogenaamde *irreducibele* representaties, ook omdat deze als bouwstenen kunnen dienen voor algemene representaties.

Voor de centrale uitbreidingen $\tilde{L}T$ is deze restrictie helaas toch niet genoeg om inzicht in hun representatietheorie te verkrijgen. Echter blijkt dat wanneer we ook een *positieve energie* conditie eisen deze studie ineens wel behapbaar wordt. In sectie 2.5 leggen we namelijk uit dat $\tilde{L}T$ slechts eindig veel irreducibele, positieve energie representaties bezit en dat deze expliciet te classificeren en construeren zijn.

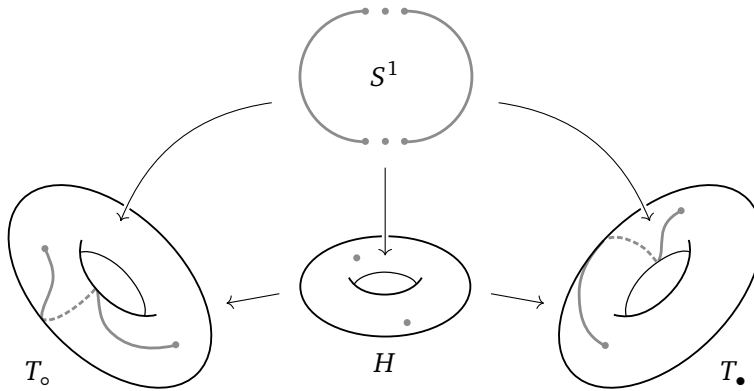
Bigekeurde toruslusgroepen

De hier boven uiteengezette theorie van toruslusgroepen, samengevat uit Hoofdstuk 2, is vrij klassiek en al eerder beschreven in de literatuur. In dit proefschrift is een nieuwe generalisatie van toruslusgroepen geïntroduceerd, die we *bigekeurde toruslusgroepen* hebben genoemd, en er is geprobeerd om analoge resultaten over ze te vinden zoals we die kennen over toruslusgroepen.

Voor het definiëren van een bigekeurde toruslusgroep is meer nodig dan een enkele torus. We fixeren in plaats daarvan drie torussen, T_\circ , H , en T_\bullet van dezelfde dimensie, samen met twee afbeeldingen, één van H naar T_\circ en één van H naar T_\bullet . Van deze afbeeldingen wordt geëist dat ze zowel de meetkundige (preciezer: de differentieerbare) structuren van de torussen respecteren, als hun groepsstructuren. We zien T_\circ en T_\bullet als respectievelijk een witte en een zwarte torus.

Net zoals een enkele torus heeft ook de zojuist genoemde lijst van data een begrip van ‘lus’, namelijk een *bigekeurde lus*. Deze is gedefinieerd als het geheel van 1) een (glad) pad op T_\circ , 2) een (glad) pad op T_\bullet , en 3) twee aangewezen punten op H zodanig dat ze worden gestuurd naar de eindpunten van de paden op T_\circ en T_\bullet onder de afbeeldingen van H naar deze torussen. Het is handig om de eenheidscirkel S^1 te zien als zijnde opgeknipt in drie stukken: een linkerhelft, een rechterhelft en hun overlappende twee punten,

en vervolgens een bigekleurde lus te beschouwen als een drietal afbeeldingen vanuit deze stukken naar de drie torussen toe.



Op dezelfde wijze als in onze eerdere uitleg over gewone lussen op een torus vormt ook de verzameling van bigekleurde lussen een groep onder de puntsgewijze optelling. Deze groep noteren we met $L(T_0, H, T_1)$ en wordt een *bigekleurde toruslusgroep* genoemd. We zien dat als we de torussen H , T_0 en T_1 gelijk aan elkaar kiezen en de twee afbeeldingen tussen hen als de identiteit nemen dat $L(T_0, H, T_1)$ dan niets anders is dan LH (onder enkele extra condities die wij hier niet noemen). Daarom is een bigekleurde toruslusgroep inderdaad een generalisatie van een gewone, en noemen wij dit tweede type achteraf beschouwd *unigekleurd*.

In Sectie 3.2 construeren wij centrale uitbreidingen $\tilde{L}(T_0, H, T_1)$ van zo een groep $L(T_0, H, T_1)$ met eigenschappen analoog aan die van unigekleurde groepen. Deze keer vereist dit niet een enkel, maar een drietal roosters Λ_0 , Γ en Λ_1 van gelijke dimensies, samen met twee afbeeldingen, één van Γ naar Λ_0 en één van Γ naar Λ_1 . Uiteraard eisen we dat deze afbeeldingen de structuren van de roosters bewaren. Samen met verdere eigenschappen bewezen in Sectie 3.3 toont het bestaan van deze centrale uitbreidingen dat bigekleurde toruslusgroepen een werkelijke generalisatie van de unigekleurde theorie toe laten.

Deze stelling wordt tenslotte kracht bijgezet in Sectie 3.4 door op een zelfde wijze als in Hoofdstuk 2 de irreducibele, positieve energie representaties van zo een groep $\tilde{L}(T_0, H, T_1)$ te classificeren en te construeren. Hier blijkt opnieuw dat er slechts eindig veel van zulke representaties bestaan.

Acknowledgements

Thank you, André, for accepting me as a student and thereby giving me the opportunity to enter the world of research. I am grateful for your trust, patience and the freedom you allowed me. Your efforts of support during trying times and while we were separated geographically and dealt with digital malfunctions were more than a student could ask from an advisor. If I gained any sense of mathematical beauty at all, it owes largely to your lectures, my attempts to view material through your lens and seeing my work slowly evolve by your guiding hand.

The next person to offer thanks to is Erik van den Ban. Erik, while your role as a promotor was initially formal, it expanded greatly in my final year. You became a second sounding board for the ideas developed with André, found crucial, technical errors and helped to repair them. Your experience with administrative matters in the end phase was also much appreciated.

I am grateful to the members of my reading committee, consisting of Marius Crainic, Christopher Douglas, Karl-Hermann Neeb, Karl-Henning Rehren and Christoph Schweigert, for their time and effort spent studying this thesis and their valuable feedback.

Cécile, Jean, Ria and Sylvia, thank you for always keeping your door open for unexpected questions and requests and greeting us with a smile. Without you the workings of the department would immediately grind to a halt. I am also indebted to Gunther Cornelissen for his advice on planning the defence.

I was lucky to share the department with fellow PhD students and postdocs who made work much more enjoyable. Thanks, guys, for the lunch breaks we had and especially Ajinkya, Anshui, Dali, Joost, Pouyan and Sebastian for being great office mates. I would also like to single out all the participants of the QFT seminar we organised. Jules, thank you for initiating it, convincing me to take part in it and being a driving force behind it. It gave me much needed practice giving talks and I regret not starting the seminar earlier.

During my visits abroad I had the joy to meet many other students, among whom I want to thank in particular Byung Do, Daniel, Dominic, Lars, Marco,

Mark, Massi, Matt and Peter for sharing their knowledge of and passion for mathematics so freely. The staff at the Hausdorff Research Institute for Mathematics deserves praise as well for their hospitality and generosity during my stay in Bonn in the summer of 2015, which created an ideal working atmosphere.

Lastly, I want to thank my parents for all their support throughout my studies, and my sister for teaching me to read.

Curriculum vitae

Shan Shah was born on 6 May 1989 in Lelystad, the Netherlands. After finishing his high school there in 2007, he obtained his bachelor diploma in Mathematics in 2010 at the University of Amsterdam. He then continued his studies with a master's program in Mathematics at the same institution, concluding in 2012 with a thesis under the supervision of dr. Tilman Bauer.

He started a PhD in Mathematics in the same year at Utrecht University under the supervision of dr. André Henriques, with prof.dr. Erik van der Ban serving as the promotor and secondary supervisor. The results of his research project are described in this thesis. Besides conducting research he also led exercise classes for bachelor students and he had the opportunity to participate in several workshops and conferences, both locally and abroad.