"Many particle systems out of equilibrium" Problems, Series 2, 2006-07.

Problem 4. Flow through a pipe

Find the stationary velocity, temperature and pressure profiles for laminar flow through a vertical cylindrical pipe under the influence of a uniform gravitational field. Use stick boundary conditions, i.e. the velocity at the wall (in rest) vanishes. In addition you may assume that the temperature gradient has no vertical component.

Problem 5. Linearized hydrodynamic equations

The hydrodynamic equations can be linearized, by expanding the flow fields around their equilibrium values, i.e. $p(\boldsymbol{r},t) = \bar{p} + \delta p(\boldsymbol{r},t)$; $\rho(\boldsymbol{r},t) = \bar{\rho} + \delta\rho(\boldsymbol{r},t)$, whereas $\boldsymbol{u}(\boldsymbol{r},t) = \delta\boldsymbol{u}(\boldsymbol{r},t)$, and keeping only terms that are of zero'th and first order in the deviations from equilibrium.

a) Show that the linearized hydrodynamic equations, in the absence of external fields, can be written as

$$\begin{aligned} \frac{d\rho}{dt} &= -\bar{\rho}\nabla\cdot\boldsymbol{u}, & (1)\\ \bar{\rho}\frac{d\boldsymbol{u}}{dt} &= -\nabla p + 2\eta\nabla\cdot(\widetilde{\nabla\boldsymbol{u}}) + (\kappa - \frac{2}{3}\eta)\nabla\nabla\cdot\boldsymbol{u}, & (2)\\ \frac{dE}{dt} &= -\frac{\bar{p}}{\bar{\rho}}\nabla\cdot\boldsymbol{u} + \frac{\lambda}{\bar{\rho}}\nabla^2 T. & (3) \end{aligned}$$

b) Transform these equations into linear equations for the velocity field $\boldsymbol{u}(\boldsymbol{r},t)$ the pressure field $p(\boldsymbol{r},t)$ and the entropy per unit of mass, $\hat{s}(\boldsymbol{r},t)$. To this end, transform the temporal derivatives and the gradients by using thermodynamic relationships of the type $dA = (\partial A/\partial B)_C\, dB + (\partial A/\partial C)_B\, dC$.

c) Fourier transform these equations, determine the eigenvalues of the resulting set up to order k^2 and the (right) eigenvectors to leading order in k. Hint: decompose the velocity field into a component parallel to $\boldsymbol{k}$ (the curl-free velocity field) and components normal to $\boldsymbol{k}$ (the divergence-free velocity field). The equations for the latter decouple. Next apply standard perturbation theory to the resulting equations.

Problem 6. Random flights

In a 1-dimensional random flight process a particle runs at constant speed (but with changing directions) along a line, hitting scatterers at a constant rate ν

(without memory effects. The scatterers just are at the line for an instant; they neither stay around nor return). We will consider the case of isotropic scattering, where the probabilities of the particle turning around or continuing in the same direction are 1/2 each. We also will assume the initial probabilities for the velocity being v or $-v$ are equal.

a) As usual, the Green function $G(x,t)$, describes the probability density for finding a particle at position x at time t, after starting out at the origin at $t=0$. Show that for this process it is given by

$$\begin{aligned} G(x,t) &= \sum_{n=0}^{\infty} \nu^n \int_0^t dt_1 \int_{t_1}^t dt_2 \cdots \int_{t_{n-1}}^t dt_n \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_n \\ & \quad G_0(x_1,t_1) G_0(x_2-x_1, t_2-t_1) \cdots G_0(x-x_n, t-t_n), \end{aligned} \tag{4}$$

with

$$G_0(x,t) = \frac{1}{2}\theta(t) e^{-\nu t}[\delta(x-vt)+\delta(x+vt)] \tag{5}$$

describing the same probability density, with the additional condition that the particle has not been scattered yet since the initial time.

b) The Fourier and Laplace transform of a function $f(x,t)$ is defined as

$$\tilde{f}(k,z) = \int_{-\infty}^{\infty} dx e^{-ikx} \int_0^{\infty} dt e^{-zt} f(x,t)$$

Show that the Fourier and Laplace transform of the Green function is given by

$$\widetilde{G}(k,z) = \frac{z+\nu}{z(z+\nu)+k^2v^2}. \tag{6}$$

Hint: by using the convolution theorems for both Fourier and Laplace transforms, first show that $\widetilde{G}(k,z)$ can be expressed in terms of a geometric series, as

$$\widetilde{G}(k,z) = \frac{\widetilde{G}_0(k,z)}{1-\nu\widetilde{G}_0(k,z)},$$

with $\widetilde{G}_0(k,z)$ the Fourier and laplace transform of $G_0(x,t)$.

c) Find an expression for the intermediate scattering function (just the Fourier transform of the Green function) in the form

$$F(k,t) = \sum_i a_i e^{-z_i(k)t}. \tag{7}$$

d) From this, one may obtain for $G(x,t)$ the approximate expression

$$G(x,t) = \sqrt{\frac{\nu v t^2}{4\pi((vt)^2-x^2)^{3/2}}} \exp -\frac{\nu}{2v}\left(vt - \sqrt{(vt)^2-x^2}\right). \tag{8}$$

by means of *saddle point integration*, also known as the *steepest descent method*. This goes as follows: in order to evaluate $\int_{-\infty}^{\infty} d\,k \exp(\alpha f(k))$, with α some large parameter, first find the stationary points of $f(k)$ in the complex plane, defined as the points for which $f'(k) = 0$. Around such a point, say k_0, $f(k)$ can be expanded as $f(k_0) + \frac{1}{2} f''(k_0)(k - k_0)^2$. Now deform the integration path such that it runs through (at least) one of the stationary points, again denoted as k_0, along the *path of steepest descent*, i.e. the line $z - z_0 = |z - z_0| \exp(i\phi)$, with $2\phi + arg(f''(k_0)) = \pi$. If the absolute value of the integrand on the new integration path attains an absolute maximum at $k = k_0$, the main contribution to the full integral for large α comes from the direct neighborhood of k_0. The integral may then be approximated by a Gaussian integral with outcome $\sqrt{2\pi/|\alpha f''(k_0)|} \exp(\alpha f(k_0))$.

e) Under which conditions is $G(x, t)$ well-approximated by the diffusion Green function

$$G_D(x, t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right)$$

and under which conditions is the approximation poor? What is D for the random flight? Finally, for which values of x is (8) obviously not correct? And what should be the correct answer there?

Problem 7. Alternative derivation of $\widetilde{G}(k, z)$

Instead of considering the random flight process of the previous problem as a process with isotropic scattering at frequency ν one may also consider it as a process with random velocity reversals, ocurring at the frequency $\nu/2$. Show, by starting from a similar expression as used in problem 6a, with now $t_1, \cdots t_n$ the subsequent times of velocity reversal, that this interpretation gives rise to the same Green function indeed.