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Long-Range Spatial Correlations in a Simple Diffusion Model

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A simple anisotropic diffusion model, according to semiphenomenological arguments, exhibits long-ranged spatial correlations in uniform stationary states.

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Spatial correlations in stationary nonequilibrium states of hydrodynamic systems are known to be long-ranged.[(1)] Their magnitudes typically are proportional to powers of gradients of the hydrodynamic fields. Experimentally this was confirmed by Law *et al.*[(2)] Recently it has been found that spatial correlations may also be long-ranged[(3)] in uniform stationary nonequilibrium states, for example, in diffusive systems with periodic boundary conditions, driven by a constant external field. It has been speculated that in fact long-range correlations ought to be generic in such systems, and in case they are absent, this will in general be due to some symmetry properties, such as detailed balance.[2]

A relatively simple class of systems on which to study these properties includes interacting particle systems with anisotropic hopping dynamics. Phenomenologically these systems are expected to be describable by a fluctuating diffusion equation of the form

$$\partial n(\mathbf{r}, t)/\partial t = \mathsf{D}: \nabla\nabla n(\mathbf{r}, t) - \nabla \cdot \mathbf{J}(\mathbf{r}, t) \tag{1}$$

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[2] An example where detailed balance is absent but nonetheless spatial correlations are short-ranged is the asymmetric simple exclusion model. Here the simplifying feature is that for each transition by which one may lose a certain configuration at some given rate, one can point out in a one-to-one fashion a transition occurring at the same rate by which one may gain that same configuration.

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Here $n(\mathbf{r}, t)$ is the position- and time-dependent density, D is the diffusion matrix, and $\mathbf{J}(\mathbf{r}, t)$ is a fluctuating current density, assumed to be distributed as a Gaussian with variance

$$\langle \mathbf{J}(\mathbf{r}, t)\, \mathbf{J}(\mathbf{r}', t')\rangle = 2\mathsf{L}\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t') \tag{2}$$

where L is the matrix of Onsager coefficients. Nonlinearities, resulting, e.g., from a density dependence of D, are excluded from these equations. Assuming their validity, one may easily show[3] that stationary equal-time density correlations are short-ranged in space if and only if the matrices D and L are proportional. Otherwise the density correlations are long-ranged and of quadrupolar structure.

Here I want to present a simple model for which this requirement of proportionality between the matrices D and L can be shown not to be fulfilled in general, and hence the occurrence of long-ranged density correlations should be expected.

The system is a hopping model on a square lattice with periodic boundary conditions behaving as a simple random walk model in the x direction and as a zero-range model in the y direction. Each site may be occupied by an arbitrary number of particles, the jump rate from a site occupied by n particles to a neighboring site in the $+$ or $-x$ direction is $n\Gamma_h$, and the jump rate from a site occupied by $n>0$ particles to a neighboring site in the $+$ or $-y$ direction equals Γ_v.

If at times t_i jumps in directions $\hat{e}_i$ occur, the total particle current is given by

$$\mathbf{J}(t) = a \sum_i \hat{e}_i \delta(t-t_i) \tag{3}$$

where a is the lattice constant. From (3) it is seen that $\mathbf{J}(t)$ is a *random* current, with zero average, as jumps in the $\pm\hat{e}_i$ directions are equally probable. Moreover, $\langle \mathbf{J}(t)\, \mathbf{J}(t')\rangle$, where the average goes over all realizations of the jumping process, is δ-correlated in time. Specifically, one obtains

$$\frac{1}{N}\langle \mathbf{J}(t)\, \mathbf{J}(t')\rangle = a^2\delta(t-t')\{2c\Gamma_h \hat{x}\hat{x} + 2(1-p_0)\,\Gamma_v\, \hat{y}\hat{y}\} \tag{4}$$

Here N is the total number of lattice sites, c is the average occupation of a lattice site, and p_0 is the probability for any given lattice site to be unoccupied. Further, $\hat{x}$ and $\hat{y}$ are the unit vectors in the x and y directions, respectively. This result can be obtained even though we do not know the stationary distribution explicitly. First of all, due to the equality of the jump rates for jumps to left and right, respectively, to above and below, irrespective of the configuration of particles, no contributions to $\langle \mathbf{J}(t)\, \mathbf{J}(t')\rangle$ will result from different jumps at t and t'. Hence, only the autocorrelation of a jump at t to itself remains. This explains the δ-function on the right-hand side of (4). Furthermore, the average jump rate per site for jumps in the $+$ or $-x$ direction equals $2c\Gamma_h$, irrespective of configuration, and that for jumps in the $+$ or $-y$ direction equals $2(1-p_0)\,\Gamma_v$, where p_0 is the only property needed of the stationary distribution.

Equation (2) implies that the Onsager coefficients are given as

$$L_{xx} = a^2 c\Gamma_h; \qquad L_{yy} = a^2(1-p_0)\,\Gamma_v \tag{5}$$

whereas $L_{xy} = L_{yx} = 0$.

On the other hand, the diffusion matrix is defined by the relation

$$\mathbf{j} = \mathbf{D} \cdot \nabla c \tag{6}$$

where $\mathbf{j}$ is the stationary current per unit cell for a nonuniform stationary state in the limit of ∇c approaching zero. For reasons of symmetry, only D_{xx} and D_{yy} are nonvanishing. In fact both quantities can be expressed quite simply in terms of the quantities introduced already. For j_x one obtains

$$\begin{aligned} j_x &= \frac{a}{N} \sum_{n,m} \langle j(n, m \to n+1, m) - j(n+1, m \to n, m) \rangle_{\mathrm{nu}} \\ &= \frac{a}{N} \sum_{n,m} \Gamma_h \{ c(n, m) - c(n+1, m) \} \\ &= -a^2 \Gamma_h \frac{\partial c}{\partial x} \end{aligned} \tag{7}$$

where $j(n, m \to n+1, m)$ describes the contribution to the current due to jumps from the site (n, m) to $(n+1, m)$. Here $\langle \cdots \rangle_{\mathrm{nu}}$ denotes the average over the nonuniform stationary state, and $c(n, m)$ is the average occupation number of site (n, m) in this state. Hence

$$D_{xx} = a^2 \Gamma_h \tag{8}$$

Similarly, for J_y one finds

$$\begin{aligned} j_y &= \frac{a}{N} \sum_{n,m} \langle j(n, m \to n, m+1) - j(n, m+1 \to n, m) \rangle_{\mathrm{nu}} \\ &= \frac{a}{N} \sum_{n,m} \Gamma_v \{ 1 - p_0(n, m) - 1 + p_0(n, m+1) \} \\ &= -a^2 \Gamma_v \frac{\partial}{\partial y} (1 - p_0) \\ &= -a^2 \Gamma_v \frac{d(1-p_0)}{dc} \frac{\partial c}{\partial y} \end{aligned} \tag{9}$$

For the last equality one has to assume that in the limit $\nabla c \to 0$, $\langle p_0 \rangle_{\text{nu}}$ approaches its value in the uniform stationary state with average occupation c. If correlations were to be short-ranged, this certainly ought to be true, but probably it will also be true for most systems having long-ranged correlations. From (9) one finds

$$D_{yy} = a^2 \Gamma_v \frac{d(1-p_0)}{dc} \tag{10}$$

According to the arguments given above, the absence of long-ranged correlations requires the equality

$$\frac{L_{xx}}{D_{xx}} = \frac{L_{yy}}{D_{yy}} \tag{11}$$

Substituting (5), (8), and (10) into (11), one obtains the relation

$$\frac{d(1-p_0)}{dc} = \frac{1-p_0}{c} \tag{12}$$

which has as solution

$$1 - p_0 = Ac \tag{13}$$

with A an arbitrary constant. However, $1 - p_0$ is bounded between 0 and 1, so (12) cannot hold generally. In fact, one would expect p_0 to behave intermediately between its behavior for independent particles (pure random walk), which is $p_0 = e^{-c}$, and the pure zero-range model, yielding $p_0 = 1/(1+c)$. This would imply that (9) would hold only in the limit $c \to 0$. Especially for large density the anisotropy, and as a consequence also the long-range character of the stationary correlation functions, would become very strong.

Generalizations of the model discussed here to different lattices, higher dimensions, or different zero-range processes for different principal directions are easy to envisage. The conjecture that correlations in general will be long-ranged for these models seems very plausible.

It would be worthwhile trying to make the above reasoning completely rigorous. This would require proving the relation

$$\lim_{\nabla c \to 0} \frac{\partial}{\partial y}(1 - p_0) = \frac{\partial p_0}{\partial c} \frac{\partial c}{\partial y}$$

under the assumption that stationary spatial correlations are short-ranged and proving the validity of the statement made below (10). Besides showing the validity of fluctuating hydrodynamics for the given model, this would require an estimate of the effect of nonlinearities in the full, nonlinear diffusion equation.

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