Monodromy of A-hypergeometric functions

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May 7, 2013

Abstract

Using Mellin-Barnes integrals we give a method to compute a relevant subgroup of the monodromy group of an A-hypergeometric system of differential equations. Presumably this group is the full monodromy group of the system

1 Introduction

At the end of the 1980's Gel'fand, Kapranov and Zelevinsky, in [11], [12], [13], defined a general class of hypergeometric functions, encompassing the classical one-variable hypergeometric functions, the Appell and Lauricella functions and Horn's functions. They are called A-hypergeometric functions and they provide a beautiful and elegant basis of a theory of hypergeometric functions in several variables. For an introduction to the subject we refer the reader to [33], [5] or the book by Saito, Sturmfels and Takayama, [29]. We briefly recall the main facts. Let $A \subset \mathbb{Z}^r$ be a finite set such that

1. The $\mathbb{Z}$-span of A is $\mathbb{Z}^r$.
2. There exists a linear form h such that $h(\mathbf{a}) = 1$ for all $\mathbf{a} \in A$.

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$. Denote $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$ (with $N > r$). Writing the vectors $\mathbf{a}_i$ in column form we get the so-called A-matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rN} \end{pmatrix}$$

For $i = 1, 2, \ldots, r$ consider the first order differential operators

$$Z_i = a_{i1}v_1\partial_1 + a_{i2}v_2\partial_2 + \cdots + a_{iN}v_N\partial_N$$

where $\partial_j = \frac{\partial}{\partial v_j}$ for all j.
Let
$$L = \{(l_1, \ldots, l_N) \in \mathbb{Z}^N |\ l_1\mathbf{a}_1 + l_2\mathbf{a}_2 + \cdots + l_N\mathbf{a}_N = \mathbf{0}\}$$
be the lattice of integer relations between the elements of A. For every $\mathbf{l} \in L$ we define the so-called box-operator
$$\Box_{\mathbf{l}} = \prod_{l_i>0} \partial_i^{l_i} - \prod_{l_i<0} \partial_i^{-l_i}$$
The system of differential equations
$$\begin{aligned} (Z_i - \alpha_i)\Phi &= 0 \quad (i = 1, \ldots, r) \\ \Box_{\mathbf{l}}\Phi &= 0 \quad \mathbf{l} \in L \end{aligned}$$
is known as the system of *A-hypergeometric differential equations* and we denote it by $H_A(\boldsymbol{\alpha})$. It turns out that in general the solution space of $H_A(\boldsymbol{\alpha})$ is finite dimensional with dimension equal to the volume of the convex hull $Q(A)$ of A. In order to be more precise we have to introduce $C(A)$, the cone generated by the $\mathbb{R}_{\geq 0}$-linear combinations of $\mathbf{a}_1, \ldots, \mathbf{a}_N$. We say that an A-hypergeometric system is *non-resonant* if the boundary of $C(A)$ has empty intersection with the shifted lattice $\boldsymbol{\alpha} + \mathbb{Z}^r$. We have the following theorem.

Theorem 1.1 (GKZ, Adolphson) *Suppose either one of the following conditions holds,*

1. *the toric ideal I_A in $\mathbb{C}[\partial_1, \ldots, \partial_N]$ generated by the box operators has the Cohen-Macaulay property.*
2. *The system $H_A(\boldsymbol{\alpha})$ is non-resonant.*

Then the rank of $H_A(\boldsymbol{\alpha})$ is finite and equals the volume of the convex hull $Q(A)$ of the points of A. The volume is normalized so that a minimal $(r-1)$-simplex with integer vertices in $h(\mathbf{x}) = 1$ has volume 1.

Theorem 1.1 is proven in [13], (corrected in [15]) and [1, Corollary 5.20].
Among the many papers written on A-hypergeometric equations there are very few papers dealing with the monodromy group of these systems in general. In the case of one-variable hypergeometric functions there is the paper by Beukers and Heckman, [8], which give an characterisation of monodromy groups as complex reflection groups. There is also a classical method to compute monodromy with respect to an explicit basis of functions using so-called Mellin-Barnes integrals, see F.C.Smith, [32]. In the special case of the fourth order equation with symplectic monodromy there is a detailed calculation in [9]. A recent paper by Golyshev and Mellit, [16], deals with the same problem using Fourier-transforms of Γ-products. A recent paper by K.Mimachi [25] uses computation of twisted cycle intersection.

For the two-variable Appell system F_1 and Lauricella's F_D, monodromy follows from the work by E.Picard [28], T.Terada [35] and Deligne-Mostow [10]. In T.Sasaki's paper [30] we find explicit monodromy generators for Appell F_1. They all use the fact that Lauricella functions of F_D-type can be written as one-dimensional twisted period integrals and monodromy is a representation of the pure braid group on $n+1$ strands (where n is the number of variables).
The two-variable Appell F_2 has been considered explicitly by M.Kato, [19]. The Appell F_3 system has the same A-set as Appell F_2, and therefore gives nothing new. Finally, the Appell system F_4 has been considered completely explicitely by K.Takano [34] and later Kaneko, [18]. In Haraoka, Ueno [17] we find some rigidity considerations on the monodromy of F_4. In the paper [24] by K.Matsumoto and M.Yoshida, the authors provide generators for the monodromy of Lauricella F_A.
Finally, the complete monodromy of the Aomoto-Gel'fand system $E(3,6)$ has been determined by K.Matsumoto, T.Sasaki, N.Takayama, M.Yoshida in [22] and further properties in [23]. See also M.Yoshida's book 'Hypergeometric Functions, my Love', [36].
In essentially all of the above studies the monodromy is computed by studying the behaviour of Euler integrals for hypergeometric functions under analytic continuation and corresponding deformation of the contours of integration. For this, knowledge of the fundamental group of the complement of the singular locus of the system of equations is required. It is the purpose of the present paper to avoid these geometric difficulties as long as possible and compute monodromy groups of A-hypergeometric systems by methods which are combinatorial in nature. We do this by starting with local monodromy groups which arise from series expansions of solutions of $H_A(\boldsymbol{\alpha})$. It is well known that such local expansions correspond one-to-one with regular triangulations of A. This is a discovery by Gel'fand, Kapranov and Zelevinsky that we shall explain in Section 2. The local monodromy groups have to be glued together to build a global monodromy group. This glue is provided by multidimensional Mellin-Barnes integrals as defined in Section 3. Unfortunately, the Mellin-Barnes integrals do not always provide a basis of solutions. But if they do (Assumption 4.5), the construction of the global group generated by the local contributions is completely combinatorial. In Section 6 we give a practical recipe for the calculation of these matrices. This algorithm is based on the theoretical considerations in the preceding sections.
Although in a good number of classical cases the method described in Section 6 works very well, it is not always garantueed to work. There are two potential obstacles:

1. There does not always exist a basis of Mellin-Barnes solutions. In many classical cases such a basis exists. For example, two variable Appell, Horn and higher Lauricella F_A, F_B, F_D. But, on the other hand, in case of Lauricella F_C and many other Aomoto-systems such a basis does not seem to exist. This is a subject of further investigation.

2. The group we calculate is the subgroup generated by the contributions of local monodromies at different points, modulo scalars. Let us call this group $lMon$. It is not clear if this group equals the complete monodromy group modulo scalars, which we denote Mon.

   The reason we consider the monodromy group modulo scalars is that for the A-hypergeometric system and their classical counterparts these groups are the same. An explanation for this can be found at the end of Section 2.

The groups we calculate are determined with respect to a basis of solutions in Mellin-Barnes integral form. In the case of one variable ${}_{n+1}F_n$ they turn out to coincide with the matrices found in [8] (see Section 8). If one would like to calculate monodromy matrices with respect to explicit bases of local power series expansions one would have to find an explicit calculation of a Mellin-Barnes integral as a linear combination of power series solutions. This is a tedious task which we like to carry out in a forthcoming paper. We remark that such a calculation has been carried out in the so-called confluent case (i.e. A does not lie in translated hyperplane) by O.N.Zhdanov and A.K.Tsikh, [37]. In her PhD-thesis from 2009, Lisa Nilsson [26] introduced (non-confluent) A-hypergeometric functions in terms of Mellin-Barnes integrals and initiated their study. It is these integrals that we shall use.

In the remainder of this paper we assume that *the shifted lattice $\boldsymbol{\alpha} + \mathbb{Z}^r$ has empty intersection with any hyperplane spanned by $r-1$ independent elements of A*. In that case we say that the system is *totally non-resonant*. Note that this is stronger than just non-resonance where only the faces of the cone spanned by the elements of A are involved. Non-resonance (and a fortiori total non-resonance) ensures that our system is irreducible, see for example [14, Thm 2.11] or [6] for a slightly more elementary proof. Non-resonance also implies that A-hypergeometric systems whose parameter vectors are the same modulo $\mathbb{Z}^r$ have isomorphic monodromy, see [6, Thm 2.1], which is actually a theorem due to B.Dwork. Total non-resonance implies T-nonresonance for every triangulation T in the terminology of Gel'fand, Kapranov and Zelevinsky. In particular this implies that the local solution expansions will not contain logarithms. So local monodromy representations act by characters. We prefer to leave the case of logarithmic local solutions for a later occasion.

The starting data of our computation will not be the set A and parameter vector $\boldsymbol{\alpha}$, but rather a dual version as follows. Let $d = N - r$. This will be the number of variables in the classical counterpart of the A-hypergeometric system (number of essential variables, e.g. $d = 2$ in the Appell cases). Choose a $\mathbb{Z}$-basis for the lattice L, which has rank d, and write the basis elements as rows of a $d \times N$ matrix B. In the literature the transpose of B is often called a *Gale dual* of A, we simply call B a *B-matrix*. The matrix B has the property that it has maximal rank d, the $\mathbb{Z}$-span of the columns is $\mathbb{Z}^d$ and $A.B^t$ is the zero matrix. We denote the columns of B by $\mathbf{b}_j,\ j = 1, \ldots, N$. Then the space $L \otimes \mathbb{R} \subset \mathbb{R}^N$ is parametrized by the N-tuple $(\mathbf{b}_1 \cdot \mathbf{s}, \ldots, \mathbf{b}_N \cdot \mathbf{s})$ with $\mathbf{s} = (s_1, \ldots, s_d) \in \mathbb{R}^d$

as parameters. In our computations we take a B-matrix as starting data and instead of the parameters α we choose $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_N) \in \mathbb{R}^N$ such that $\gamma_1 \mathbf{a}_1 + \cdots + \gamma_N \mathbf{a}_N = \boldsymbol{\alpha}$. Notice that there is some ambiguity in the choice of $\boldsymbol{\gamma}$ which we will fix later. The reason we take B and $\boldsymbol{\gamma}$ as starting data is that they are easily read off from the classical power series expansions. In the next section we see an example of this.

2 Power series solutions

Consider the system $H_A(\boldsymbol{\alpha})$ and a formal solution

$$\Phi_{\boldsymbol{\gamma}} = \sum_{\mathbf{l} \in L} \frac{v_1^{l_1+\gamma_1} \cdots v_N^{l_N+\gamma_N}}{\Gamma(l_1+\gamma_1+1) \cdots \Gamma(l_N+\gamma_N+1)}$$

where $\boldsymbol{\gamma}$ is chosen such that $\boldsymbol{\alpha} = \gamma_1 \mathbf{a}_1 + \cdots + \gamma_N \mathbf{a}_N$. This expansion was introduced in [13]. It is a Laurent series multiplied by generally non-integral powers of the variables v_i. We call such a series a twisted Laurent series. As is well-known we have a freedom of choice in $\boldsymbol{\gamma}$ by shifts over $L \otimes \mathbb{R}$. We shall use this freedom in the following way where, again, we denote the columns of the B-matrix by $\mathbf{b}_i$. Choose a subset $I \subset \{1, 2, \ldots, N\}$ with $|I| = d = N - r$ such that $\mathbf{b}_i$ with $i \in I$ are linearly independent. It is known that $|\det(\mathbf{b}_i)_{i \in I}| = |\det(\mathbf{a}_j)_{j \notin I}|$ and we denote this quantity by Δ_I. Choose $\boldsymbol{\gamma}$ such that $\gamma_i \in \mathbb{Z}$ for all $i \in I$. There are precisely Δ_I such choices for $\boldsymbol{\gamma}$ which are distinct modulo $L \otimes \mathbb{R}$. The series Φ now reads

$$\Phi_{\boldsymbol{\gamma}} = \sum_{\mathbf{l} \in L} \prod_{i \in I} \frac{v_i^{l_i+\gamma_i}}{\Gamma(l_i+\gamma_i+1)} \times \prod_{j \notin I} \frac{v_j^{l_j+\gamma_i}}{\Gamma(l_j+\gamma_j+1)}$$

Since $\gamma_i \in \mathbb{Z}$ for all $i \in I$ this summation extends over all $\mathbf{l}$ with $\gamma_i + l_i \geq 0$ for all $i \in I$. The other terms vanish because $1/\Gamma(\gamma_i + l_i + 1) = 0$ whenever $\gamma_i + l_i$ is a negative integer for some $i \in I$. Hence $\Phi_{\boldsymbol{\gamma}}$ is now a twisted power series. Let us fix such a choice of $\boldsymbol{\gamma}$. It is not hard to see that a set of series $\Phi_{\boldsymbol{\gamma}}$ with $\boldsymbol{\gamma}$-values which are distinct modulo $L \otimes \mathbb{R}$, is linearly independent over $\mathbb{C}$.

Let now $\rho_1, \ldots, \rho_N$ be any N-tuple with the property that $\rho_1 l_1 + \cdots + \rho_N l_N > 0$ for any non-zero $\mathbf{l} \in L$ with $l_i \geq 0$ for all $i \in I$. Then, according to Theorem [5, Proposition 16.2] or [33, Section 3.3,3.4], the series $\Phi_{\boldsymbol{\gamma}}$ converges for all $v_1, \ldots, v_N$ with $\forall i : |v_i| = t^{\rho_i}$ and $t \in \mathbb{R}_{>0}$ sufficiently small. We call such an N-tuple $\rho_1, \ldots, \rho_N$ a *convergence direction* of $\Phi_{\boldsymbol{\gamma}}$.

There is one important assumption we need in order to make this approach work. Namely the garantee that none of the arguments $\gamma_j + l_j$ is a negative integer when

$j \notin I$. Otherwise we might even end up with a trivial series solution. Notice that

$$\boldsymbol{\alpha} = \sum_{j=1}^{N} \gamma_j \mathbf{a}_j \equiv \sum_{j \notin I} \gamma_j \mathbf{a}_j (\text{mod } \mathbb{Z}^r)$$

So if $\gamma_j \in \mathbb{Z}$ for some $j \notin I$, the point $\boldsymbol{\alpha}$ lies modulo $\mathbb{Z}^r$ in a space spanned by the $r-1$ remaining vectors $\mathbf{a}_i$. Under the assumption of total non-resonance this situation cannot occur.

From now on we assume that $H_A(\boldsymbol{\alpha})$ is totally non-resonant. We denote the set of all sets I such that $\Delta_I = |\det(\mathbf{b}_i)_{i\in I}| \neq 0$ by $\mathcal{I}$. When $s = \Delta_I > 1$ we must take s copies of I in this list. To each $I \in \mathcal{I}$ there corresponds a choice of $\boldsymbol{\gamma}$ and we see to it that all these choices are distinct modulo $L \otimes \mathbb{R}$. So to an index set I which occurs s times there correspond s choices of $\boldsymbol{\gamma}$ that are distinct modulo $L \otimes \mathbb{R}$. The corresponding powerseries solutions are denoted by Φ_I.

Choose $I \in \mathcal{I}$ and a convergence direction $(\rho_1, \ldots, \rho_N)$ such that $\rho_1 l_1 + \cdots + \rho_N l_N > 0$ for all non-zero $\mathbf{l} \in L$ with $\forall i \in I : l_i \geq 0$. Note that if $(\rho_1, \ldots, \rho_N)$ is a convergence direction, then after adding an element of the $\mathbb{R}$-row span of A, it is still a convergence condition since $A\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in L \otimes \mathbb{R}$. Consider the element $\boldsymbol{\rho} = \sum_{i=1}^{N} \rho_i \mathbf{b}_i$ in the column span of B. By shifting over the row span of A we can see to it that $\rho_i = 0$ for all $i \notin I$. Hence $\boldsymbol{\rho} = \sum_{i\in I} \rho_i \mathbf{b}_i$. The convergence condition is now equivalent to saying that the new ρ_i are positive. So if we denote

$$\mathbf{b}_I = \left\{ \sum_{i\in I} \lambda_i \mathbf{b}_i \middle| \lambda_i > 0 \right\},$$

the convergence condition can be restated as $\boldsymbol{\rho} \in \mathbf{b}_I$. By a slight abuse of language we call the vector $\boldsymbol{\rho}$ also a convergence direction.

Conversely, fix an element $\boldsymbol{\rho}$ in the span of all $\mathbf{b}_i$ which does not lie on the boundary of any $\mathbf{b}_I$. Define $\mathcal{I}_{\boldsymbol{\rho}} = \{I | \rho \in \mathbf{b}_I\}$. Then, by the theory of Gel'fand, Kapranov and Zelevinsky the powerseries Φ_I with $I \in \mathcal{I}_{\boldsymbol{\rho}}$ form a basis of solutions with a common open region of convergence. We call such a set a basis of local solutions of $H_A(\boldsymbol{\alpha})$. It also follows from the theory that the sets $\mathcal{I}_{\boldsymbol{\rho}}$ are in one-to-one correspondence with the regular triangulations of the set A. This correspondence is given by $\mathcal{I}_{\boldsymbol{\rho}} \mapsto \{I^c | I \in \mathcal{I}_{\boldsymbol{\rho}}\}$.

The intersections of the simplicial cones $\mathbf{b}_I$ define a subdivision of $\mathbb{R}^d$ into open convex polyhedral cones whose closure of the union is $\mathbb{R}^d$. This is a polyhedral fan which is called the *secondary fan*. The open cones in the secondary fan are in one-to-one correspondence with the bases of local series solutions. As an example let us take the system Appell F_2. The standard Appell F_2-series reads

$$F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y) = \sum_{m,n\geq 0} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{m!n!(\gamma)_m(\gamma')_n} \, x^m y^n.$$

We hope no confusion arise with the existing notations $\boldsymbol{\alpha}, \boldsymbol{\gamma}$. Using the identity

$$\Gamma(z)\Gamma(1-z) = \pi / \sin \pi z$$

we see that the series is proportional to

$$\sum_{m,n\geq 0} \frac{x^m y^n}{\Gamma(-\alpha-m-n+1)\Gamma(-\beta-m+1)\Gamma(-\beta'-n+1)\Gamma(\gamma+m)\Gamma(\gamma'+n)\Gamma(m+1)\Gamma(n+1)}.$$

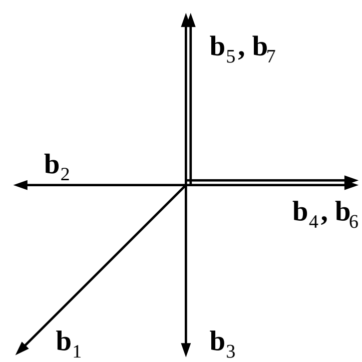
The basisvectors $(-1,-1,0,1,0,1,0)$ and $(-1,0,-1,0,1,0,1)$ of L are given to us naturally because these are the coefficient vectors of m and n respectively in the Γ-factors of the expansion just given. This follows from the shape of the canonical solution $\Phi_{\boldsymbol{\gamma}}$. So our B-matrix reads

$$B = \begin{pmatrix} -1 & -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

A parameter vector $\boldsymbol{\gamma}$ can also be read off from the Γ-expansion, namely

$$\boldsymbol{\gamma} = (-\alpha, -\beta, -\beta', \gamma-1, \gamma'-1, 0, 0).$$

The column vectors of B are depicted here,



For example, consider the vector $(-0.5, 1)$ in this picture. We see that it is contained in the positive cones of the following pairs: $\{\mathbf{b}_2, \mathbf{b}_5\}, \{\mathbf{b}_2, \mathbf{b}_7\}, \{\mathbf{b}_1, \mathbf{b}_5\}, \{\mathbf{b}_1, \mathbf{b}_7\}$. Taking the complimentary sets of indices of each pair we get

$$\{1,3,4,6,7\}, \{1,3,4,5,6\}, \{2,3,4,5,6\}, \{2,3,4,5,6\}.$$

These form the index sets of the simplices of a triangulation of the set A. Take the alternative parameter vector

$$\boldsymbol{\gamma} = (\gamma' - \beta - \alpha - 1, 0, \gamma' - 1 - \beta', \gamma - \beta - 1, 0, -\beta, 1 - \gamma'),$$

which differs from the original choice by an element of $L \otimes \mathbb{R}$. The coordinates on positions 2,5 are made zero, corresponding to the choice $\mathbf{b}_2, \mathbf{b}_5$. The formal solution for this new parameter vector reads

$$\sum_{m,n} \frac{v_1^{\gamma'-\beta-\alpha-1-m-n} v_2^{-m} v_3^{\gamma'-\beta'-1-n} v_4^{\gamma-\beta-1+m} v_5^n v_6^{-\beta+m} v_7^{1-\gamma'+n}}{\Gamma(\gamma'-\beta-\alpha-m-n)\Gamma(-m+1)\Gamma(\gamma'-\beta'-n)\Gamma(\gamma-\beta+m)\Gamma(n+1)\Gamma(-\beta+m)\Gamma(2-\gamma'+n)}.$$

Since $1/\Gamma(n+1) = 0$ when $n < 0$ and $1/\Gamma(-m+1) = 0$ when $m > 0$ we see that our summation runs over $m \le 0$ and $n \ge 0$. Replace m by $-m$ to get $v_1^{\gamma'-\beta-\alpha-1} v_3^{\gamma'-\beta'-1} v_4^{\gamma-\beta-1} v_6^{-\beta} v_7^{1-\gamma'}$ times

$$\sum_{m,n\ge 0} \frac{(v_1 v_2 v_4^{-1} v_6^{-1})^m (v_5 v_7 v_1^{-1} v_3^{-1})^n}{\Gamma(\gamma'-\beta-\alpha+m-n)\Gamma(m+1)\Gamma(\gamma'-\beta'-n)\Gamma(\gamma-\beta-m)\Gamma(n+1)\Gamma(-\beta-m)\Gamma(2-\gamma'+n)},$$

a twisted powerseries in $v_1 v_2 v_4^{-1} v_6^{-1}$ and $v_5 v_7 v_1^{-1} v_6^{-1}$, which has $(-0.5, 1)$ as convergence direction. In the same way we can construct three other (twisted) powerseries expansions and thus obtain a basis of local powerseries solutions of our system.

To return to our general story, suppose we have a basis of local series solutions and suppose they converge in an open neighbourhood of a set defined by $|v_1| = r_1, \ldots, |v_N| = r_N$ with $r_i > 0$ for all i. Let $\boldsymbol{\gamma}^{(1)}, \ldots, \boldsymbol{\gamma}^{(D)}$ be the choices of $\boldsymbol{\gamma}$ used in the construction of the local series solutions $\Phi_1, \ldots, \Phi_D$. Given any N-tuple of integers $n_1, \ldots, n_N$ we define the closed path $c(n_1, \ldots, n_N)$ by

$$(r_1 e^{2\pi i n_1 t}, \ldots, r_N e^{2\pi i n_N t}), \quad t \in [0, 1].$$

Taking $(r_1, \ldots, r_N)$ as a base point, the series Φ_j changes into

$$\exp(2\pi i (n_1 \gamma_1^{(j)} + \cdots + n_N \gamma_N^{(j)})) = \exp(2\pi i \, \mathbf{n} \cdot \boldsymbol{\gamma}^{(j)})$$

times Φ_j after analytic continuation along $c(n_1, \ldots, n_N)$. The group of these substitutions is called a *local monodromy group* with respect to $\Phi_1, \ldots, \Phi_D$. It is important to note that it is generated by d elements and a group of scalar elements generated by the scalars $\exp(2\pi i \alpha_j)$, $j = 1, \ldots, r$. The explanation is as follows. The r rows of the matrix A span a lattice. The basis of rows can be completed to a basis of $\mathbb{Z}^N$ by $d = N - r$ extra integer vectors $\mathbf{n}_1, \ldots, \mathbf{n}_d$, say. So all $\mathbf{n} \in \mathbb{Z}^N$ are $\mathbb{Z}$-linear combinations of these vectors. Suppose that $\mathbf{n}$ is the j-th row of A. Since $\mathbf{n} \cdot \boldsymbol{\gamma}^{(i)} = \alpha_j$ for all $i = 1, \ldots, D$, the local monodromy transformation is the scalar element given by $\exp(2\pi i \alpha_j)$. The remaining vectors $\mathbf{n}_1, \ldots, \mathbf{n}_d$ provide us with d generators. In our implementation in Section 6 we shall make a sensible choice for these generators.

The fact that we have d generators corresponds to the fact that the number of essential variables (modulo homogeneities) is d. The scalar group is simply the difference between the full A-hypergeometric system and its classical counterpart. In this paper we shall adopt the convention that we compute the monodromy modulo scalars. Hence it suffices to compute the d generators of the local monodromies.

3 Mellin-Barnes integrals

Let notations be as in the previous sections. Consider the parametrization of $L \otimes \mathbb{R}$ by the N-tuple $(\mathbf{b}_1 \cdot \mathbf{s}, \ldots, \mathbf{b}_N \cdot \mathbf{s})$ with $\mathbf{s} = (s_1, \ldots, s_d) \in \mathbb{R}^d$ as parameters. Choose $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_d) \in \mathbb{R}^d$.
We now complexify the parameters s_i and consider the integral

$$M(v_1, \ldots, v_N) = \int_{\boldsymbol{\sigma}+\sqrt{-1}\ \mathbb{R}^d} \prod_{i=1}^{N} \Gamma(-\gamma_i - \mathbf{b}_i \cdot \mathbf{s}) v_i^{\gamma_i + \mathbf{b}_i \cdot \mathbf{s}} d\mathbf{s} \qquad (MB)$$

where $d\mathbf{s} = ds_1 \wedge \cdots \wedge ds_d$ and the integration takes place over $-\infty < \mathrm{Im}(s_i) < \infty$ and $\mathrm{Re}(s_i) = \sigma_i$ for $i = 1, \ldots, d$. This is an example of a so-called *Mellin-Barnes integral.* It will be crucial in the determination of the monodromy of A-hypergeometric systems.
We prove the following Theorem,

Theorem 3.1 *Assume that $\gamma_i < -\mathbf{b}_i \cdot \boldsymbol{\sigma}$ for $i = 1, 2, \ldots, N$. Then the Mellin-Barnes integral $M(v_1, \ldots, v_N)$ satisfies the set of A-hypergeometric equations $H_A(\boldsymbol{\alpha})$.*

This will be done under the assumption that the Mellin-Barnes integral converges absolutely. We come to the matter of convergence in the next section.

Proof: The Mellin-Barnes integral clearly has the property

$$M(\mathbf{t}^{\mathbf{a}_1} v_1, \ldots, \mathbf{t}^{\mathbf{a}_N} v_N) = \mathbf{t}^{\boldsymbol{\alpha}} M(v_1, \ldots, v_N)$$

for all $\mathbf{t} \in (\mathbb{C}^*)^N$. So $M(\mathbf{v})$ satisfies the hypergeometric homogeneity equations.
Now let $\lambda \in L$ and put $\lambda = \lambda_+ - \lambda_-$ where $\lambda_\pm$ have non-negative coefficients and disjoint support. Define $|\lambda| = \sum_{i=1}^{N} |\lambda_i|$. Then

$$\begin{aligned} & \Box_\lambda M(v_1, \ldots, v_N) \\ = \ & (-1)^{|\lambda|/2} \int_{\boldsymbol{\sigma}+\sqrt{-1}\mathbb{R}^d} \prod_{i=1}^{N} \Gamma(-\gamma_i - \mathbf{b}_i \cdot \mathbf{s} + \lambda_{+,i}) v_i^{\gamma_i + \mathbf{b}_i \cdot \mathbf{s} - \lambda_{+,i}} d\mathbf{s} \\ - \ & (-1)^{|\lambda|/2} \int_{\boldsymbol{\sigma}+\sqrt{-1}\mathbb{R}^d} \prod_{i=1}^{N} \Gamma(-\gamma_i - \mathbf{b}_i \cdot \mathbf{s} + \lambda_{-,i}) v_i^{\gamma_i + \mathbf{b}_i \cdot \mathbf{s} - \lambda_{-,i}} d\mathbf{s}. \end{aligned}$$

Choose $\mathbf{s}_\lambda$ such that $\mathbf{b}_i \cdot \mathbf{s}_\lambda = \lambda_i$ for $i = 1, \ldots, N$. Then the second integral is actually integration over $\mathbf{s}_\lambda + \sqrt{-1}\mathbb{R}^d$ of the integrand of the first integral.
Because of the assumption $\gamma_i < -\mathbf{b}_i \cdot \boldsymbol{\sigma}$ we see that $-\gamma_i - t\mathbf{b}_i \cdot (\mathbf{s}_\lambda + \boldsymbol{\gamma}) + \lambda_{+,i} > 0$ for all $t \in [0, 1]$ and $i = 1, \ldots, N$. Hence the $d+1$-dimensional domain $\{t(\mathbf{s}_\lambda + \boldsymbol{\gamma}) + i\mathbb{R}^d | 0 \leq t \leq 1\}$ does not contain any poles of the integrand and a homotopy argument gives that the two integrals are equal and cancel.

$\Box$

Not all systems $H_A(\boldsymbol{\alpha})$ allow a choice of negative γ_i. However, under the assumption of non-resonance it is known that two hypergeometric systems $H_A(\boldsymbol{\alpha})$ and $H_A(\boldsymbol{\alpha}')$ have the same monodromy if $\boldsymbol{\alpha} - \boldsymbol{\alpha}' \in \mathbb{Z}^r$ (see [6, Theorem 2.1]). We call such systems contiguous. Thus we can always replace an irreducible A-hypergeometric system by a contiguous one which does allow a choice of $\gamma_i < -\mathbf{b}_i \cdot \boldsymbol{\sigma}$ for all i. In concrete cases we can also play with the value of $\boldsymbol{\sigma}$. From now on we make this assumption, i.e all our Mellin-Barnes integrals are solution of an A-hypergeometric system. Of course there is also the question whether or not $M(v_1, \ldots, v_N)$ is a trivial function. By Proposition 4.5 we will find that it is non-trivial.

4 Convergence of the Mellin-Barnes integral

We find from [2] the following estimate. Suppose $s = a + bi$ with $a_1 < a < a_2$ and $|b| \to \infty$. Then

$$|\Gamma(a+bi)| = \sqrt{2\pi}|b|^{a-1/2}e^{-\pi|b|/2}[1 + O(1/|b|)].$$

Notice also that for any $v \in \mathbb{C}^*$ we have $|v^{a+bi}| = |v|^a e^{-b\arg(v)}$. Write $s_j = \sigma_j + i\tau_j$ for $j = 1, \ldots, N-r$. Let us denote $\theta_j = \arg(v_j)$ and $l_j(\boldsymbol{\tau}) = l_j(\tau_1, \ldots, \tau_d)$. The integrand in the Mellin-Barnes integral can now be estimated by

$$\left|\prod_{i=1}^{N} \Gamma(-\gamma_i - \mathbf{b}_i \cdot \mathbf{s}))v_i^{\gamma_i + \mathbf{b}_i \cdot \mathbf{s}}\right| \le c_1 \max_j |\tau_j|^{c_2} \exp\left(-\sum_{j=1}^{N} \pi|\mathbf{b}_j \cdot \boldsymbol{\tau}|/2 - \theta_j \mathbf{b}_j \cdot \boldsymbol{\tau}\right)$$

where c_1, c_2 are positive numbers depending only on γ_j, v_j, σ_j. In order to ensure convergence of the integral we must have that

$$\sum_{j=1}^{N} \pi|\mathbf{b}_j \cdot \boldsymbol{\tau}|/2 + \theta_j \mathbf{b}_j \cdot \boldsymbol{\tau} > 0 \qquad \text{(C)}$$

for every non-zero $\boldsymbol{\tau} \in \mathbb{R}^d$. We apply the following Lemma.

Lemma 4.1 *Given N vectors $\mathbf{p}_1, \ldots, \mathbf{p}_N$ in $\mathbb{R}^d$. Suppose they have rank d. Let $\mathbf{q} \in \mathbb{R}^d$. Then the following statements are equivalent,*

i) For all non-zero $\mathbf{x} \in \mathbb{R}^d$:

$$|\mathbf{q} \cdot \mathbf{x}| < \sum_{i=1}^{N} |\mathbf{p}_i \cdot \mathbf{x}|$$

ii) There exist $\mu_1, \ldots, \mu_N$ with $-1 < \mu_i < 1$ such that

$$\mathbf{q} = \mu_1 \mathbf{p}_1 + \cdots + \mu_N \mathbf{p}_N.$$

Proof: First suppose that $\mathbf{q} = \mu_1\mathbf{p}_1 + \cdots + \mu_N\mathbf{p}_N$. Then, for all non-zero $\mathbf{x} \in \mathbb{R}^d$,

$$\begin{aligned} |\mathbf{q}\cdot\mathbf{x}| &= |\sum_{i=1}^{N} \mu_i \mathbf{p}_i \cdot \mathbf{x}| \\ &\le \sum_{i=1}^{N} |\mu_i||\mathbf{p}_i\cdot\mathbf{x}| < \sum_{i=1}^{N} |\mathbf{p}_i\cdot\mathbf{x}| \end{aligned}$$

To show the converse statement consider the set

$$V = \left\{ \sum_{i=1}^{N} \mu_i \mathbf{p}_i \;\middle|\; -1 < \mu_i < 1 \right\}.$$

This is a convex set. Suppose $\mathbf{q} \notin V$. Then there exists a linear form h such that $h(\mathbf{q}) > h(\mathbf{p})$ for all $\mathbf{p} \in V$. In other words, there exists $\mathbf{x} \in \mathbb{R}^d$ such that $\mathbf{q}\cdot\mathbf{x} > \sum_{i=1}^{N} \lambda_i \mathbf{p}_i \cdot \mathbf{x}$ for all $-1 < \lambda_i < 1$. In particular,

$$|\mathbf{q}\cdot\mathbf{x}| \ge \sum_{i=1}^{N} |\mathbf{p}_i\cdot\mathbf{x}|$$

contradicting our assumption. Hence $\mathbf{q} \in V$.

□

Application of Lemma 4.1 with $\mathbf{q} = \sum_{j=1}^{N} \theta_j \mathbf{b}_j$ and $\mathbf{p}_j = \pi\mathbf{b}_j/2$ to inequality (C) on page 10 yields the following criterion.

Corollary 4.2 *Let notations be as above. Then the Mellin-Barnes integral converges absolutely if there exist $\mu_i \in (-1, 1)$ such that*

$$\sum_{i=1}^{N} \frac{\theta_i}{2\pi}\mathbf{b}_i = \frac{1}{4}\sum_{i=1}^{N} \mu_i \mathbf{b}_i.$$

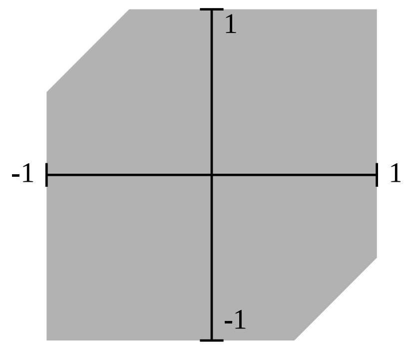
Let us define $Z_B = \{\frac{1}{4}\sum_{i=1}^{N} \mu_i \mathbf{b}_i | \mu_i \in (-1, 1)\}$. This is a so-called zonotope in d-dimensional space. The convergence condition for the Mellin-Barnes integral now reads

$$\sum_{i=1}^{N} \frac{\theta_i}{2\pi}\mathbf{b}_i \in Z_B.$$

As an example let us again take the system Appell F_2. Recall that our B-matrix reads

$$B = \begin{pmatrix} -1 & -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}^t.$$

As before, a parameter vector $\boldsymbol{\gamma}$ can also be read off from the Γ-expansion, namely $\boldsymbol{\gamma} = (-\alpha, -\beta, -\beta', \gamma - 1, \gamma' - 1, 0, 0)$. When $\alpha, \beta, \beta' > 0$ and $\gamma, \gamma' < 1$ we see that $\boldsymbol{\gamma}$ has negative components, except for the last two. By making a suitable choice for $\boldsymbol{\sigma} \in \mathbb{R}^2$ we can see to it that $\gamma_i < -\mathbf{b}_i \cdot \boldsymbol{\gamma}$ for all i. Thus the corresponding Mellin-Barnes integral is indeed a solution of the F_2-system. The zonotope Z_B can be pictured as



and the convergence condition reads

$$\frac{1}{2\pi}(-\theta_1 - \theta_2 + \theta_4 + \theta_6, -\theta_1 - \theta_3 + \theta_5 + \theta_7) \in Z_B.$$

Note that the four points $(\pm 1/2, \pm 1/2)$ are contained in Z_B. They correspond to the arguments $\boldsymbol{\theta} = 2\pi(0, 0, 0, 0, 0, \pm 1/2, \pm 1/2)$. These argument choices represent the same point in $v_1, \ldots, v_7$-space. Hence we have four Mellin-Barnes solutions of the system F_2 around one point. According to Proposition 4.3 these integrals are linearly independent, and hence form a basis of local solutions of the F_2-system. We say that we have a *Mellin-Barnes basis* of solutions.

Proposition 4.3 *Let $\mathbf{v}_0 = (v_1^{(0)}, \ldots, v_N^{(0)}) \in (\mathbb{C}^*)^N$ and let Θ be a finite set of N-tuples $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_N)$ such that $v_j^{(0)} = |v_j^{(0)}| \exp(i\theta_j)$ and the sums $\frac{1}{2\pi}\sum_{j=1}^{d} \theta_j \mathbf{b}_j$ are distinct elements of Z_B. To each $\boldsymbol{\theta} \in \Theta$ denote the corresponding determination of the Mellin-Barnes integral in the neighbourhood of $\mathbf{v}_0$ by $M_{\boldsymbol{\theta}}$.*
Then the functions $M_{\boldsymbol{\theta}}$ are linearly independent over $\mathbb{C}$.

The proof of this Lemma depends on a d-dimensional version or if one wants, repeated application, of the following Theorem.

Theorem 4.4 (Mellin inversion theorem) *Let $\phi(z)$ be function on $\mathbb{C}$ satisfying the follwoing properties*

(a) ϕ is analytic in a vertical strip of the form $\alpha < x = \mathrm{Re}(z) < \beta$ where $\alpha, \beta \in \mathbb{R}$.

(b) $\int_{-\infty}^{\infty} |\phi(x + iy)| dy$ converges for all $x \in (\alpha, \beta)$.

(c) $\phi(z) \to 0$ *uniformly as* $|y| \to \infty$ *in* $\alpha + \epsilon < x < \beta - \epsilon$ *for all* $\epsilon > 0$.

Denote for all $t > 0$,

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-z} \phi(z) dz.$$

Then,

$$\phi(z) = \int_0^\infty t^{z-1} f(t) dt.$$

For a proof of this theorem see [21, Appendix 4, p341-342]. A treatment of the multidimensional case can also be found for example in [3].

Proof of Proposition 4.3. Suppose we have a non-trivial relation $\sum_{\boldsymbol{\theta} \in \Theta} \lambda_{\boldsymbol{\theta}} M_{\boldsymbol{\theta}} = 0$. Let us use the notation $(\boldsymbol{\theta} \cdot B)(\mathbf{s}) = \theta_1 \mathbf{b}_1 \cdot \mathbf{s} + \cdots + \theta_N \mathbf{b}_N \cdot \mathbf{s}$. The relation can be written as

$$0 = \int_{\mathbf{s} \in \boldsymbol{\sigma} + \sqrt{-1}\mathbb{R}^d} \left(\sum_{\boldsymbol{\theta} \in \Theta} \lambda_{\boldsymbol{\theta}} e^{(\boldsymbol{\theta} \cdot B)(\mathbf{s})} \right) |v_1|^{\mathbf{b}_1 \cdot \mathbf{s}} \cdots |v_N|^{\mathbf{b}_N \cdot \mathbf{s}} \prod_{i=1}^N \Gamma(-\gamma_i - \mathbf{b}_i \cdot \mathbf{s}) d\mathbf{s}$$

Let us now write $x_j = |v_1|^{b_{1j}} \cdots |v_N|^{b_{Nj}}$ where b_{ij} are the entries of the B-matrix. Then

$$|v_1|^{\mathbf{b}_1 \cdot \mathbf{s}} \cdots |v_N|^{\mathbf{b}_N \cdot \mathbf{s}} = x_1^{s_1} \cdots x_d^{s_d}$$

By repeated use of the Mellin inversion Theorem 4.4 we conclude that the vanishing of the integral implies the identical vanishing of

$$\sum_{\boldsymbol{\theta} \in \Theta} \lambda_{\boldsymbol{\theta}} e^{(\boldsymbol{\theta} \cdot B)(\mathbf{s})}$$

Since the exponentials are all distinct linear forms in $s_1, \ldots, s_d$ this implies that $\lambda_{\boldsymbol{\theta}} = 0$ for all $\boldsymbol{\theta} \in \Theta$.

□

Another proof of Proposition 4.3 can be found in [26, Lemma 5.5]. However, I hesitate somewhat about its completeness and decided to give the proof above.

Since we assume that $\gamma_i < 0$ for all i, all Mellin-Barnes integrals are solutions of the corresponding A-hypergeometric system. It would be very convenient if such a basis of solutions given by Mellin-Barnes integrals would always exist. It turns out that with the exception of Appell F_4 all $d = 2$ systems Appell F_1, F_2, F_3 and Horn $G_1, G_2, G_3, H_1, \ldots, H_7$ this is the case. A theoretical framework for a result like this may be provided in the PhD-thesis of Lisa Nilsson [26] suggesting that there does indeed exist such a basis if the complement of the so-called coamoeba of the A-resultant is non-empty. In [26] this is elaborated for the case $d = 2$.

Let us now make the following assumption on our system $H_A(\boldsymbol{\alpha})$.

Assumption 4.5 *There exists a point* $\mathbf{v}_0 \in (\mathbb{C}^*)^N$ *with an open neighbourhood in which there exists a Mellin-Barnes basis of solutions.*

For the practical determination of a Mellin-Barnes basis we use the following Proposition.

Proposition 4.6 *Let* $H_A(\boldsymbol{\alpha})$ *be a non-resonant system of rank* D*. The system allows a Mellin-Barnes basis of solutions if and only if the zonotope* Z_B *contains* D *distinct points* $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_D$ *whose coordinates differ by integers. Note that* Z_B *is an open set in* $\mathbb{R}^d$*.*

Proof: From the discussion above it follows that the existence of a Mellin-Barnes basis corresponds to the choice of D N-tuples $(\theta_1, \ldots, \theta_N)$, representing argument choices of a given point. Hence the differences between these N-tuples have coordinates which are integer multiples of 2π. The sums $\frac{1}{2\pi}\sum_{i=1}^{N} \theta_i \mathbf{b}_i$ are distinct, hence the D Mellin-Barnes basis elements correspond to D points $\boldsymbol{\tau}_i \in Z_B$ whose coordinates also differ by integers. Suppose conversely we have D points $\boldsymbol{\tau}_i \in Z_B$ whose coordinates differ by integers. Since the $\mathbb{Z}$-span of the columns $\mathbf{b}_i$ is $\mathbb{Z}^d$ we can find for every i integers $n_{i1}, \ldots, n_{iN}$ such that $\tau_i - \tau_1 = n_{i1}\mathbf{b}_1 + \cdots + n_{iN}\mathbf{b}_N$. So if $(\theta_1, \ldots, \theta_N)$ is an argument choice for τ_1, then the N-tuples $(\theta_1 + 2\pi n_{i1}, \ldots, \theta_N + 2\pi n_{iN})$ represent argument choices for τ_i with $i = 1, 2, \ldots, D$.

□

For later use, we consider the vector $\boldsymbol{\theta}_j$ of arguments and the vectors $\boldsymbol{\tau}_j$ as column vectors. Then $2\pi\boldsymbol{\tau}_j = B\boldsymbol{\theta}_j$ for $j = 1, 2, \ldots, D$.

5 Monodromy computation

Suppose our system $H_A(\boldsymbol{\alpha})$ allows a Mellin-Barnes basis of solutions (Assumption 4.5). Denote the basis elements by $M_1, \ldots, M_D$ and the corresponding points in Z_B by $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_D$. It is the goal of this section to compute the local monodromy groups with respect to the basis $M_1, \ldots, M_D$. To this end we shall determine the transition matrices from $M_1, \ldots, M_D$ to each of the bases of local series expansions.
To each point $\boldsymbol{\tau}_i$ there corresponds a (not necessarily unique) choice of arguments $\boldsymbol{\theta}_i = (\mathrm{Arg}(v_1), \ldots, \mathrm{Arg}(v_N))$ for $i = 1, \ldots, D$. We assume that the arguments are chosen such that the differences $\boldsymbol{\theta}_i - \boldsymbol{\theta}_1$ have all their components equal to integer multiples of 2π (see Proposition 4.6). Let $\mathbf{v}_0 \in (\mathbb{C}^*)^N$ be a point whose coordinates have arguments $\boldsymbol{\tau}_1$. In particular we have a basis of Mellin-Barnes solutions around $\mathbf{v}_0$. The Mellin-Barnes integral corresponding to the argument vector $\boldsymbol{\theta}_i$ is denoted by M_i.

Let $f_1, \ldots, f_D$ be a basis of series expansions for some local basis. Suppose that the realm of convergence of these local series expansions contains the torus R: $|v_i| = r_i$ for $i = 1, \ldots, N$. Choose a point $\mathbf{v}_0' \in R$ with the same argument values as $\mathbf{v}_0$ and let δ be a path from $\mathbf{v}_0$ to $\mathbf{v}_0'$ while keeping the arguments fixed. In a neighbourhood of $\mathbf{v}_0'$ we also have a Mellin-Barnes basis of solutions which are simply the analytic continuation of $M_1, \ldots, M_D$ along δ. For any N-tuple of integers $\mathbf{n} = (n_1, \ldots, n_N)$ we consider the loop

$$c = c(\mathbf{n}) = c(n_1, \ldots, n_N) : \; (e^{2\pi i n_1 t} v_1^{(0)'}, \ldots, e^{2\pi i n_N t} v_N^{(0)'}), \; t \in [0,1].$$

Note that after analytic continuation of M_1 along the path $c((\boldsymbol{\theta}_j - \boldsymbol{\theta}_1)/2\pi)$ we end up with the Mellin-Barnes solution M_j for every j. We denote this path by c_j.
Let us denote a basis of local series expansions by $f_1, \ldots, f_D$ and the corresponding choices of $\boldsymbol{\gamma}$ by $\boldsymbol{\gamma}^{(1)}, \ldots, \boldsymbol{\gamma}^{(D)}$. We regard the latter as row vectors. Then there exist scalars μ_i such that

$$M_1 = \mu_1 f_1 + \cdots + \mu_D f_D$$

in a neighbourhood of $\mathbf{v}_0'$. After continuation along c_j the integral M_1 changes into M_j for every j. Under the paths c_j the local expansions f_i are multiplied by scalars. The space spanned by $M_1, \ldots, M_D$ is D-dimensional. The space spanned by the images of $\mu_1 f_1 + \cdots + \mu_D f_D$ under $c_j(\mathbf{v}_0'), j = 1, 2, \ldots, D$ is at most equal to the number of non-zero μ_i. Hence we conclude that $\mu_i \neq 0$ for all i. Let us renormalise the f_i such that

$$M_1 = f_1 + \cdots + f_D.$$

Then after continuation along c_j we get

$$M_j = e^{i\boldsymbol{\gamma}^{(1)}(\boldsymbol{\theta}_j - \boldsymbol{\theta}_1)} f_1 + \cdots + e^{i\boldsymbol{\gamma}^{(D)}(\boldsymbol{\theta}_j - \boldsymbol{\theta}_1)} f_D$$

for every j. Define

$$X_\rho = \begin{pmatrix} 1 & \cdots & 1 \\ e^{i\boldsymbol{\gamma}^{(1)}(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1)} & \cdots & e^{i\boldsymbol{\gamma}^{(D)}(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1)} \\ \vdots & & \vdots \\ e^{i\boldsymbol{\gamma}^{(1)}(\boldsymbol{\theta}_D - \boldsymbol{\theta}_1)} & \cdots & e^{i\boldsymbol{\gamma}^{(D)}(\boldsymbol{\theta}_D - \boldsymbol{\theta}_1)} \end{pmatrix}.$$

Then

$$\begin{pmatrix} M_1 \\ \vdots \\ M_D \end{pmatrix} = X_\rho \begin{pmatrix} f_1 \\ \vdots \\ f_D \end{pmatrix}$$

hence X_ρ is the desired transition matrix. Let us now consider any closed path of the form $\delta^{-1} c(\mathbf{n}) \delta, \mathbf{n} \in \mathbb{Z}^N$ beginning and ending in $\mathbf{v}_0$. Continuation of $M_1, \ldots, M_D$ along δ is trivial since the Mellin-Barnes integrals converge throughout. However these integrals do not converge anymore if we continue along $c(\mathbf{n})$. For that we have to change to the local basis $f_i, i = 1, \ldots, D$. Analytic continuation along $c(\mathbf{n})$ changes them into $e^{2\pi i \mathbf{n}\cdot\boldsymbol{\gamma}^{(i)}} f_i$ for $i = 1, \ldots, D$. Express these solutions in terms of the M_i again and continue back along δ^{-1}. The monodromy matrix can be computed as follows. Let $\chi_\rho(\mathbf{n})$ be the diagonal $D \times D$-matrix with entries $e^{2\pi i \gamma^{(i)}\mathbf{n}}$, $i = 1, \ldots, D$. It is the monodromy matrix with respect to $f_1, \ldots, f_D$. With respect to $M_1, \ldots, M_D$ this monodromy element has matrix $X_\rho \chi_\rho(\mathbf{n}) X_\rho^{-1}$. Thus we see that all local monodromies can be written with respect to a fixed Mellin-Barnes basis.

6 An implementation

The considerations in the previous sections, together with some practical tricks, lead to an algorithm to compute monodromy matrices, which we describe in this section.

We start with a totally non-resonant hypergeometric system $H_A(\boldsymbol{\alpha})$ and we assume that there exists a Mellin-Barnes basis. The starting data are a $d \times N$ B-matrix B and a parametervector $\boldsymbol{\gamma}_0$ (in row form) such that $A\boldsymbol{\gamma}_0 = \boldsymbol{\alpha}$. In general both B and $\boldsymbol{\gamma}_0$ can easily be read off from an explicit series solution of a hypergeometric system. For example, from the expansion of Appell F_2 as on page 7. We also assume we know the rank D of the system.

Step 1. Using the B-matrix we determine the zonotope Z_B and a find D distinct points in it, whose coordinates differ by integers. Since we assumed the existence of a Mellin-Barnes basis these points exist. Call the points $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_D$. From the proof of Proposition 4.6 we know that to each τ_i there exists a column vector of arguments $\theta_i \in \mathbb{R}^n$ such that $2\pi\tau_i = B\theta_i$. However, we do not compute these angle vectors.

Step 2. Construct the set $\mathcal{I}$ of all subsets I of cardinality d of the columns $\{\mathbf{b}_1, \ldots, \mathbf{b}_N\}$ with $\delta_I = |\det_{i\in I}(\mathbf{b}_i)| \neq 0$. As a fine point, if $\Delta_I > 1$ we include Δ_I copies of I in $\mathcal{I}$. For each I there exists a parametervector $\boldsymbol{\gamma}^I$ in the following way. Denote the rows of the B-matrix B by $\mathbf{l}_1, \ldots, \mathbf{l}_d$, recall that this is a basis of the lattice L. In case $\Delta_I = 1$ we take the uniquely determined real numbers $\mu_1, \ldots, \mu_d$ such that $\boldsymbol{\gamma}_0 + \mu_1\mathbf{l}_1 + \cdots + \mu_d\mathbf{l}_d$ has i-th coordinate 0 for all $i \in I$ and call this sum γ^I. In case $\Delta_I > 1$ we make Δ_I choices for $(\mu_1, \ldots, \mu_d)$, distinct modulo $\mathbb{Z}^d$, such that $\boldsymbol{\gamma}_0 + \mu_1\mathbf{l}_1 + \cdots + \mu_d\mathbf{l}_d$ has integer coordinates on the i-th position for all $i \in I$. In this way we get Δ_I different parametervectors $\boldsymbol{\gamma}^I$ (in row form) for a given $I \in \mathcal{I}$. However, in the computation we only retain the (row vectors) $\boldsymbol{\mu}^I = (\mu_1, \ldots, \mu_d)^t$ for each I. They have the property that $\boldsymbol{\gamma}^I - \boldsymbol{\gamma}_0 = \boldsymbol{\mu}^I B$ for all I.

Step 3. To every I we associate a column vector X_I of length D given by

$$X_I = (1, \exp(2\pi i \boldsymbol{\mu}^I(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1)), \ldots, \exp(2\pi i \boldsymbol{\mu}^I(\boldsymbol{\tau}_D - \boldsymbol{\tau}_1)))^t. \tag{1}$$

Since, for each $j = 1, 2, \ldots, D$ we have

$$\begin{aligned} 2\pi\boldsymbol{\mu}^I(\boldsymbol{\tau}_j - \boldsymbol{\tau}_1) &= \mu^I B(\theta_j - \theta_1) \\ &= (\boldsymbol{\gamma}^I - \boldsymbol{\gamma}_0)(\theta_j - \theta_1) \end{aligned}$$

the j-th components of X_I differs by a factor

$$\exp(-i\boldsymbol{\gamma}_0(\theta_j - \theta_1))$$

from the similar components in the columns of the transition matrices $X_{\boldsymbol{\rho}}$. The only effect is that the transition matrices built out of our present X_I will give us the transition of the Mellin-Barnes basis to a renormalized local basis. This will have no effect on the monodromy computation.

Step 4. For every cone in the secondary fan (specified by a convergence direction ρ inside that cone) we determine the sets $I \in \mathcal{I}$ such that the cone or, equivalently, ρ lies in the positive real cone spanned by the vectors $\{\mathbf{b}_i\}_{i \in I}$. Call this set of sets $\mathcal{I}_\rho$. The theory of Gel'fand, Kapranov and Zelevinsky tells us that $\mathcal{I}_\rho$ contains precisely D sets (when we count possible repetitions of a set with $\Delta_I > 1$). Let X_ρ be the $D \times D$-matrix whose columns are the vectors X_I with $I \in \mathcal{I}_\rho$. The matrices X_ρ are the transition matrices from the Mellin-Barnes basis to the local power series basis, all of whose elements contain ρ as a convergence direction.

Step 5. For every cone in the secondary fan (specified by a convergence direction ρ) we determine the characters of the local monodromies of the corresponding power series solutions. Let I_0 be such that $|\det(\mathbf{b}_j)_{j \in I_0}| = 1$ We choose a set of d generators for the local monodromy as follows. For every $j = 1, \ldots, d$ we define $\mathbf{n}_j \in \mathbb{Z}^N$ such that $\mathbf{n}_j$ has support in I_0 and $B\mathbf{n}_j = \mathbf{e}_j$, the j-th standard basis vector in $\mathbb{R}^d$. Since the support is in I_0 we have $\boldsymbol{\gamma}_0\mathbf{n}_j = 0$. Furthermore,

$$\boldsymbol{\gamma}^I\mathbf{n}_j = (\boldsymbol{\gamma}^I - \boldsymbol{\gamma}_0)\mathbf{n}_j = (\boldsymbol{\mu}^I - \boldsymbol{\mu}^{I_0})B\mathbf{n}_j = \mu^I_j - \mu^{I_0}_j$$

for $j = 1, \ldots, d$. Hence the characters corresponding to the path $c(\mathbf{n}_j)$ read

$$\exp(2\pi i\boldsymbol{\gamma}^I\mathbf{n}_j) = \exp(2\pi i(\mu^I_j - \mu^{I_0}_j)).$$

Step 6. This is the final step in which we compute d monodromy matrices for every cone of the secondary fan. For a cone, specified by a convergence direction ρ, and a loop $c(\mathbf{n}_j)$ (defined in step 5) we construct the matrix X_ρ as in Step 4, and a diagonal matrix $\chi_{\boldsymbol{\rho},j}$ with entries $\exp(2\pi i\mu^I_j)$, $I \in \mathcal{I}_\rho$, as in Step 5. We see to it that both in X_ρ and $\chi_{\boldsymbol{\rho},j}$ we keep the same ordering of the set $\mathcal{I}_\rho$. Then construct the matrix

$$M_{\boldsymbol{\rho},j} = X_\rho\chi_{\rho,j}X_\rho^{-1}.$$

Let F be the number of open cones in the secondary fan. Then we get dF monodromy matrices in this way. They generate a subgroup of the monodromy group whose projectivization (quotient by scalars) we denote by $lMon$. As remarked before, computation of local monodromies by other loops will only add scalar matrices and therefore does not change $lMon$.

7 An example, Appell F_2

Recall that a B-matrix is given by

$$B = \begin{pmatrix} -1 & -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

and a parameter vector $\boldsymbol{\gamma}_0 = (-\alpha, -\beta, -\beta', \gamma-1, \gamma'-1, 0, 0)$. We trust that no confusion will arise with the existing notations $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$. Two powerseries solution expansions have already been given on pages 6 and 7. The set $\mathcal{I}$ consists of 15 elements, $\{1,3\}$, $\{1,4\}$, $\{1,6\}$, $\{1,5\}$, $\{1,7\}$, $\{1,2\}$, $\{2,3\}$, $\{2,5\}$, $\{2,7\}$, $\{3,4\}$, $\{3,6\}$, $\{4,5\}$, $\{4,7\}$, $\{5,6\}$, $\{6,7\}$. Here is a table with the corresponding values of μ^J.

	J	$\boldsymbol{\mu}^J$
1	$\{1,3\}$	$-\alpha+\beta', -\beta'$
2	$\{1,4\}$	$1-\gamma, -1-\alpha+\gamma$
3	$\{1,6\}$	$0, -\alpha$
4	$\{1,5\}$	$-1-\alpha+\gamma', 1-\gamma'$
5	$\{1,7\}$	$-1+\gamma', -\alpha, 0$
6	$\{1,2\}$	$-\beta, -\alpha+\beta$
7	$\{2,3\}$	$-\beta, -\beta'$

8	$\{2,5\}$	$-\beta, 1-\gamma'$
9	$\{2,7\}$	$-\beta, 0$
10	$\{3,4\}$	$1-\gamma, -\beta'$
11	$\{3,6\}$	$0, -\beta'$
12	$\{4,5\}$	$1-\gamma, 1-\gamma'$
13	$\{4,7\}$	$1-\gamma, 0$
14	$\{5,6\}$	$0, 1-\gamma'$
15	$\{6,7\}$	$0, 0$

As noted earlier, the zonotope Z_B contains the four points $(\pm 1/2, \pm 1/2)$. Define

$$\boldsymbol{\tau}_1 = (-1/2, -1/2)^t,\ \boldsymbol{\tau}_2 = (1/2, -1/2)^t,\ \boldsymbol{\tau}_3 = (-1/2, 1/2)^t,\ \boldsymbol{\tau}_4 = (1/2, 1/2)^t.$$

Then the vectors X_J, as defined in Step 3 of our algorithm on page 16, read

$$1, e(\mu_1^J), e(\mu_2^J), e(\mu_1^J + \mu_2^J)$$

where we use the notations $e(x) = e^{2\pi i x}$ and $a = e(\alpha), b = e(\beta), b' = e(\beta'), c = e(\gamma), c' = e(\gamma')$. Here is the list of all X_J with the same ordering as in the previous table,

	J	X_J			
1	$\{1,3\}$	1	b'/a	$1/b'$	$1/a$
2	$\{1,4\}$	1	$1/c$	c/a	$1/a$
3	$\{1,6\}$	1	1	$1/a$	$1/a$
4	$\{1,5\}$	1	c'/a	$1/c')$	$1/a$
5	$\{1,7\}$	1	$1/a$	1	$1/a$
6	$\{1,2\}$	1	$1/b$	b/a	$1/a$
7	$\{2,3\}$	1	$1/b$	$1/b'$	$1/bb'$
8	$\{2,5\}$	1	$1/b$	$1/c'$	$1/bc'$
9	$\{2,7\}$	1	$1/b$	1	$1/b$
10	$\{3,4\}$	1	$1/c$	$1/b'$	$1/cb'$
11	$\{3,6\}$	1	1	$1/b'$	$1/b'$
12	$\{4,5\}$	1	$1/c$	$1/c'$	$1/cc'$
13	$\{4,7\}$	1	$1/c$	1	$1/c$
14	$\{5,6\}$	1	1	$1/c'$	$1/c'$
15	$\{6,7\}$	1	1	1	1

To write down local monodromies we use Step 5 of our algorithm. The characters $e(\mu_1)$ are said to correspond to path I and the characters $e(\mu_2)$ correspond to path II. We do not need to write down a separate table for them since they are simply the second and third component of the vectors X_J.

As an example for the action of path I on a local basis we take the four local basis solutions with convergence direction $-0.5, 1$, as before. The transition matrix X_ρ consists of the vectors X_J with numbering 4,5,8,9 in Table 7. The transition matrix reads

$$X_\rho = \begin{pmatrix} 1 & 1 & 1 & 1 \\ c'/a & 1/a & 1/b & 1/b \\ 1/c' & 1 & 1/c' & 1 \\ 1/a & 1/a & 1/bc' & 1/b \end{pmatrix}$$

For the path I we get the monodromy matrix

$$X_\rho \begin{pmatrix} c'/a & 0 & 0 & 0 \\ 0 & 1/a & 0 & 0 \\ 0 & 0 & 1/b & 0 \\ 0 & 0 & 0 & 1/b \end{pmatrix} X_\rho^{-1}$$

which equals

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ (-1+c')/ab & 1/b+1/a+c'/a & c'/ab & -c'/a \\ 0 & 0 & 0 & 1 \\ -1/ab & 1/a & 0 & 1/b \end{pmatrix}$$

with respect to the Mellin-Barnes basis. For the path II we get

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1/c' & 0 & 1+1/c' & 0 \\ 0 & -1/c' & 0 & 1+1/c' \end{pmatrix}.$$

The calculation so far has been carried out for the convergence direction $(-0.5, 1)$. In fact we get the same matrices for every convergence direction in the cone spanned by

$\mathbf{b}_2, \mathbf{b}_5$ of the secondary fan. We can proceed in the same way with the other four cones. In each case we find two monodromy matrices. After removing duplicate matrices we end up with six monodromy matrices. They are given in the Appendix of this paper, together with a comparison of them with the five generators given by M.Kato in [19]. It turns out that the group generated by our six generators is conjugate to the group computed in [19].

8 An example, Clausen $_3F_2$

In this section we apply our method to the case of one variable

$$_3F_2\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{matrix}\middle| z\right)$$

which, up to a constant factor, is defined by the series

$$\sum_{n\geq 0}\frac{\Gamma(\alpha_1+n)\Gamma(\alpha_2+n)\Gamma(\alpha_3+n)}{\Gamma(\beta_1+n)\Gamma(\beta_2+n)n!}\ z^n.$$

Using the identity $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$ we see that the series is proportional to

$$\sum_n\frac{(-z)^n}{\Gamma(1-\alpha_1-n)\Gamma(1-\alpha_2-n)\Gamma(1-\alpha_3-n)\Gamma(\beta_1+n)\Gamma(\beta_2+n)\Gamma(1+n)}.$$

So the B-matrix is given by

$$B=(-1,-1,-1,1,1,1)^t$$

and Z_B is simply the open interval $(-3/2,3/2)$. In it we can take the three points $\tau_1=-1,\tau_2=0,\tau_3=1$ and so we see that we have a Mellin-Barnes basis of solutions. For the set I_0 we take $\{6\}$ and

$$\boldsymbol{\gamma}_0=(-\alpha_1,-\alpha_2,\alpha_3,\beta_1,\beta_2,0).$$

We consider the components modulo $\mathbb{Z}$. The set of columns of B has 6 subsets of cardinality 1 and the corresponding values of μ_1 are

$$\alpha_1,\alpha_2,\alpha_3,-\beta_1,-\beta_2,0.$$

Letting $a_i=e(\alpha_i)$ and $b_j=e(\beta_j)$ we get for the vectors X_J,

$$(1,a_1,a_1^2),\ (1,a_2,a_2^2),\ (1,a_3,a_3^2)$$

$$(1, b_1, b_1^2),\ (1, b_2, b_2^2),\ (1, 1, 1).$$

There is only one loop to consider for every local basis. Consider the convergence direction -1. This lies in the positive cones spanned by $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ respectively. The transition matrix reads

$$X_\rho = \begin{pmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{pmatrix}$$

and the diagonal character matrix

$$\chi_\rho = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}.$$

We get

$$X_\rho \chi_\rho X_\rho^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 a_2 a_3 & -a_2 a_3 - a_2 a_1 - a_3 a_1 & a_1 + a_2 + a_3 \end{pmatrix}.$$

This is precisely the matrix representation for the monodromy matrix around $z = \infty$ for ${}_3F_2$ as given in [8]. We get a similar result for the monodromy matrix around $z = 0$ (with $b_1, b_2, 1$ instead of a_1, a_2, a_3)

9 Existence of Mellin-Barnes bases

In this section we show that certain families of hypergeometric equations satify Assumption 4.5, and some don't (the case of Lauricell F_C).

9.1 Lauricella F_A

The Lauricella system F_A in n variables is a system of rank 2^n. From the powerseries

$$F_A(a, \mathbf{b}, \mathbf{c}|\mathbf{x}) = \sum_{\mathbf{m} \geq 0} \frac{(a)_{|\mathbf{m}|} (\mathbf{b})_{\mathbf{m}}}{(\mathbf{c})_{\mathbf{m}} \mathbf{m}!} \mathbf{x}^{\mathbf{m}}$$

in $\mathbf{x} = (x_1, \ldots, x_n)$ we see that an $n \times (3n+2)$ B-matrix is given by

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 & -1 & 0 & \cdots 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 & -1 & \cdots & 0 & 0 & -1 & \cdots 0 \\ \vdots & & & & & & & & & & & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots -1 \end{pmatrix}$$

The B-zonotope is thus given by the points

$$\lambda(\mathbf{e}_1 + \cdots \mathbf{e}_n) + \sum_{i=1}^{m} \mu_i \mathbf{e}_i$$

where $|\lambda| < 1/4$ and $|\mu_i| < 3/4$ for $i = 1, \ldots, n$. Let us choose $\epsilon > 0$ sufficiently small. Consider the 2^n points

$$(3/4 - 2\epsilon)(\mathbf{e}_1 + \cdots + \mathbf{e}_n) - k_1\mathbf{e}_1 - \cdots k_n\mathbf{e}_n$$

where $k_i \in \{0, 1\}$ for all i. Each such point equals

$$(1/4 - \epsilon)(\mathbf{e}_1 + \cdots + \mathbf{e}_n) + (1/2 - k_1 - \epsilon)\mathbf{e}_1 + \cdots + (1/2 - k_n - \epsilon)\mathbf{e}_n$$

which is clearly contained in the B-zonotope.

9.2 Lauricella F_B

The Lauricella system F_B is also a system of rank 2^n. From the powerseries

$$F_B(\mathbf{a}, \mathbf{b}, c|\mathbf{x}) = \sum_{\mathbf{m} \geq 0} \frac{(\mathbf{a})_{\mathbf{m}}(\mathbf{b})_{\mathbf{m}}}{(c)_{|\mathbf{m}|}\mathbf{m}!}\mathbf{x}^{\mathbf{m}}$$

in $\mathbf{x} = (x_1, \ldots, x_n)$ we see that an $n \times (3n+2)$ B-matrix is given by

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 & -1 & 0 & -1 & \cdots 0 \\ \vdots & & & & & & & & & & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots -1 \end{pmatrix}$$

Note the B-zonotope is the same as in the case of Lauricella F_A. Hence there exists a Mellin-Barnes basis of solutions.

9.3 Lauricella F_D

The Lauricella system F_D in n variables is a system of rank $n+1$. From the powerseries

$$F_D(a, \mathbf{b}, c|\mathbf{x}) = \sum_{\mathbf{m} \geq 0} \frac{(a)_{|\mathbf{m}|}(\mathbf{b})_{\mathbf{m}}}{(c)_{|\mathbf{m}|}\mathbf{m}!}\mathbf{x}^{\mathbf{m}}$$

in $\mathbf{x} = (x_1, \ldots, x_n)$ we deduce an $n \times (2n+2)$ B-matrix

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & -1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 & -1 & 0 & -1 & \cdots & 0 \\ \vdots & & & & & & & & & \vdots \\ 1 & 0 & 0 & \cdots & 1 & -1 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

Hence the B-zonotope consists of the points

$$\lambda_0(\mathbf{e}_1 + \cdots + \mathbf{e}_n) + \lambda_1\mathbf{e}_1 + \cdots + \lambda_n\mathbf{e}_n$$

where $|\lambda_i| < 1/2$ for $i = 0, 1, \ldots, n$. Choose $\epsilon > 0$ sufficiently small and consider the $n+1$ points

$$-\epsilon(n\mathbf{e}_1 + (n-1)\mathbf{e}_2 + \cdots + 2\mathbf{e}_{n-1} + \mathbf{e}_n) + \sum_{i=0}^{k} \mathbf{e}_i$$

for $k = 0, 1, 2 \ldots, n$. Each such point can be rewritten as

$$(1/2 - (n-k-1/2)\epsilon)(\mathbf{e}_1 + \cdots + \mathbf{e}_n) + \sum_{j=1}^{n}(\pm 1/2 - (k-j-1/2)\epsilon)\mathbf{e}_j$$

where $\pm 1/2$ is $1/2$ if $k > j + 1/2$ and $-1/2$ if $k < j + 1/2$. Hence they are contained in the B-zonotope and we have found a Mellin-Barnes basis for Lauricella F_D.

9.4 Lauricella F_C

The Lauricella system F_C in n variables is system of rank 2^n. From the powerseries

$$F_C(a, b, \mathbf{c}|\mathbf{x}) = \sum_{\mathbf{m} \geq 0} \frac{(a)_{|\mathbf{m}|}(\mathbf{b})_{|\mathbf{m}|}}{(c)_{\mathbf{m}}\mathbf{m}!}\mathbf{x}^{\mathbf{m}}$$

in $\mathbf{x} = (x_1, \ldots, x_n$ we deduce an $n \times (2n+2)$ B-matrix

$$\begin{pmatrix} 1 & 1 & -1 & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & -1 & \cdots & 0 & 0 & -1 & \cdots & 0 \\ \vdots & & & & & & & & & \vdots \\ 1 & 1 & 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

Note that the B-zonotope is the same as for Lauricella F_D, but this time we have to find a Mellin-Barnes basis of 2^n solutions. Clearly this is impossible if $n > 1$.

9.5 Aomoto-Gel'fand system $E(3,6)$

This system forms the subject of the second part of M.Yoshida's book [36]. It is an Aomoto system which can be reinterpreted as an A-hypergeometric system. It has four essential variables ($d = 4$) and rank 6. system that corresponds to configurations of six points (or lines) in $\mathbb{P}^2$. We start by giving the A-matrix of the system,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

and the parameters $(\alpha_1, \alpha_2, 2-\alpha_4, 2-\alpha_5, 2-\alpha_6)$. The integrand of the Euler integral as defined in [7, p 607] reads

$$\frac{s^{\alpha_1-1}t^{\alpha_2-1}t_1^{1-\alpha_4}t_2^{1-\alpha_5}t_3^{1-\alpha_6}}{1-t_1(s+t+1)-t_2(s+v_1t+v_3)-t_3(s+v_2t+v_4)} ds \wedge dt \wedge dt_1 \wedge dt_2 \wedge dt_3.$$

Perform the substitutions $t_1 \to t_1/(s+t+1), t_2 \to t_2/(s+v_1t+v_3)$ and $t_3 \to t_3/(s+v_2t+v_4)$. We obtain the integrand

$$\frac{t_1^{1-\alpha_4}t_2^{1-\alpha_5}t_3^{1-\alpha_6}}{1-t_1-t_2-t_3}\prod_{i=1}^{6}(L_i)^{\alpha_i-1} ds \wedge dt \wedge dt_1 \wedge dt_2 \wedge dt_3$$

where

$$L_1 = s,\ L_2 = t,\ L_3 = 1,\ L_4 = s+t+1,$$
$$L_5 = s+v_1t+v_3,\ L_6 = s+v_2t+v_4$$

and $\alpha_3 = 3-\alpha_1-\alpha_2-\alpha_4-\alpha_5-\alpha_6$. Integration with respect to t_1, t_2, t_3 leaves us with a 2-form which is the integrand given on page 221 of [36], but with v_i instead of x^i. Thus we see that our A-matrix corresponds to a Gel'fand-Aomoto system which is associated to configurations of six lines in $\mathbb{P}^2$. The system is a four variable system of rank 6. It is irreducible if and only if none of the α_i is an integer, see [23, Prop 2]. A possible B-matrix reads

$$B = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}^t$$

The set $\mathcal{I}$ consists of 81 sets, and hence 81 distinct local solutions. The number of local solution bases is 108. In a straightforward manner one can check that the B-zonotope Z_B contains the points

$$p, p+(0,0,0,1), p+(1,0,0,0), p+(1,0,1,1), p+(1,1,0,1), p+(1,1,1,1)$$

where $p = (-0.9, -0.4, -0.5, -0.7)$. Hence the system $E(3,6)$ has a Mellin-Barnes basis of solutions. Computation of a set of generators for the monodromy group $lMon$ is now straightforward. We get 82 matrices, but have not made an attempt to compare with the 20 generators of Mon found in [22].

10 Hermitian forms

In the cases where we carried out the algorithm given above, it turns out that whenever $\boldsymbol{\alpha} \in \mathbb{R}^r$, and the system is totally nonresonant, there exists a unique (up to a constant factor) hermitean form which is invariant under the group $lMon$. Subsequent studies lead us to the following conjecture.

Conjecture 10.1 *Let $H_A(\boldsymbol{\alpha})$ be a non-resonant A-hypergeometric system with $\boldsymbol{\alpha} \in \mathbb{R}^r$. Then there exists a non-trivial unique (up to scalars) Hermitean form, invariant under the monodromy group. More concretely, there exist a Hermitean $D \times D$-matrix H such that $\overline{g}^t H g = H$ for all elements g of the monodromy group. Here D denotes the rank of $H_A(\boldsymbol{\alpha})$.*
Moreover, when the system is totally non-resonant, the signature of H is determined by the signs of the numbers

$$\prod_{i \notin I} \sin(\pi \gamma_i^I)$$

as I runs through the elements of $\mathcal{I}_{\boldsymbol{\rho}}$ for some convergence direction ρ.

We want to deal with this matter in another paper. However, we do like to add that a detailed calculation shows that the signatures thus obtained are in accordance with the results on $E(3,6)$ in [23, Prop 1] (except for a small printing error). Note that signature $(5,1)$ does not occur. Similarly, calculations for Lauricella F_D give us results which are in accordance with Picard [28], Terada [35] and Deligne-Mostow [10].

11 Appendix

Here we reproduce the six matrices obtained from the monodromy calculation of Appell F_2.

$$M_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(1+c)/ab') & c/ab' & 1/b' + 1/a + c/a & -c/a \\ -1/ab' & 0 & 1/a & 1/b' \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -b/ab' - 1/c' & b/c' & 1 + b/a + 1/c' & b(-1 + 1/b' - 1/c') \\ -1/ab' & 0 & 1/a & 1/b' \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1/c' & 0 & 1 + 1/c' & 0 \\ 0 & -1/c' & 0 & 1 + 1/c' \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -b'/ab - 1/c & 1 + b'/a + 1/c & b'/c & b'(-1 + 1/b - 1/c) \\ 0 & 0 & 0 & 1 \\ -1/ab & 1/a & 0 & 1/b \end{pmatrix},$$

$$M_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1/c & 1 + 1/c & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1/c & 1 + 1/c \end{pmatrix},$$

$$M_6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -(1 + c')/ab & 1/b + 1/a + c'/a & c'/ab & -c'/a \\ 0 & 0 & 0 & 1 \\ -1/ab & 1/a & 0 & 1/b \end{pmatrix}.$$

Now let g_1, g_2, g_3, g_4, g_5 be the monodromy matrices defined in formulas (2.7), (2.8), (2.9), (2.10), (2.11) in M.Kato's paper [19], where our symbols $a, b, b', c, '$ are Kato's symbols $e(a), e(b), e(b'), e(c), e(c')$. Define the conjugation matrix

$$S = \begin{pmatrix} -1 & c & c' & -cc' \\ -1 & 1 & c' & -c' \\ -1 & c & 1 & -c \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$

Then the relations between the M_i and g_j are given by

$$M_1 = S^{-1} g_2 g_3 g_5 S \quad M_2 = S^{-1} g_2 g_5 S, \quad M_3 = S^{-1} g_2 S$$

$$M_4 = S^{-1} g_1 g_4 S, \quad M_5 = S^{-1} g_1 S, \quad M_6 = S^{-1} g_1 g_3 g_4 S.$$

From these relations it follows that the group we computed and the group computed in [19] are conjugate.

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