1 Euler integral, differential forms

1.1 Integral monodromy and geometry

Recall from the previous section that the hypergeometric equation for ${}_2F_1(1/2,1/2,1|z)$ has a monodromy group which is conjugate to $\Gamma(2)$, the subgroup of $SL(2,\mathbb{Z})$ consisting of matrices equal to the identity matrix modulo 2. Another explanation of the integrality of this monodromy comes from geometric considerations that we will explain here. Recall that from the Euler integral representation of hypergeometric functions we have

$$F(z) := {}_2F_1(1/2,1/2,1|z) = \frac{1}{\pi}\int_0^1 \frac{dx}{\sqrt{x(1-x)(1-zx)}}.$$

If we assume that z is real and $0 < z < 1$ the sign of the integrand is chosen positive. Consider the second integral

$$G(z) := \frac{1}{\pi}\int_1^{1/z} \frac{dx}{\sqrt{x(x-1)(zx-1)}},$$

whose integrand is chosen in $i\mathbb{R}_{>0}$ say, if $0 < z < 1$.
Let us first show that $G(z)$ is another solution of the hypergeometric equation. We consider analytic continuation of $F(z)$ along the closed path $|z-1| = |z_0-1|$, traversed anti-clockwise and starting at $z = z_0 \in (0,1)$. The point $1/z$ then decribes a circle in clockwise direction, starting at $1/z_0$ and passing between 0 and 1. As soon as $1/z$ reaches the interval $(0,1)$ we have to deform the integration path between $(0,1)$ in order to ensure analytic continuation. In the end we find an integral along the following path
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The final path consists of three parts. The part from 0 to 1, then from 1 to $1/z_0$ and finally from $1/z_0$ to 1 with an integrand having opposite sign. So $F(z)$, after analytic continuation, has changed into $\epsilon F(z) + 2\epsilon' G(z)$, where $\epsilon, \epsilon' \in \{\pm 1\}$ come from the sign of the integrand. Let $\mathcal{L}$ be the differential operator of the hypergeometric equation, i.e. $\mathcal{L}(F) = 0$. After continuation we obtain $0 = \epsilon(F) + \epsilon'\mathcal{L}(G) = \epsilon'(G)$. Thus we see that $G(z)$ is another solution of the hypergeometric equation $\mathcal{L}(y) = 0$. When continuing the path between 1 and $1/z_0$, the point 1 stays fixed and $1/z_0$ describes a closed loop around 1, but not around 0. This means that $G(z)$ changes into $\epsilon'' G(z)$ for some $\epsilon''\{\pm 1\}$, depending on the sign of the integrand. We know that the eigenvalues of the monodromy around $z = 1$ are $1, 1$. So $\epsilon = \epsilon'' = 1$. If one wants, ϵ' can be determined but we do not do this here. The monodromy substitution of the path around $z = 1$ is now given by

$$F(z) \to F(z) \pm 2G(z), \qquad G(z) \to G(z).$$

Now we consider analytic continuation of $F(z), G(z)$ along the closed path $|z| = |z_0|$, traversed anti-clockwise and starting at $z = z_0 \in (0, 1)$. It is clear that $F(z)$ will not change. However the integration path between 0 and $1/z_0$ will change from the simple interval $[1, 1/z_0]$ to
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The integration path can be split in three parts, namely the part running from 1 to 0, the part from 0 to 1 where the sign of the integrand is opposite and the part from 1 to $1/z_0$. So $G(z)$ continues analytically to $2\epsilon' F(z) + \epsilon G(z)$, where again $\epsilon, \epsilon' \in \{\pm 1\}$ depend on the sign of the integrand. Because the eigenvalues of the monodromy around $z = 0$ are $1, 1$ we conclude $\epsilon = 1$ and the monodromy substitution is given by

$$F(z) \to F(z), \qquad G(z) \to G(z) \pm 2F(z).$$

We finally get the two generating matrices for the monodromy,

$$\begin{pmatrix} 1 & \pm 2 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ \pm 2 & 1 \end{pmatrix}.$$

The integral we just considered is an example of a so-called period of a differential form on the elliptic curve E_z given by $y^2 = x(1-x)(1-zx)$. To explain this we first note that $y(x) = \sqrt{x(1-x)(1-zx)}$ with $z \neq 0, 1$ is a one-valued analytic function of x in the complex x-plane with the line segments between $0, 1$ and between $1/z, \infty$ deleted. The integral over $(0, 1)$ can be replaced by an integration over the closed contour Γ around the line segment $[0, 1]$.
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Notice that

$$2\int_0^1 \frac{dx}{y} = \int_\Gamma \frac{dx}{y}.$$

In the space of points $(x, y) \in \mathbb{C}^2$ that satisfy $y^2 = x(1-x)(1-zx)$ the curve paramerized by $(x, y(x))$ with $x \in \Gamma$ is a closed contour.
The expression $\frac{dx}{y}$ is a differential form on E_z. To describe these we need to consider the local parameters on E_z. We always assume $z \neq 0, 1$. Choose a point x_0, y_0 on E_z.

If $x_0 \neq 0, 1, 1/z$ we can write $y(x)$ as a power series $y = y_0 + c_1 t + \cdots$ in $t = x - x_0$ with $y(x_0) = y_0$ and we say that t is a local parameter at (x_0, y_0). If $x_0 = 0$, say, this is not possible because $y = \sqrt{x} + O(x)$ when x is near 0. But then we can write x as powerseries in y^2, namely $x = y^2 + O(y^4)$. We then take $t = y$ as local parameter at $(0, 0)$. Similarly $t = y$ is a local parameter at $(1, 0)$ and $(1/z, 0)$. Finally, $t = x/y$ can be taken as local parameter at the point at ∞. Substitute $y = x/t$ in the equation for E_z to obtain $x^2/t^2 = x(1-x)(1-zx)$. This implies $1/x = t^2(1-1/x)(z-1/x)$ and hence $1/x$ cab be written as a power series in t^2, i.e. $1/x = zt^2 + c_4 t^4 + \cdots$. Then $1/y = t(zt^2 + O(t^4))$. The point corresponding $t = 0$ is called the point at ∞. The curve E_z, together with the point at infinity, is a compact complex Riemann surface. A bit like the Riemann sphere which is a compactification of $\mathbb{C}$ with a point at infinity.
The differential form $\omega = \frac{dx}{y}$ may acquire a pole if $y = 0$. Let us rewrite the form in terms of the local parameter $t = y$. We get $\omega_z = \frac{1}{y} d(y^2 + O(y^4)) = (2 + O(y^2))dy$. We say that ω_z has no pole at $(0, 0)$ and for the same reason it has on poles at $(1, 0)$ and $(1/z, 0)$. We rewrite ω_z around ∞ with its local paramater and get $\omega_z = (2/z + O(t))dt$. Again we have no pole at $t = 0$. So ω_z is a holomorphic differential form on all points of E_z.
The rational function field of E_z is the field defined by $\mathbb{C}(z)(x)[y]/(y^2 - x(1-x)(1-zx))$. Notation $\mathbb{C}(E_z)$. Any element can be written in the form $A(x) + B(x)y$ with $A, B \in \mathbb{C}(x)$. A rational differential form on E_z is a form $\Omega = F(x, y)dx$ where $F \in \mathbb{C}(E_z)$. To study it locally at a point P on E_z we rewrite the form as a series expansion in the local parameter at P. In general we get an expansion of the form

$$\Omega = \left(\frac{c_{-k}}{t^k} + \cdots + \frac{c_{-1}}{t} + c_0 + c_1 t + \cdots \right) dt \tag{1}$$

with $c_i \in \mathbb{C}$. If no negative powers occur we say that the form is holomorphic at P. The coefficient c_{-1} is called the *residue* of Ω. An important property of the residue of a differential form is that it is independent of the choice of local parameter at P.
A differential form on E_z is said to be of the *first kind* if it holomorphic at every point on E_z. It is said to be a form of the *second kind* if its residues are zero at every point. So they include the forms of the first kind. All remaining forms are of the *third kind*. An important class of forms of the second kind is given by the *exact differentials* $d(G(x, y))$ with $G \in \mathbb{C}(E_z)$.

Definition 1.2 *The vector space of differential forms of the second kind on E_z modulo the exact differentials is called the algebraic De Rham cohomology on E_z. Notation: $H^1_{\mathrm{DR}}(E_z)$. The index 1 refers to the fact that we work with differential one-forms.*

Theorem 1.3 *The De Rham cohomology of E_z has dimension 2 and is generated by $\frac{dx}{y}$ and $\frac{xdx}{y}$.*

A proof of this theorem will be given in a more general setting in Theorem 1.5.
A consequence of this theorem is the following. Start with the holomorphic form ω_z. Take the partial derivative with respect to z. A small calculation gives $\partial_z(\omega_z) = \frac{x/2}{1-zx}\,\omega_z$

which is a form of the second kind. Similarly $\partial_z^2(\omega_z)$ is a form of the second kind. Since $H^1_{\mathrm{DR}}(E_z)$ has dimension 2 there exist rational functions $p, q, r \in \mathbb{C}(z)$ such that $p\partial_z^2(\omega_z) + q\partial_z(\omega_z) + r\omega_z$ is an exact form. Now integrate over the closed contour. The integral of an exact form over a closed contour is zero. So we obtain

$$p(z)\partial_z^2 \int_\Gamma \frac{dx}{y} + q(z)\partial_z \int_\Gamma \frac{dx}{y} + r(z) \int_\Gamma \frac{dx}{y} = 0.$$

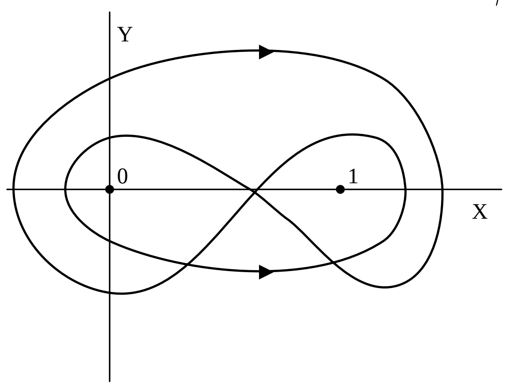
In other words, we obtained a second order differential equation for ${}_2F_1(1/2, 1/2, 1|z)$, which turns out to be the hypergeometric equation.

1.4 Cohomological approach

Recall the Euler integral

$$F(a, b, c|z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \quad (c > b > 0).$$

The restriction $c > b > 0$ is included to ensure convergence of the integral at 0 and 1. We can drop this condition if we take the Pochhammer contour γ given by



as integration path. Notice that the integrand acquires the same value after analytic continuation along γ. We get, after some computation,

$${}_2F_1(a, b, c|z) = -\frac{e^{\pi i c}}{4\pi^2}\Gamma(c)\Gamma(1-b)\Gamma(1+b-c) \int_\gamma t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt.$$

When $a, b, c \in \mathbb{Q}$ we can interpret the integrand as a differential form on an algebraic curve. Let N be the common denominator of a, b, c. Define $A = N(1-b), B = N(1+b-c), C = Na$ and assume $A, B < C > 0$. Consider the algebraic curve $C_z : y^N = x^A(1-x)^B(1-zx)^C$. Our Euler integral can be rewritten as a constant times

$$\int_\gamma \frac{dx}{y},$$

where the Pochhammer contour γ is a now a closed curve on C_z. Let us assume $\gcd(A, N) = \gcd(B, N) = 1, \gcd(C, N) = 1, \gcd(A+B+C, N) = 1$. This comes down to $a, b, c-a, c-b$

having exact denominator N, which is somewhat stronger than just $a, b, c-a, c-b \notin \mathbb{Z}$, the familiar irreducibility condition. Under this stronger condition one can show that the space of differential forms of C_z of the second kind is spanned by forms $x^p(1-x)^q(1-zx)^r dx/y^k$ with $0 < k < N, \gcd(k, N) = 1$ and $p, q, r \in \mathbb{Z}$. It turns out that the dimension of $H^1_{\rm DR}(C_z)$ equals $2\phi(N)$, where ϕ is Euler's ϕ-function. However, this space can be split up into 2-dimensional spaces via the action of the automorphism $\psi : C_z \to C_z$ given by $\psi : (x, y) \mapsto (x, \zeta_N y)$ where $\zeta_N = e^{2\pi i/N}$. This automorphism acts via its pullback on differential forms via $\psi^* : R(x)dx/y^k \to \zeta_N^{-k} R(x)dx/y^k$. Thus we see that $H^1_{\rm DR}(C_z)$ splits up into eigenspaces under the action of ψ^*. The eigenspace corresponding to ζ_N^{-k} corresponds to forms with given k. Denote this by $H^1_{\rm DR}(C_z)^{(k)}$. It will follow from Theorem 1.5 that the dimension of $H^1_{\rm DR}(C_z)^{(k)}$ is 2. In the same way as in our elliptic curve example it now follows that the periods of these forms satisfy a second order linear differential equation with coefficients in $\mathbb{C}(z)$.

In order to compute the dimension of $H^1_{\rm DR}(C_z)^{(k)}$ we adopt a more general point of view. Choose $\rho, \sigma, \tau \in \mathbb{R}$ (or $\mathbb{C}$ if you like) such that $\rho\,\sigma, \tau, \rho+\sigma+\tau \notin \mathbb{Z}$. Let $z \in \mathbb{C}, z \neq 0, 1$. Write $y = x^\rho(1-x)^\sigma(1-zx)^\tau$ and consider differential forms of the shape $R(x)\frac{dx}{y}$ with $R(x) \in \mathbb{C}(x)$ having no poles outside $0, 1, 1/z, \infty$. Such a form is called *exact* if it can be written as $d(S(x)/y)$ with $S(x) \in \mathbb{C}(x)$ having no poles outside $0, 1, 1/z, \infty$. The space of such forms modulo exact ones is called the *twisted De Rham cohomology* with parameters ρ, σ, τ. Notation: $H^1_{\rm twist}(\rho, \sigma, \tau)$.

Theorem 1.5 *With the notation as above, the space $H^1_{\rm twist}(\rho, \sigma, \tau)$ has dimension two and is generated by the forms dx/y and xdx/y.*

Proof Notice that for any $S(x) \in \mathbb{C}(x)$ we have

$$d\left(\frac{S(x)}{y}\right) = \left(S'(x) + S(x)\left(-\frac{\rho}{x} + \frac{\sigma}{1-x} + \frac{z\tau}{1-zx}\right)\right)\frac{dx}{y}. \tag{2}$$

We show that any $R(x)dx/y$ is modulo exact forms equivalent to $(A+Bx)dx/y$ for some $A, B \in \mathbb{C}$.

Suppose that $R(x)$ has a pole of order k in $x = 0$ with principal part $r/x^k + O(1/x^{k-1})$. Here $O(1/x^{k-1})$ denotes a Laurent expansion in $x = 0$ with pole order $\leq k-1$. Notice that

$$d\left(\frac{(1-x)(1-zx)}{x^{k-1}y}\right) = (-(k-1+\rho) + Ax + Bx^2)\frac{dx}{x^k y}$$

for certain $A, B \in \mathbb{C}$. Hence

$$R(x)\frac{dx}{y} + \frac{r}{k-1+\rho} d\left(\frac{1}{x^{k-1}y}\right) = \tilde{R}(x)\frac{dx}{y},$$

where $\tilde{R}(x)$ has pole order $\leq k-1$ in $x = 0$, while the poles at the other finite points do not increase in order. Note that $k-1+\rho \neq 0$ because $\rho \notin \mathbb{Z}$. By induction on k we can

reduce to the case when $R(x)$ has no poles in $x = 0$. Similarly we can reduce the poles at $1, 1/z$ and conclude that we can restrict to $R(x) \in \mathbb{C}[x]$.
Suppose that $R(x)$ is a polynomial of degree k with $k \geq 2$. Write $R(x) = rx^k + O(x^{k-1})$. A straightforward computation shows that

$$d\left(\frac{x(1-x)(1-zx)x^{k-2}}{y}\right) = (z(k+1-\rho-\sigma-\tau)x^k + Ax^{k-1} + Bx^{k-2})\frac{dx}{y},$$

for certain $A, B \in \mathbb{C}$. Hence

$$R(x)\frac{dx}{y} - \frac{r}{z(k+1-\rho-\sigma-\tau)}d\left(\frac{x(1-x)(1-zx)x^{k-2}}{y}\right) = \tilde{R}(x)\frac{dx}{y},$$

where $\tilde{R}(x)$ is a polynomial of degree $\leq k-1$. Hence $R(x)$ can be reduced to a polynomial of the form $A + Bx$.
It remains to show that a form $(A + Bx)dx/y$ cannot be exact unless $A = B = 0$. Suppose there exists a rational function $S(x)$ such that $d(S(x)/y) = (A+Bx)dx/y$. From the calculations above it follows that if $S(x)$ has a pole in $0, 1, 1/z$ then so does $d(S(x)/y)$. Hence $S(x)$ must be a polynomial. From equation (2) it follows that $S(x)$ must have zeros at $0, 1, 1/z$. But then the calculation at $x = \infty$ above shows that if S is non-trivial of degree ≥ 3, the form $d(S(x)/y)$ has a pole of order at least 2 at $x = \infty$. Thus S is trivial and $A = B = 0$.

□

1.6 Exercises

1. Let $\omega = dx/\sqrt{x(1-x)(1-zx)}$. Determine polynomials $p_2(z), p_1(z), p_0(z)$ such that $p_2(z)\partial_z^2\omega + p_1(z)\partial_z\omega + p_0(z)\omega$ is exact.

2. A twisted form $R(x)dx/y$ with $y = x^\rho(1-x)^\sigma(1-zx)^\tau$ is called holomorphic at 0 if it has an expansion $x^{-\rho}(c_kx^k + c_{k+1}x^{k+1} + \cdots)$ such that $c_k \neq 0$ and $k - \rho > -1$. Similarly at $x = 1, 1/z, \infty$. For example at $x = 1$ we replace x by $1 + t$ to determine the local expansion at $t = 0$ (or in terms of $x - 1$ if you like). At ∞ we replace x by $1/t$.

   What are the conditions on ρ, σ, τ such that both dx/y and xdx/y are holomorphic everywhere? How do these conditions read in terms of the hypergeometric parameters a, b, c where $\rho = 1 - b, \sigma = b + 1 - c, \tau = a$.