Exercises Hypergeometric Functions, Sep 21, 2015

1. Give a basis of local solutions around $z = 0$ of the second order equation
$$(\theta + b_1)(\theta + b_2)f - z(\theta + a_1)(\theta + a_2)f = 0$$
where a_1, a_2, b_1, b_2 are fixed parameters and $b_1 \neq b_2 (\mathrm{mod}\ \mathbb{Z})$.

2. Prove that
$${}_2F_1(a, b; c|z) = (1-z)^{c-a-b}\, {}_2F_1(c-a, c-b; c|z)$$
using Riemann schemes.

3. (a) Let f_1, f_2 be independent solutions of a second order linear differential equation. Prove that this equation is given by
$$f'' - \frac{1}{W}\begin{vmatrix} f_1 & f_2 \\ f_1'' & f_2'' \end{vmatrix} f' + \frac{1}{W}\begin{vmatrix} f_1' & f_2' \\ f_1'' & f_2'' \end{vmatrix} f = 0, \quad W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}.$$
The determinant W is called the Wronskian determninant.

   (b) Let f_1, f_2 be solutions of a second order linear differential equation, holomorphic around a point $z_0 \in \mathbb{C}$. Suppose that $f_1 = c_0 + c_1(z - z_0) + \cdots$ and $f_2 = d_1(z - z_0) + \cdots$ with $c_0, d_1 \neq 0$. Prove that z_0 is a regular point of the differential equation.

4. In this exercise we prove the relation
$${}_2F_1(a, b, a + b + 1/2|4t - 4t^2) = {}_2F_1(2a, 2b, a + b + 1/2|t).$$
   (a) What are the local exponents of the hypergeometric equation for ${}_2F_1(a, b, a + b + 1/2|z_)$?

   (b) We substitute $z = 4t - 4t^2$ in this equation and get a new differential equation in t. Without explicitly computing this equation, determine its singular points and the corresponding local exponents.

   (c) Deduce that the new equation is again hypergeometric and prove the above mentioned identity.