Exercises Hypergeometric Functions, Nov 23 2015

1. Let $d \in \mathbb{N}$.

   (a) Prove that for any $a \in \mathbb{R}$ and any $n \in \mathbb{N}$:

   $$d^{dn} \prod_{k=0}^{d-1} (\alpha + k/d)_n = (d\alpha)_{dn}.$$

   (b) Prove that $(dn)! = d^{dn} \prod_{k=1}^{d} (k/d)_n$ for any $n \in \mathbb{N}$.

   (c) Show that the power series

   $$\sum_{k \geq 0} \frac{(30k)!k!}{(15k)!(10k)!(6k)!} z^k$$

   can be expressed in terms of a hypergeometric function of order 8. Show that the latter function is algebraic.

   (d) Show that the coefficients of the above series are all integers (you may use that the number of prime factors p in $n!$ equals $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \cdots$).

2. Let α, β be a pair of multisets (i.e. repetition of elements is allowed) of hypergeometrische parameters of size m and suppose that there exist multisets γ, δ such $\alpha = \cup_{k=0}^{d-1} (\gamma + k/d)$ en $\beta = \cup_{k=0}^{d-1} (\delta + k/d)$ We use the notation $\{\gamma_1, \ldots, \gamma_k\} + 1/d = \{\gamma_1 + 1/d, \ldots, \gamma_k + 1/d\}$.

   (a) Express the m-th order hypergeometric functions $F(\alpha; \beta|z)$ in terms of the m/d-th order hypergeometric function $F(\gamma; \delta|z)$.

   (b) Show that if $F(\gamma; \delta|z)$ is algebraic, then so is $F(\alpha; \beta|z)$.