Exercises Hypergeometric Functions, Sep 28, 2015

We recall the definition of a Schwarzian derivative,

$$S(w,z) = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2.$$

1. Prove that
$$_2F_1(a,b,c|\frac{z}{z-1}) = (1-z)^a\, {}_2F_1(a,c-b,c|z).$$
(Hint: use Riemann schemes)

2. When $c \notin \mathbb{Z}$ a basis of solutions of the hypergeometric equation is given by
$$f(z) = {}_2F_1(a,b,c|z), \quad g(z) = z^{1-c}\, {}_2F_1(a+1-c, b+1-c, 2-c|z).$$
In this exercise we study the solutions when $c \in \mathbb{Z}$.

   (a) Show that for $c = 1$ the non-holomorphic solution near $z = 0$ is given by
$$f(z)\log z + \sum_{k\geq 1} \frac{(a)_k(b)_k}{(k!)^2}\left(\sum_{j=1}^{k-1} \frac{1}{a+j} + \frac{1}{b+j} - \frac{2}{j+1}\right) z^k.$$
(Hint: the difference quotient $\frac{g(z)-f(z)}{c-1}$ is a solution of the hypergeometric equation for any c close to 1. Take the limit as $c \to 1$)

   (b) Find a basis of solutions around $z = 0$ for the hypergeometric equation when $c = 0$.

3. Consider the function
$$f(z) = \int_0^z \frac{dt}{\sqrt{1-t^2}}$$
on the complex upper half plane $\mathcal{H}$. We take $\sqrt{1-t^2}$ positive realvalued on the real segment $(-1,1)$.

   (a) Prove that $f(z)$ maps $\overline{\mathcal{H}}$ (the upper half plane plus real line) one-to-one onto the half strip S in the complex plane given by $|\Re(z)| \leq a, \Im(z) \geq 0$ where $a = \int_0^1 \frac{dt}{\sqrt{1-t^2}}$. In particular, $f(-1) = -a, f(1) = a, f(\infty) = \infty$. (Hint: use the Schwarzian derivative).

   (b) Let $g : S \to \overline{\mathcal{H}}$ be the inverse of f. Show that it can be continued analytically to $\mathbb{C}$ and show that $g(z+4a) = g(z)$ for all $z \in \mathbb{C}$.

4. Consider the function
$$f(z) = \int_\infty^z \frac{dt}{\sqrt{t(1-t^2)}}$$
on the complex upper half plane $\mathcal{H}$. We take $\sqrt{t(1-t^2)}$ positive realvalued on the real segment $(-\infty,-1)$.

(a) Prove that $f(z)$ maps the real line to the boundary of a square S with vertices $0, b, b+bi, bi$ minus 0. Here $b = \int_{\infty}^{-1} \frac{dt}{\sqrt{t(1-t^2)}}$.

(b) (much work) Prove that $f(z)$ maps $\overline{\mathcal{H}}$ one-to-one onto the square S minus 0. (Hint: use the Schwarzian derivative and the local expansion of $f(z)$ near $z = \infty$, or $z = 0$ if you prefer)

(c) Let $g : S \to \overline{\mathcal{H}}$ be te inverse of f. Show that it can be continued meromorphically to $\mathbb{C}$ with second order poles in the points $2mb, 2nbi$ with $m, n \in \mathbb{Z}$.

(d) Show that $g(z + 2b) = g(z + 2bi) = g(z)$ for all $z \in \mathbb{C}$.