Homework Hypergeometric Functions, part II

1. Let $0 \le \lambda, \mu, \nu < 1$. Prove that there exists a curvilinear triangle with vertex angles $\lambda\pi, \mu\pi, \nu\pi$. Show also that it is unique up to a conformal transformation.

2. Suppose a, b, c are hypergeometric parameters such that none of $a, b, a-c, b-c$ is an integer. In this assignment we use the notation $f_{a,b,c} = {}_2F_1(a, b, c|z)$. We wil show that for any choice of integer triples k_i, l_i, m_i with $i = 1, 2$ there exists a $\mathbb{C}(z)$-linear relation between
$$f_{a,b,c},\ f_{a+k_1,b+l_1,c+m_1},\ f_{a+k_2,b+l_2,c+m_2}.$$
These relations are known as Gauss's contiguity relations. We proceed in steps.
   (a) In class we showed that $af_{a+1,b,c} = \theta(f_{a,b,c}) + af_{a,b,c}$. Find rational functions $A, B \in \mathbb{C}(z)$ such that $f_{a+2,b,c} = A\theta(f_{a,b,c}) + Bf_{a,b,c}$.
   (b) Determine a $\mathbb{C}(z)$-linear relation between $f_{a+2,b,c}, f_{a+1,b,c}, f_{a,b,c}$.
   (c) Determine a $\mathbb{C}(z)$-linear relation between $f_{a,b,c}, f_{a,b,c+1}, f_{a,b,c+2}$.
   (d) Prove the existence of Gauss's contiguity relations.

3. Let $L(a, b, c)$ be the hypergeometric differential equation with parameters a, b, c. In this assignment we show
$$a, b, c-a, c-b \notin \mathbb{Z} \iff \text{the monodromy of } L(a, b, c) \text{ acts irreducibly.}$$
By the latter we mean that there is no non-trivial solution f of $L(a, b, c)$ which is an eigenvector for all monodromy transformations. In other words, the solution space does not contain a one-dimensional subspace that is stable under monodromy.

   Let us suppose that there is an f which is eigenvector for all monodromy transformations.

   (i) Prove that there is a rational function $R(z)$ such that $f'/f = R$. Moreover, show that R has poles of order at most 1.
   (ii) Prove that there exist exponents μ_0, μ_1 such that $f = z^{\mu_0}(z-1)^{\mu_1}P(z)$, where P is a nontrivial polynomial. (Hint: solve the equation $f' = Rf$ and use that f also satisfies $L(a, b, c)$)
   (iii) Conclude that at least one of $a, b, c-a, c-b$ is an integer.

   Suppose conversely that at least one of $a, b, c-a, c-b$ is an integer. To fix ideas assume that $a \in \mathbb{Z}$, the other cases being similar.

   (iv) Suppose $a \in \mathbb{Z}_{\le 0}$. Show that the hypergeometric function with parameters a, b, c is fixed under monodromy, i.e. the monodromy group acts reducibly.
   (v) Suppose that $a \in \mathbb{Z}_{>0}$. Show that $L(a, b, c)$ has a solution which is an eigenvector for all monodromy substitutions.