Homework Hypergeometric Functions, part IV

1. Let $0 < \lambda, \mu < 1$ be given. Consider the curvilear triangle Δ with vertices $0, 1, e^{2\pi i\lambda}$ and angles $2\lambda\pi, \mu\pi, \mu\pi$ respectively. So there are two edges which are line segments, between 0 and 1, and between 0 and $e^{2\pi i\lambda}$. Take the angular bisector in 0 and use it to cut Δ into two pieces, which are triangles with angles $\lambda\pi, \mu\pi$ and $\pi/2$. Denote the piece containg the vertices $0, 1$ by $\tilde{\Delta}$.

   (i) Sketch a picture of the triangles and compute the vertex P of the triangle $\tilde{\Delta}$ with angle $\pi/2$.

   Let η be the Schwarz map which maps the upper half plane $\mathcal{H}$ to Δ such that $\eta(0) = 0, \eta(1) = 1, \eta(\infty) = e^{2\pi i\lambda}$. Let $\tilde{\eta}$ be the Schwarz map which map the upper half plane $\mathcal{H}$ to $\tilde{\Delta}$ such that $\tilde{\eta}(0) = 0, \tilde{\eta}(1) = 1, \tilde{\eta}(\infty) = P$.

   (ii) Consider $\tilde{\eta}^{-1} : \tilde{\Delta} \to \mathcal{H}$ and extend it to an an analytic function on Δ. Show that $\tilde{\eta}^{-1} \circ \eta$ extends to a rational function $R(z)$ on $\mathbb{C} \cup \infty$. Furthermore, prove that $R(z) = p\left(\frac{z}{z-q}\right)^2$ for some constants p, q. Determine p, q.

   We know that $\eta, \tilde{\eta}$ can be given by quotients of hypergeometric functions of the form
   $$\eta(z) = r\,\frac{z^{1-c}\,{}_2F_1(a+1-c, b+1-c, 2-c|z)}{{}_2F_1(a, b, c|z)},$$
   $$\eta(z) = \tilde{r}\,\frac{z^{1-\tilde{c}}\,{}_2F_1(\tilde{a}+1-\tilde{c}, \tilde{b}+1-\tilde{c}, 2-\tilde{c}|z)}{{}_2F_1(\tilde{a}, \tilde{b}, \tilde{c}|z)},$$
   where $r, \tilde{r}$ are some non-zero normalization constants.

   (iii) Determine $a, b, c, \tilde{a}, \tilde{b}, \tilde{c}$ in terms of λ, μ.

   From item (ii) we know that $\eta(z) = \tilde{\eta}(R(z))$, so there is a relation between the Schwarz quotients with parameters a, b, c and parameters $\tilde{a}, \tilde{b}, \tilde{c}$. We know like to establish a relation between the hypergeometric functions themselves.

   (iv) Take the derivative of the relation $\eta(z) = \tilde{\eta}(R(z))$ and deduce a relation of the form
   $${}_2F_1(\tilde{a}, \tilde{b}, \tilde{c}|R(z)) = S(z)^\rho\,{}_2F_1(a, b, c|z)$$
   where $S(z)$ is a rational function and ρ a constant. Determine $S(z)$ and ρ. (Hint: use that the derivative of f/g is $(f'g - fg')/f^2$ and note we have a Wronskian determinant in the numerator).