Homework Hypergeometric Functions, part VI

1. A twisted form $R(x)dx/y$ with $y = x^\rho(1-x)^\sigma(1-zx)^\tau$ with $\rho,\sigma,\tau \in \mathbb{R}$ and $\rho,\sigma,\tau,\rho+\sigma+\tau \notin \mathbb{Z}$ is called holomorphic at 0 if it has an expansion $x^{-\rho}(c_k x^k + c_{k+1}x^{k+1} + \cdots)dx$ such that $c_k \neq 0$ and $k-\rho > -1$. Similarly at $x = 1, 1/z, \infty$. For example at $x = 1$ we replace x by $1+t$ to determine the local expansion at $t = 0$ (or in terms of $x-1$ if you like). At ∞ we replace x by $1/t$.

   (a) What are the conditions on ρ,σ,τ such that

      i. both dx/y and xdx/y are holomorphic everywhere

      ii. precisely one of dx/y and xdx/y is holomorphic everywhere

      iii. none of $dx/y, xdx/y$ is holomorphic everywhere

   (b) How do the above three conditions read in terms of the hypergeometric parameters a,b,c where $\rho = 1-b, \sigma = b+1-c, \tau = a$?

   (c) Determine in each of these cases the signature of the monodromy of the corresponding Euler integrals.

2. We consider a third order hypergeometric equation with parameters $\alpha_1, \alpha_2, 1/2$ and $\beta_1, \beta_2, 1$ with $\alpha_i, \beta_j \in \mathbb{R}$. We assume that $\alpha_1+\alpha_2, \beta_1+\beta_2 \in \mathbb{Z}$ and that the monodromy acts irreducibly.

   (i) Show that the monodromy can be defined over $\mathbb{R}$ (i.e. there exists a basis such that the monodromy matrices have entries in $\mathbb{R}$).

   (ii) Choose a basis $y_1(z), y_2(z), y_3(z)$ in the space of solutions and consider the group of monodromy matrices G with respect to that basis. Show that there exists a symmetric 3×3-matrix Q such that $g^tQg = Q$ for all $g \in G$.

   (iii) Show that there exists a quadratic form $F(x_1,x_2,x_3) = \sum_{1\le i\le j\le 3} f_{ij}x_ix_j$ with $f_{ij} \in \mathbb{C}$, not all zero, such that $F(y_1(z), y_2(z), y_3(z))$ is a rational function $R(z)$ in z. Moreover, show that $R(z)$ has a pole of order at most $2\max(\beta_1-1, \beta_2-1)$ at $z = 0$, a pole of order at most $2(\alpha_1+\alpha_2+1/2-\beta_1-\beta_2)$ at $z = 1$ and no other poles in $\mathbb{C}$. Show that at ∞ we have $R(z) = c/z^k + O(1/z^{k+1})$ with $k \ge 2\min(\alpha_1, \alpha_2, 1/2)$.

   (iv) Assume that $\alpha_1+\alpha_2 = 1$ and $\beta_1+\beta_2 = 2$. Formulate a necessary condition on α_i, β_j to ensure that $R(z) \equiv 0$.

   We now restrict to the hypergeometric equation with parameters $\alpha_i = 1/2, \beta_i = 1$ for $i = 1,2,3$. There exists a basis of solutions of the form

$$\begin{aligned} y_1 &= f_1 \\ y_2 &= f_1 \log z + f_2 \\ y_3 &= f_1(\log z)^2/2 + f_2 \log z + f_3 \end{aligned}$$

where f_1 is the hypergeometric function and f_2, f_3 are power series expansions in z with $f_2(0) = f_3(0) = 0$.

(v) Show that $y_2^2 - 2y_1y_3 \equiv 0$.

(vi) Show that the Yukawa coupling $y_1^3 W(y_1, y_2, y_3)/W(y_1, y_2)^3$ is a constant. Here W denotes the Wronskian determinant.