Gauss' hypergeometric function

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Abstract

We give a basic introduction to the properties of Gauss' hypergeometric functions, with an emphasis on the determination of the monodromy group of the Gaussian hyperegeometric equation. Initially this document started as an informal introduction to Gauss' hypergeometric functions for those who want to have a quick idea of some main facts on hypergeometric functions. It is the startig of a book I intend to write on 1-variable hypergeometric functions. As time progressed this informal note attracted increasing attention. Therefore I would like to add a point of WARNING here: right now the manuscript needs to be double-checked on possible errors again. At the moment I do not want to consider it as a solid reference. With this provision in mind you are welcome to read it (and let me know if you find errors)

1 Definition, first properties

Let $a, b, c \in \mathbf{R}$ and $c \notin \mathbf{Z}_{\leq 0}$. Define *Gauss' hypergeometric function* by

$$F(a,b,c|z) = \sum \frac{(a)_n (b)_n}{(c)_n n!} z^n. \tag{1}$$

The *Pochhammer symbol* $(x)_n$ is defined by $(x)_0 = 1$ and $(x)_n = x(x+1)\cdots(x+n-1)$. The radius of convergence of (1) is 1 unless a or b is a non-positive integer, in which cases we have a polynomial.

Examples.

$$\begin{aligned}
(1-z)^{-a} &= F(a,1,1|z) \\
\log\frac{1+z}{1-z} &= 2zF(1/2,1,3/2|z^2) \\
\arcsin z &= zF(1/2,1/2,3/2|z^2) \\
K(z) &= \frac{\pi}{2}F(1/2,1/2,1,z^2) \\
P_n(z) &= 2^n F(-n,n+1,1|(1+z)/2) \\
T_n(z) &= (-1)^n F(-n,n,1/2|(1+z)/2)
\end{aligned}$$

Here $K(z)$ is the Jacobi's elliptic integral of the first kind given by

$$K(z) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-z^2x^2)}}.$$

The polynomials P_n, T_n given by $P_n = (1/n!)(d/dz)^n(1-z^2)^n$ and $T_n(\cos z) = \cos(nz)$ are known as the *Legendre* and *Chebyshev* polynomials respectively. They are examples of orthogonal polynomials.

One easily verifies that (1) satisfies the linear differential equation

$$z(D+a)(D+b)F = D(D+c-1)F, \qquad D = z\frac{d}{dz}.$$

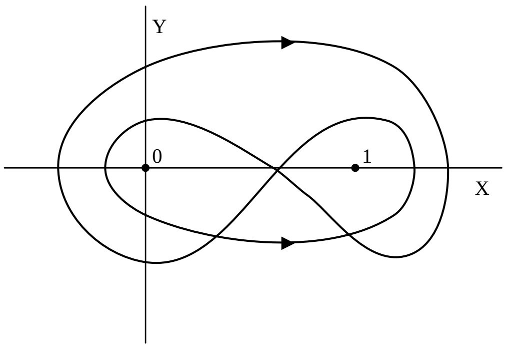
Written more explicitly,

$$z(z-1)F'' + ((a+b+1)z - c)F' + abF = 0. \tag{2}$$

There exist various ways to study the analytic continuation of (1), via *Euler integrals*, *Kummer's solutions* and *Riemann's* approach. The latter will be discussed in later sections. The Euler integral reads

$$F(a,b,c|z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt \quad (c > b > 0)$$

and allows choices of z with $|z| > 1$. The restriction $c > b > 0$ is included to ensure convergence of the integral at 0 and 1. We can drop this condition if we take the Pochhammer contour γ given by



as integration path. Notice that the integrand acquires the same value after analytic continuation along γ.

It is a straightforward exercise to show that for any $b, c-b \notin \mathbf{Z}$ we have

$$F(a,b,c|z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \frac{1}{(1-e^{2\pi i b})(1-e^{2\pi i(c-b)})} \int_\gamma t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt$$

Kummer gave the following 24 solutions to (2)

$$\begin{aligned}
& F(a,b,c|z) \\
= \;& (1-z)^{c-a-b}F(c-a,c-b,c|z) \\
= \;& (1-z)^{-a}F(a,c-b,c|z/(z-1)) \\
= \;& (1-z)^{-b}F(a-c,b,c|z/(z-1))
\end{aligned}$$

$$\begin{aligned}
& z^{1-c}F(a-c+1,b-c+1,2-c|z) \\
= \;& z^{1-c}(1-z)^{c-a-b}F(1-a,1-b,2-c|z) \\
= \;& z^{1-c}(1-z)^{c-a-1}F(a-c+1,1-b,2-c|z/(z-1)) \\
= \;& z^{1-c}(1-z)^{c-b-1}F(1-a,b-c+1,2-c|z/(z-1))
\end{aligned}$$

$$
\begin{aligned}
& F(a,b,a+b-c+1|1-z) \\
= \; & x^{1-c}F(a-c+1,b-c+1,a+b-c+1|1-z) \\
= \; & z^{-a}F(a,a-c+1,a+b-c+1|1-1/z) \\
= \; & z^{-b}F(b-c+1,b,a+b-c+1|1-1/z)
\end{aligned}
$$

$$
\begin{aligned}
& (1-z)^{c-a-b}F(c-a,c-b,c-a-b+1|1-z) \\
= \; & (1-z)^{c-a-b}z^{1-c}F(1-a,1-b,c-a-b+1|1-z) \\
= \; & (1-z)^{c-a-b}z^{a-c}F(1-a,c-a,c-a-b+1|1-1/z) \\
= \; & (1-z)^{c-a-b}z^{b-c}F(c-b,1-b,c-a-b+1|1-1/z)
\end{aligned}
$$

$$
\begin{aligned}
& z^{-a}F(a,a-c+1,a-b+1|1/z) \\
= \; & z^{-a}(1-1/z)^{c-a-b}F(1-b,c-b,a-b+1|1/z) \\
= \; & z^{-a}(1-1/z)^{c-a-1}F(a-c+1,1-b,2-c|1/(1-z)) \\
= \; & z^{-a}(1-1/z)^{-a}F(a,c-b,a-b+1|1/(1-z))
\end{aligned}
$$

$$
\begin{aligned}
& z^{-b}F(b,b-c+1,b-a+1|1/z) \\
= \; & z^{-b}(1-1/z)^{c-a-b}F(1-a,c-a,b-a+1|1/z) \\
= \; & z^{-b}(1-1/z)^{c-b-1}F(b-c+1,1-a,2-c|1/(1-z)) \\
= \; & z^{-b}(1-1/z)^{-b}F(b,c-a,b-a+1|1/(1-z))
\end{aligned}
$$

Strictly speaking, the above six 4-tuples of functions are only distinct when $c, c-a-b, a-b \notin \mathbf{Z}$. If one of these numbers is an integer we find that there are other solutions containing logarithms. For example, when $c=1$ we find that z^{1-c} becomes $\log z$ and a second solution near $z=0$ reads

$$
(\log z)F(a,b,1|z) + \sum_{n=1}^{\infty} \frac{(a)_n(b_n)}{(n!)^2} z^n \left[\sum_{k=1}^{n} \left(\frac{1}{a+k-1} + \frac{1}{b+k-1} - \frac{2}{k} \right) \right].
$$

Notice that this solution can be obtained by taking the difference of solutions $z^{1-c}F(a-c+1,b-c+1,2-c|z) - F(a,b,c|z)$, divide it by $c-1$ and take the limit as $c \to 1$.

Later it will turn out that Riemann's approach to hypergeometric functions gives a remarkably transparent insight into these formulas as well as the quadratic transformations of Kummer and Goursat.

Examples of such transformations are

$$
F(a,b,a+b+1/2|4z-4z^2) = F(2a,2b,a+b+1/2|z)
$$

and

$$
F(a,b,a+b+1/2|z^2/(4z-4)) = (1-z)^a F(2a,a+b,2a+2b|z).
$$

Finally we mention the 6 *contiguous* functions

$$
F(a\pm 1,b,c|z), \qquad F(a,b\pm 1,c|z), \qquad F(a,b,c\pm 1|z).
$$

Gauss found that $F(a,b,c|z)$ and any two contiguous functions satisfy a linear relation with coefficients which are linear polynomials in z or constants, for example,

$$
(c-a)F(a-1,b,c|z) + (2a-c-az+bz)F(a,b,c|z) + a(z-1)F(a+1,b,c|z) = 0.
$$

Notice also that $F'(a,b,c|z) = (ab/c)F(a+1,b+1,c+1|z)$. These observations are part of the following theorem.

Theorem 1.1 *Suppose $a, b \not\equiv 0, c(\text{mod } \mathbf{Z})$ and $c \notin \mathbf{Z}$. Then any function $F(a+k, b+l, c+m|z)$ with $k, l, m \in \mathbf{Z}$ equals a linear combination of F, F' with rational functions as coefficients.*

Proof. One easily verifies that

$$\begin{aligned}
F(a+1, b, c|z) &= \frac{1}{a}(z\frac{d}{dz} + a)F(a, b, c|z) \\
F(a-1, b, c|z) &= \frac{1}{c-a}(z(1-z)\frac{d}{dz} - bz + c - a)F(a, b, c|z)
\end{aligned}$$

and similarly for $F(a, b+1, c|z), F(a, b-1, c|z)$. Furthermore,

$$\begin{aligned}
F(a, b, c+1|z) &= \frac{c}{(c-a)(c-b)}(z(1-z)\frac{d}{dz} + c - a - b)F(a, b, c|z) \\
F(a, b, c-1|z) &= \frac{1}{c-1}(z\frac{d}{dz} + c - 1)F(a, b, c|z)
\end{aligned}$$

Hence there exists a linear differential operator $\mathcal{L}_{k,l,m} \in \mathbf{C}(z)[\frac{d}{dz}]$ such that $F(a+k, b+l, c+m|z) = \mathcal{L}_{k,l,m}F(a, b, c|z)$. Since F satifies a second order linear differential equation, $\mathcal{L}_{k,l,m}F$ can be written as a $\mathbf{C}(z)$-linear combination of F and F'. $\square$

In general we shall call any function $F(a+k, b+l, c+m|z)$ with $k, l, m \in \mathbf{Z}$ contiguous with $F(a, b, c|z)$. Thus we see that, under the assumptions of Theorem 1.1, any three contiguous functions satisfy a $\mathbf{C}(z)$-linear relation.

For many more identities and formulas we refer to [AS] and [E].

2 Ordinary linear differential equations, local theory

Consider the linear differential equation of order n,

$$y^{(n)} + p_1(z)y^{(n-1)} + \cdots + p_{n-1}(z)y' + p_n(z)y = 0, \tag{3}$$

where the p_i are analytic in a neighbourhood of $z = 0$, except for a possible pole at 0. In this section we recall, without proof, a number of facts from the local theory of ordinary linear differential equations. Most of it can be found in standard text books such as Poole, Ince, Hille.

Lemma 2.1 (Wronski) *Let $f_1, \ldots, f_m$ be meromorphic functions on some open subset $G \subset \mathbf{C}$. There exists a $\mathbf{C}$-linear relation between these function if and only if $W(f_1, \ldots, f_m) = 0$, where*

$$W(f_1, \ldots, f_m) = \begin{vmatrix} f_1 & \ldots & f_m \\ f_1' & \ldots & f_m' \\ \vdots & & \vdots \\ f_1^{(m-1)} & \ldots & f_m^{(m-1)} \end{vmatrix}$$

is the Wronskian determinant of $f_1, \ldots, f_m$.

If $z = 0$ is not a pole of any p_i it is called a *regular point* of (3), otherwise it is called a *singular point* of (3). The point $z = 0$ is called a *regular singularity* if p_i has a pole of order at most i for $i = 1, \ldots, n$.

Theorem 2.2 (Cauchy) *Suppose* 0 *is a regular point of* (3). *Then the vector space of solutions of* (3) *is spanned by* n $\mathbf{C}$*-linear independent Taylor series solutions* $f_1, \ldots, f_n$ *in* z *with positive radius of convergence.*

Moreover, the f_i *can be chosen such that* $f_i(z)/z^{i-1}$ *has a non-zero limit as* $z \to 0$ *for* $i = 1, 2, \ldots, n$.

Finally, the Wronskian determinant $W(f_1, \ldots, f_n)$ *satisfies the equation* $W' = -p_1(z)W$.

As an important remark we note that it may happen that there is a basis of holomorphic solutions near $z = 0$ but 0 may still be a singular point. In that case we call 0 an *apparent singularity*. An example is given by the differential equation $(D-1)(D-3)y = 0$ which obviously has the holomorphic solutions z, z^3. However, working $(D-1)(D-3)y = 0$ out, we find $y'' - \frac{3}{z}y' + \frac{3}{z^2}y = 0$, hence $z = 0$ is a singularity. However, we do have the following theorem which we shall repeatedly apply.

Theorem 2.3 *Suppose there exists a basis of power series solutions* $f_1, \ldots, f_n$ *such that* f_i/z^{i-1} *has a non-zero limit as* $z \to 0$ *for* $i = 1, \ldots, n$. *Then* $z = 0$ *is a regular point.*

Suppose that $z = 0$ is regular or a regular singularity. We can rewrite (3) by multiplication with z^n and using the rule $z^r(d/dz)^r = D(D-1)\cdots(D-r+1)$ where $D = z\frac{d}{dz}$. We obtain

$$D^n y + q_1(z)D^{n-1}y + \cdots + q_{n-1}(z)Dy + q_n(z)y = 0. \tag{4}$$

The condition of regular singularity sees to it that the functions $q_i(z)$ are holomorphic near $z = 0$. The *indicial equation* of (3) at $z = 0$ is defined as

$$X^n + q_1(0)X^{n-1} + \cdots + q_{n-1}(0)X + q_n(0) = 0.$$

Suppose we introduce a local parameter t at 0 given by $z = c_1 t + c_2 t^2 + c_3 t^3 + \cdots$ with $c_1 \neq 0$. The differential equation can be rewritten in the new variable t. We obtain, writing $D_t = t\frac{d}{dt}$,

$$D_t^n y + \tilde{q}_1(t)D_t^{n-1}y + \cdots + \tilde{q}_{n-1}(t)D_t y + \tilde{q}_n(t)y = 0.$$

with new functions $\tilde{q}_i(t)$ holomorphic at $t = 0$.

One can show that $\tilde{q}_i(0) = q_i(0)$ for $i = 1, \ldots, n$, hence the indicial equation does not depend on the choice of local parameter at 0. The roots of the indicial equation are called the *local exponents* at $z = 0$.

Remark 2.4 *Notice that if we replace* y *by* $z^\mu w$, *the differential equation for* w *reads*

$$(D+\mu)^n w + q_1(z)(D+\mu)^{n-1}w + \cdots + q_{n-1}(z)(D+\mu)w + q_n(z)w = 0.$$

In particular, the local exponents have all decreased by μ.

Remark 2.5 *Show that the local exponents at a regular point read* $0, 1, \ldots, n-1$. *Theorem 2.3 can be rephrased by saying that if there is a basis of holomorphic solutions around* $z = 0$, *and if the local exponents are* $0, 1, \ldots, n-1$, *then* $z = 0$ *is a regular point of* (3).

In the following theorem we shall consider expressions of the form z^A where A is a constant $n \times n$ matrix. This is short hand for

$$z^A = \exp(A \log z) = \sum_{k \geq 0} \frac{1}{k!} A^k (\log z)^k.$$

In particular z^A is an $n\times n$ matrix of multivalued functions around $z=0$. Examples are,

$$z^{\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2\end{pmatrix}} = \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2}\end{pmatrix}, \qquad z^{\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}} = \begin{pmatrix} 1 & \log z \\ 0 & 1\end{pmatrix}.$$

Theorem 2.6 (Fuchs) *Let $z=0$ be a regular singularity of (3). Let ρ be a local exponent at 0 such that none of the numbers $\rho+1,\rho+2,\ldots$ is a local exponent. Then there exists a holomorphic power series $g(z)$ with non-zero constant term such that $z^\rho g(z)$ is a solution of (3).*

Let $\rho_1,\ldots,\rho_n$ be the set of local exponents ordered in such a way that exponents which differ by an integer occur in decreasing order. Then there exists a nilpotent $n\times n$ matrix N, and functions $g_1,\ldots,g_n$, analytic near $z=0$ with $g_i(0)\ne 0$, such that $(z^{\rho_1}g_1,\ldots,z^{\rho_n}g_n)z^N$ is a basis of solutions of (3). Moreover, $N_{ij}\ne 0$ implies $i\ne j$ and $\rho_i-\rho_j\in\mathbf{Z}_{\ge 0}$.

Example 2.7 *Consider the linear differential equation*

$$(z^3+11z^2-z)y''+(3z^2+22z-1)y'+(z+3)y=0$$

. The local exponents at $z=0$ are $0,0$ and a basis for the local solutions is given by

$$f_1(z)=1+3z+19z^2+147z^3+1251z^4+\cdots$$

$$f_2(z)=f_1(z)\log z+5z+\frac{75}{2}z^2+\frac{1855}{6}z^3+\frac{10875}{4}z^4+\cdots.$$

3 Fuchsian linear differential equations

Consider the linear differential equation

$$y^{(n)}+p_1(z)y^{(n-1)}+\cdots+p_{n-1}(z)y'+p_n(z)y=0, \qquad p_i(z)\in\mathbf{C}(z) \tag{5}$$

To study this differential equation near any point $P\in\mathbf{P}^1$ we choose a local parameter $t\in\mathbf{C}(z)$ at this point (usually $t=z-P$ if $P\in\mathbf{C}$ and $t=1/z$ if $P=\infty$), and rewrite the equation with respect to the new variable t. We call the point P a regular point or a regular singularity if this is so for the equation in t at $t=0$. It is not difficult to verify that a point $P\in\mathbf{C}$ is regular if and only if the p_i have no pole at P. It is a regular point or a regular singularity if and only if $\lim_{z\to P}(z-P)^ip_i(z)$ exists for $i=1,\ldots,n$. The point ∞ is regular or a regular singularity if and only if $\lim_{z\to\infty}z^ip_i(z)$ exists for $i=1,\ldots,n$.

Let $P\in\mathbf{P}^1$ be any point which is regular or a regular singularity. Let t be a local parameter around this point and rewrite the equation with respect to the variable t. The corresponding indicial equation will be called the indicial equation of (5) at P. The roots of the indicial equation at P are called the *local exponents* of (5) at P.

As a shortcut to compute indicial equations we use the following lemma.

Lemma 3.1 *Let $P\in\mathbf{C}$ be a regular point or regular singularity of (5). Let*

$$a_i=\lim_{z\to P}(z-P)^ip_i(z)$$

for $i=1,\ldots,n$. The indicial equation at P is given by

$$X(X-1)\cdots(X-n+1)+a_1X(X-1)\cdots(X-n+2)+\cdots+a_{n-1}X+a_n=0.$$

When ∞ is regular or a regular singularity, let $a_i=\lim_{z\to\infty}z^ip_i(z)$ for $i=1,\ldots,n$. The indicial equation at ∞ is given by

$$X(X+1)\cdots(X+n-1)-a_1X(X+1)\cdots(X+n-2)+\cdots$$
$$+(-1)^{n-1}a_{n-1}X+(-1)^na_n=0.$$

Proof. Exercise

From Cauchy's theorem of the previous section follows automatically

Theorem 3.2 (Cauchy) *Suppose $P \in \mathbf{C}$ is a regular point of* (5). *Then there exist n $\mathbf{C}$-linear independent Taylor series solutions $f_1, \ldots, f_n$ in $z - P$ with positive radius of convergence. Moreover, any Taylor series solution of* (5) *is a $\mathbf{C}$-linear combination of $f_1, \ldots, f_n$.*

Corollary 3.3 *Any analytic solution of* (5) *near a regular point can be continued analytically along any path in $\mathbf{C}$ not meeting any singularity.*

Let S be the set of singularities of (5) and let $z_0 \in \mathbf{P}^1 \setminus S$. Let $f_1, \ldots, f_n$ be an independent set of analytic solutions around z_0. Let $\gamma \in \pi_1(\mathbf{P}^1 \setminus S, z_0)$. After analytic continuation of $f_1, \ldots, f_n$ along γ we obtain continuations $\tilde{f}_1, \ldots, \tilde{f}_n$, which are again solutions of our equation. Hence there exists a square matrix $M(\gamma) \in GL(n, \mathbf{C})$ such that $(\tilde{f}_1, \ldots, \tilde{f}_n)^t = M(\gamma)(f_1, \ldots, f_n)^t$. The map $\rho : \pi_1(\mathbf{P}^1 \setminus S) \to GL(n, \mathbf{C})$ given by $\rho : \gamma \mapsto M(\gamma)$ is a group homomorphism and its image is called the *monodromy group* of (3). Notice also that after analytic continuation along γ we have $W(f_1, \ldots, f_n) \to \det(M(\gamma))W(f_1, \ldots, f_n)$.

Definition 3.4 *The equation* (5) *is called Fuchsian if all points on $\mathbf{P}^1$ are regular or a regular singularity.*

Theorem 3.5 (Fuchs' relation) *Suppose* (5) *is a Fuchsian equation. Let $\rho_1(P), \ldots, \rho_n(P)$ the set of local exponents at any $P \in \mathbf{P}^1$. Then,*

$$\sum_{P \in \mathbf{P}^1} (\rho_1(P) + \cdots + \rho_n(P) - \binom{n}{2}) = -2\binom{n}{2}$$

Since the local exponents at a regular point are always $0, 1, \ldots, n-1$ the terms in the summation are zero when P is a regular point. So, in fact, the summation in this theorem is a finite sum.

Proof. From the explicit shape of the indicial equations, given in the Lemma above, we infer that for $P \in \mathbf{C}$,

$$\rho_1(P) + \cdots + \rho_n(P) = \binom{n}{2} - \mathrm{res}_P(p_1(z)dz)$$

and

$$\rho_1(\infty) + \cdots + \rho_n(\infty) = -\binom{n}{2} - \mathrm{res}_\infty(p_1(z)dz).$$

Substract $\binom{n}{2}$ on both sides and add over all $P \in \mathbf{P}^1$. Using the fact that $\sum_{P \in \mathbf{P}^1} \mathrm{res}_P(p_1(z)dz) = 0$ yields our theorem. $\square$

The hypergeometric equation (1) is an example of a Fuchsian equation. Its singularities are $0, 1, \infty$ and the local exponents are given by the following scheme (Riemann scheme),

0	1	∞
0	0	a
$1-c$	$c-a-b$	b

It also turns out that Fuchsian equations with three singular points can characterised easily.

Theorem 3.6 *Let $A, B, C \in \mathbf{P}^1$ be distinct points. Let $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ be any complex numbers which satisfy $\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$. Then there exists a unique Fuchsian equation of order two with rational function coefficients, no singularities other than A, B, C and having local exponents given by the following Riemann scheme,*

$$\begin{array}{ccc} A & B & C \\ \hline \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{array}$$

Proof. Exercise

The solutions of this Fuchsian equation are related to the hypergeometric function as follows. Via a Möbius transformation we can map A, B, C to any three distinct points of $\mathbf{P}^1$. Let us take the mapping $A, B, C \to 0, 1, \infty$. So we have to deal with the Fuchsian equation having Riemann scheme

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{array}$$

If we multiply the solutions of the latter equation by z^μ we obtain a set of functions that satisfy the Fuchsian equation with Riemann scheme

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline \alpha + \mu & \beta & \gamma - \mu \\ \alpha' + \mu & \beta' & \gamma' - \mu \end{array}$$

A fortiori, after multiplication of the solutions with $z^{-\alpha'}(1-z)^{-\beta'}$ we obtain a Fuchsian equation with a scheme of the form

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline \alpha'' & \beta'' & \gamma'' \\ 0 & 0 & 1 - \alpha'' - \beta'' - \gamma'' \end{array}$$

This scheme corresponds to a hypergeometric equation with suitable parameters. The 24 solutions of Kummer can now be characterised very easily. Suppose we apply the above procedure to the hypergeometric equation itself. There exist 6 ways to map the set $\{0, 1, \infty\}$ to itself. Having chosen such a map, there exist four ways to multiply by $z^{-\lambda}(1-z)^{-\mu}$ since there are four choices for the pair (λ,μ) of local exponents at 0 and 1. Choose the hypergeometric function (with suitable parameters) as a solution of the final equation, then we obtain the $4 \times 6 = 24$ solutions given by Kummer.

It is also very simple to prove for example the quadratic relation

$$F(a, b, a+b+1/2|t^2/(4t-4)) = (1-t)^a F(2a, a+b, 2a+2b|t).$$

Substitute $z = t^2/(4t-4)$ in the hypergeometric equation with parameters $a, b, a+b+1/2$. we obtain a new Fuchsian equation. The map $t \to z = t^2/(4t-4)$ ramifies above $0, 1$ in $t = 0, 2$ respectively. Above $z = 1$ we have the point $t = 2$, above $z = 0$ the point $t = 0$ and above $z = \infty$ the two points $t = 1, \infty$. Notice that our equation has local exponents $0, 1/2$ in $z = 1$. Hence the new equation has local exponents $0, 1$ in $t = 2$, with regular solutions, and $t = 2$ turns out to be a regular point. At $t = 0$ we get the local exponents $0, 2(1/2 - a - b)$ and in $t = 1, \infty$, the points above $z = \infty$, we have the local exponents a, b and a, b. Thus our equation in t has again three singular points and Riemann scheme

$$\begin{array}{ccc} 0 & 1 & \infty \\ \hline 0 & a & a \\ 1 - 2a - 2b & b & b \end{array}$$

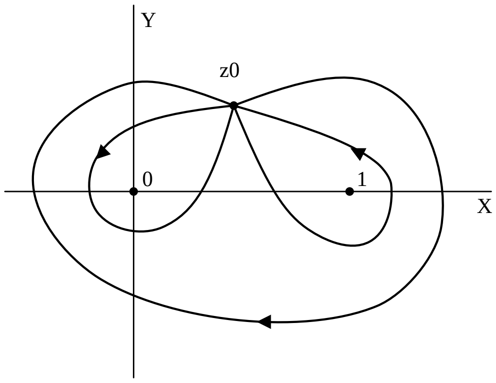
By the method sketched above, one easily sees that $(1-t)^a F(2a, a+b, 2a+2b|t)$ is a solution of this equation. Moreover, this is the unique (up to a constant factor) solution holomorphic near $t=0$. At the same time $F(a, b, a+b+1/2|t^2/(4t-4))$ is a solution, and by the uniquess equality follows.

Example 3.7 *In a similar way one can show the equality*

$$F(a, b, a+b+1/2|4z-4z^2) = F(2a, 2b, a+b+1/2|z).$$

3.8 Monodromy of the hypergeometric function

Let us now turn to the monodromy of the hypergeometric equation. Consider the three loops g_0, g_1, g_∞ which satisfy the relation $g_0 g_1 g_\infty = 1$.



We denote the corresponding monodromy matrices by M_0, M_1, M_∞. They also satisfy $M_0 M_1 M_\infty = 1$ and M_0, M_∞ generate the monodromy group. Since the local exponents at $0, 1, \infty$ are $0, 1-c$, $0, c-a-b$ and a, b respectively, the eigenvalues of the matrices M_0, M_1 and M_∞ are $1, \exp(2\pi i(1-c))$, $1, \exp(2\pi i(c-a-b))$ and $\exp(2\pi ia), \exp(2\pi ib)$ respectively. The monodromy group can be considered as being generated by M_0, M_∞ and we know that $M_\infty M_0 = M_1^{-1}$ has eigenvalue 1. This scant information already suffices to draw some important conclusions.

Lemma 3.9 *Let $A, B \in GL(2, \mathbf{C})$. Suppose that AB^{-1} has eigenvalue 1. Then there exists a common eigenvector of A, B if and only if A, B have a common eigenvalue.*

Proof. Notice that $\ker(A-B)$ has dimension at least 1. If the dimension were 2 we would have $A = B$ and our lemma would be trivial. So we can assume $\dim(\ker(A-B)) = 1$. In this proof we let $v \in \ker(A-B)$, $v \neq 0$.

Suppose there exists a common eigenvector, w say, of A, B with eigenvalues λ_A, λ_B. If these eigenvalues are equal, we are done. Suppose they are not equal. Then w, v span $\mathbf{C}^2$. Choose α, β such that $Av = \alpha v + \beta w$. Since $Av = Bv$ we also have $Bv = \alpha v + \beta w$. Hence with respect to the basis v, w the matrices of A, B read

$$\begin{pmatrix} \alpha & \beta \\ 0 & \lambda_A \end{pmatrix} \qquad \begin{pmatrix} \alpha & \beta \\ 0 & \lambda_B \end{pmatrix}$$

Hence they have the common eigenvalue α.

Suppose A, B have a common eigenvalue λ. If v is an eigenvectore of A we are done, since $Av = Bv$ implies that it is also an eigenvector of B. So suppose v is not an eigenvector of A. Consider the vector $w = (A - \lambda)v$. Since $A - \lambda$ has non-trivial kernel we have $< w >_{\mathbf{C}} = (A - \lambda)\mathbf{C}^2$. In particular, $(A - \lambda)w$ is a scalar multiple of w, i.e. w is an eigenvector of A. We also have $w = (B - \lambda)v$ and a similar argument shows that w is an eigenvector of B. Hence A, B have a common eigenvector. $\square$

Corollary 3.10 *The monodromy group of* (2) *acts reducibly on the space of solutions if and only if at least one of the numbers $a, b, c - a, c - b$ is integral.*

Proof. This follows by application of the previous lemma to the case $A = M_\infty, B = M_0^{-1}$. Since $M_1^{-1} = M_\infty M_0$ the condition that AB^{-1} has eigenvalue 1 is fullfilled. Knowing the eigenvalues of M_0, M_∞ one easily checks that equality of eigenvalues comes down to the non-empty intersection of the sets $\{0, c\}$ and $\{a, b\}$ considered modulo $\mathbf{Z}$.

Definition 3.11 *A hypergeometric equation is called reducible if its monodromy group is reducible. A hypergeometric equation is called abelian if its monodromy group is abelian.*

Typical examples of abelian equations are (2) with $a = c = 0$ having solutions $1, (1 - z)^{-(b+1)}$ and $a = b = 1, c = 2$ having solutions $1/z, \log(1-z)/z$. Here is a simple necessary condition for abelian equations, which has the pleasant property that it depends only on $a, b, c (\mathrm{mod}\ \mathbf{Z})$.

Lemma 3.12 *If* (2) *is abelian then at least two of the numbers $a, b, c-a, c-b$ are integral.*

Proof. Abelian monodromy implies reducibility of the monodromy, hence at least one of the four numbers is integral. Let us say $a \in \mathbf{Z}$, the other cases can be dealt with similarly. It suffices to show that in at least one of the points $0, 1, \infty$ the local exponent difference of (2) is integral. Then clearly, $1 - c \in \mathbf{Z}$ implies $c - a \in \mathbf{Z}$, $c - a - b \in \mathbf{Z}$ implies $c - b \in \mathbf{Z}$ and $a - b \in \mathbf{Z}$ implies $b \in \mathbf{Z}$.

Suppose that all local exponent differences are non-integral. In particular the eigenvalues of each of the generating monodromy elements M_0, M_1, M_∞ are distinct. Then abelian monodromy implies that the monodromy group acts on the solution space in a completely reducible way as a sum of two one-dimensional representations. In particular the generators of these representations are functions of the form

$$z^\lambda (1-z)^\mu q(z) \qquad z^{\lambda'} (1-z)^{\mu'} p(z)$$

where $p(z), q(z)$ are polynomials with the property that they do not vanish at $z = 0$ or 1. The local exponents can be read off immediately, λ, λ' at 0, μ, μ' at 1 and $-\lambda - \mu - \deg(q), -\lambda' - \mu' - \deg(p)$ at ∞. The sum of the local exponents must be 1, hence $-\deg(p) - \deg(q) = 1$. Clearly this is a contradiction. $\square$

Lemma 3.13 *Suppose that $A, B \in GL(2, \mathbf{C})$ have disjoint sets of eigenvalues and suppose that AB^{-1} has eigenvalue 1. Then, letting $X^2 + a_1 X + a_2$ and $X^2 + b_1 X + b_2$ be the characteristic polynomials of A, B, we have up to common conjugation,*

$$A = \begin{pmatrix} 0 & -a_2 \\ 1 & -a_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -b_2 \\ 1 & -b_1 \end{pmatrix}.$$

Proof. Choose $v \in \ker(A - B)$ and $w = Av = Bv$. Since A, B have disjoint eigenvalue sets, v is not an eigenvector of A and B. Hence w, v form a basis of $\mathbf{C}^2$. With respect to this basis A, B automatically obtain the form given in our Lemma. $\square$

Corollary 3.14 *Suppose that* (2) *is irreducible. Then, up to conjugation, the monodromy group depends only on the values of* a, b, c *modulo* $\mathbf{Z}$.

Let us now assume that $a, b, c \in \mathbf{R}$, which is the case most frequently studied. The eigenvalues of M_0, M_1, M_∞ then lie on the unit circle.

Definition 3.15 *Let* R, S *be two disjoint finite subsets of the unit circle of equal cardinality. The sets* R, S *are said to interlace if every segment on the unit circle, connecting two points of* R*, contains a point of* S.

Lemma 3.16 *Let* A, B *be non-commuting elements of* $GL(2, \mathbf{C})$. *Suppose that the eigenvalues of* A, B *have absolute value 1 and that* AB^{-1} *has eigenvalue 1. Let* G *be the group generated by* A, B. *Then there exists a unique (up to a constant factor) non-trivial hermitian form* F *on* $\mathbf{C}^2$ *such that* $F(g(x), g(y)) = F(x, y)$ *for every* $g \in G$ *and every pair* $x, y \in \mathbf{C}^2$. *Moreover,*

$$F \text{ degenerate} \iff A, B \text{ have common eigenvalues}$$

Supposing A, B *have disjoint eigenvalue sets, we have in addition,*

$$F \text{ definite} \iff \text{eigenvalues of } A, B \text{ interlace}$$

$$F \text{ indefinite} \iff \text{eigenvalues of } A, B \text{ do not interlace}$$

We call these three cases the euclidean, spherical and hyperbolic case respectively.

Proof. Let $v \in \ker(A - B)$ and $w = Av$. Suppose first that v, w form a basis of $\mathbf{C}^2$. Of course, with respect to this basis A and B have the form given in the previous lemma. In particular we see that A, B cannot have the same characteristic equation, since this would imply that $A = B$.

We have to find a hermitean form F such that

$$\begin{aligned} F(gv, gv) &= F(v, v) \qquad F(gv, gw) = F(v, w) \\ F(gw, gv) &= F(w, v), \qquad F(gw, gw) = F(w, w) \end{aligned}$$

for every $g \in G$. It suffices to take $g = A, B$. Let $X^2 + a_1 X + a_2$ and $X^2 + b_1 X + b_2$ be the characteristic polynomials of A, B. Since the roots are on the unit circle we have $a_2 \bar{a}_2 = 1, a_2 \bar{a}_1 = a_1$ and similarly for b_1, b_2.

Let us first take $g = A$. Then $F(Av, Av) = F(v, v)$ implies

$$F(w, w) = F(v, v).$$

The conditions $F(Av, Aw) = F(v, w)$ and $F(Aw, Av) = F(w, v)$ imply $F(w, A^2 v) = F(v, w)$ and $F(A^2 v, w) = F(w, v)$. Hence, using $A^2 = -a_1 A - a_2$,

$$\begin{aligned} -\bar{a}_1 F(w, w) - \bar{a}_2 F(w, v) &= F(v, w) && (6) \\ -a_1 F(w, w) - a_2 F(v, w) &= F(w, v) && (7) \end{aligned}$$

Because of the relations $a_2 = \bar{a}_2^{-1}$ and $a_2 \bar{a}_1 = a_1$ these equations are actually the same. The condition $F(Aw, Aw) = F(w, w)$ yields $F(A^2 v, A^2 v) = F(w, w)$ and hence

$$|a_1|^2 F(w, w) + a_1 \bar{a}_2 F(w, v) + \bar{a}_1 a_2 F(v, w) + |a_2|^2 F(v, v) = F(w, w).$$

Using $|a_2|^2 = 1$, $a_2 \bar{a}_1 = a_1$ and $F(w, w) = F(v, v)$ this is equivalent to

$$a_1 \bar{a}_1 F(w, w) + a_1 \bar{a}_2 F(w, v) + a_1 F(v, w) = 0$$

which is precisely (6) times a_1. Hence A-invariance of F is equivalent to

$$F(v,v)=F(w,w), \qquad F(w,v)+a_1F(w,w)+a_2F(v,w)=0.$$

Invariance of F with respect to B yields the additional condition

$$F(w,v)+b_1F(w,w)+b_2F(v,w)=0.$$

Since A and B do not have the same characteristic equation the solutionspace for F is one-dimensional. When $a_2=b_2$ a solution is given by

$$F(w,w)=F(v,v)=0, \qquad F(w,v)=(-a_2)^{1/2}, \qquad F(v,w)=(-a_2)^{-1/2},$$

when $a_2\neq b_2$ a solution is given by

$$F(w,w)=F(v,v)=1, \qquad F(w,v)=\epsilon, \qquad F(v,w)=\bar{\epsilon}, \qquad \epsilon=\frac{a_1-b_1}{b_2-a_2}.$$

We formally take $\epsilon=\infty$ if $a_2=b_2$. In both cases cases we see that F is definite, degenerate, indefinite according to the conditions $|\epsilon|<1$, $|\epsilon|=1$, $|\epsilon|>1$. It now a straightforward excercise to see that these inequalities correspond to interlacing, coinciding or non-interlacing of the eigenvalues of A and B.

We are left with the case when v is an eigenvector of A and B. Let α be the eigenvalue. If both A and B have only eigenvalues α they automatically commute, which case is excluded. So either A or B has an eigenvalue different from α. Let us say that A has the distinct eigenvalues α,α'. Let w be an eigenvector corresponding to α'. Then, with respect to v,w the matrix of B must have the form

$$\begin{pmatrix} \alpha & b_{12} \\ 0 & \beta \end{pmatrix}.$$

with $b_{12}\neq 0$. It is now straightforward to verify that $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is the unique invariant hermitean matrix. Moreover it is degenerate, which it should be as A,B have a common eigenvector. □

Definition 3.17 *With the assumptions as in the previous lemma let G be the group generated by A and B. Then G is called hyperbolic, euclidean, spheric if F is indefinite, degenerate, definite respectively.*

Corollary 3.18 *Let $\{x\}$ denote the fractional part of x (x minus largest integer $\leq x$). Suppose that* (2) *is irreducible. Let F be the invariant hermitean form for the monodromy group. In particular, the sets $\{\{a\},\{b\}\}$ and $\{0,\{c\}\}$ are disjoint. If $\{c\}$ is between $\{a\}$ and $\{b\}$ then F is positive definite (spherical case). If $\{c\}$ is not between $\{a\}$ and $\{b\}$ then F is indefinite (hyperbolic case).*

The most pittoresque way to describe the monodromy group is by using *Schwarz' triangles.*

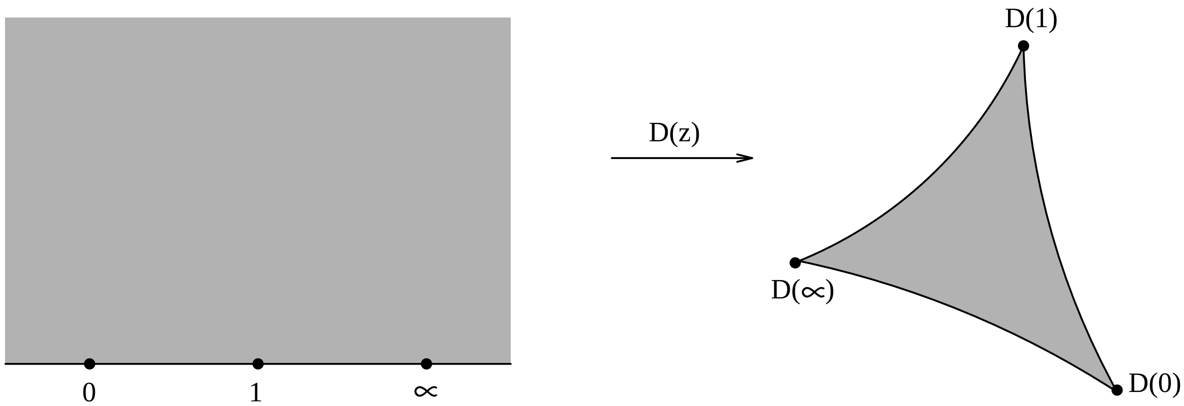
First a little geometry.

Definition 3.19 *A curvilinear triangle is a connected open subset of $\mathbf{C}\cup\infty=\mathbf{P}^1$ whose boundary is the union of three open segments of a circle or straight line and three points. The segments are called the edges of the triangles, the points are called the vertices.*

It is an exercise to prove that, given the vertices and the corresponding angles ($< \pi$), a curvilinear triangle exists and is uniquely determined This can be seen best by taking the vertices to be $0, 1, \infty$. Then the edges connected to ∞ are actually straight lines.

More generally, a curvilinear triangle in $\mathbf{C} \cup \infty = \mathbf{P}^1$ is determined by its angles (in clockwise ordering) up to a Möbius transformation.

Let z_0 be a point in the upper half plane $\mathcal{H} = \{z \in \mathbf{C} | \Im(z) > 0\}$ and let f, g be two independent solutions of the hypergeometric equation near z_0. The quotient $D(z) = f/g$, considered as a map from $\mathcal{H}$ to $\mathbf{P}^1$, is called the *Schwarz map* and we have the following picture and theorem.



Theorem 3.20 (Schwarz) *Let $\lambda = |1-c|, \mu = |c-a-b|, \nu = |a-b|$ and Suppose $0 \le \lambda, \mu, \nu < 1$. Then the map $D(z) = f/g$ maps $\mathcal{H} \cup \mathbf{R}$ one-to-one onto a curvilinear triangle. The vertices correspond to the points $D(0), D(1), D(\infty)$ and the corresponding angles are $\lambda\pi, \mu\pi, \nu\pi$.*

As to the proof of Schwarz' theorem, the following three ingredients are important.

- The map $D(z)$ is locally bijective in every point of $\mathcal{H}$. Notice that $D'(z) = (f'g - fg')/g^2$. The determinant $f'g - fg'$ is the *Wronskian determinant* of our equation and equals $z^{-c}(1-z)^{c-a-b-1}$. In particular it is non-zero in $\mathcal{H}$. When g has a zero at some point z_1 we simply consider $1/D(z)$ instead. Since f and g cannot vanish at the same time in a regular point, we have $f(z_1) \ne 0$.
- The map $D(z)$ maps the segments $(\infty, 0), (0, 1), (1, \infty)$ to segments of circles or straight lines. For example, since $a, b, c \in \mathbf{R}$ we have two real solutions on $(0, 1)$ (see Kummer's solutions). Call them $\tilde{f}, \tilde{g}$. Clearly, the function $\tilde{D}(z) = \tilde{f}/\tilde{g}$ maps $(0, 1)$ on a segment of $\mathbf{R}$. Since f, g are $\mathbf{C}$-linear combinations of $\tilde{f}, \tilde{g}$ we see that $D(z)$ is a Möbius transform of $\tilde{D}(z)$. Hence $D(z)$ maps $(0, 1)$ to a segment of a circle or a straight line.
- The map $D(z)$ maps a small neighbourhood of 0 to a sector with angle $|1-c| = \lambda$ and similarly for $1, \infty$. This follows from the fact that near $z = 0$ the functions f, g are $\mathbf{C}$-linear combinations of $F(a, b, c|z)$ and $z^{1-c}F(a-c+1, b-c+1, 2-c|z)$.

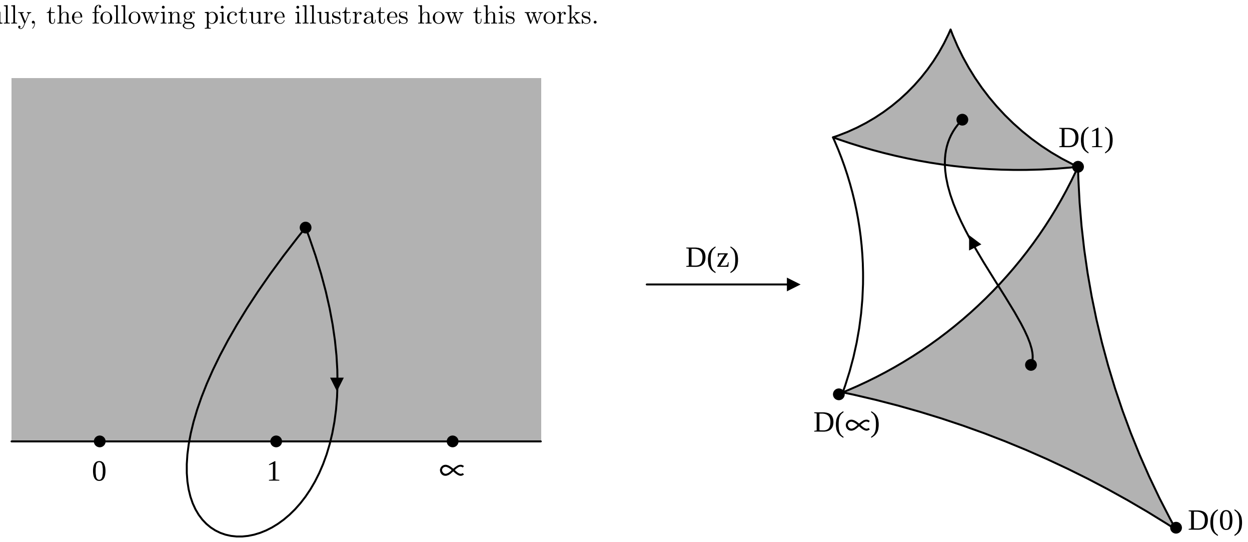
For the exact determination of the image of the Schwarz map we need the following additional result.

Proposition 3.21 (Gauss) *Suppose that $a, b, c \in \mathbf{R}$, $c \notin \mathbf{Z}_{\le 0}$ and $c > a + b$. Prove that*

$$F(a, b, c|1) = \frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(c-a-b)}.$$

This can be proven by evaluation of Euler's integral using the Euler Beta-function.

To study the analytic continuation of $D(z)$ we use *Schwarz' reflection principle.* Hopefully, the following picture illustrates how this works.



The monodromy group modulo scalars arises as follows. Let W be the group generated by the reflections in the edges of the curvilinear triangle. The monodromy group is the subgroup of W consisting of all elements which are product of an even number of reflections. In the following section we shall study precisely such groups.

3.22 Triangle groups

In this section we let S be either the Poincaré disk $\{z \in \mathbf{C} | \, |z| < 1\}$, $\mathbf{C}$ or $\mathbf{P}^1$. equipped with the hyperbolic, euclidean and spherical metric respectively.

Definition 3.23 *A (geodesic) triangle is an connected open subset of S, of finite volume, whose boundary in S is a union of three open segments of a geodesic and at most three points. The segments are called the edges of the triangles, the points are called the vertices.*

We first point out that under very mild conditions any curvilinear triangle can be thought of as a geodesic triangle.

Lemma 3.24 *Let λ, μ, ν be real numbers in the interval $[0, 1)$. There exists a geodesic triangle with angles $\lambda\pi, \mu\pi, \nu\pi$ if and only if $\lambda + \mu + \nu < 1 + 2\min(\lambda, \mu, \nu)$.*

Proof. Suppose first that $\lambda + \mu + \nu < 1$. Our condition is then trivially satisfied. For any such curvilinear triangle we can take the common orthogonal circle of the three edges, which will become the boundary of a Poincaré disk. The edges are then automatically geodesics.

Suppose that $\lambda + \mu + \nu = 1$. Our condition is equivalent to saying that all angles are positive. In this case geodesic triangles are planar triangles in the euclidean geometry with finite area. The latter property is equivalent to positivity of all angles.

Suppose that $\lambda+\mu+\nu > 1$. From spherical geometry it follows that a spherical triangle exists if and only if our condition is satisfied.

We let $W(\Delta)$ be the group of isometries of S generated by the 3 reflections through the edges of a geodesic triangle Δ. First we look at subgroups generated by reflection in two intersecting geodesics.

Lemma 3.25 *Let ρ, σ be two geodesics intersecting in a point P with an angle $\pi\lambda$. Let r, s be the reflections in ρ, σ respectively. Then the group D generated by r, s is a dihedral group consisting of rotations $(rs)^n$ around P with angles $2n\pi\lambda$, $n \in \mathbf{Z}$ and reflections in the lines $(rs)^n(\rho), (rs)^n(\sigma)$. In particular D is finite of order $2m$ if and only if $\lambda = q/m$ for some $q \in \mathbf{Z}$ with $q \neq 0$ and $\gcd(m, q) = 1$. Furthermore, D is discrete if and only if λ is either zero or a rational number.*

Theorem 3.26 *For any geodesic triangle Δ we have $S = \cup_{\gamma \in W(\Delta)} \gamma(\overline{\Delta})$, where $\overline{\Delta}$ denotes the closure of Δ in S.*

Proof. First of all we note that there exists a positive d_0 with the following property. For any point P whose distance to Δ is less than d_0 there exists $\gamma \in W(\Delta)$ such that $P \in \gamma(\overline{\Delta})$. For γ we can simply take a suitable element from one of the dihedral reflection groups around the vertices.

A fortiori, any point P with distance less than d_0 from $\cup_{\gamma \in W(\Delta)} \gamma(\overline{\Delta})$ belongs to this set.

As a consequence the set $\cup_{\gamma \in W(\Delta)} \gamma(\overline{\Delta})$ is open and closed in S, hence our theorem follows. $\square$

Definition 3.27 *An elementary triangle is a geodesic triangle whose vertex angles are all of the form π/n, $n \in \mathbf{Z}_{\geq 2} \cup \infty$.*

Theorem 3.28 *Let Δ be an elementary triangle. Then, for any $\gamma \in W(\Delta), \gamma \neq \mathrm{Id}$ we have $\gamma(\Delta) \cap \Delta = \emptyset$.*

Proof. This is a special case of the theorem of Coxeter-Tits on representations of Coxeter groups. See Humphreys book on Reflection groups and Coxeter groups [H].

A group G of isometries acting on S is said to act discretely if there exists a point $P \in S$ and a positive d_0 such that distance$(P, g(P)) > d_0$ whenever $g \neq \mathrm{Id}$. In particular it follows from the previous theorem that triangle groups generated by elementary triangles act discretely. The following theorem characterises all groups $W(\Delta)$ which act discretely on the symmetric space S.

Theorem 3.29 *Suppose $W = W(\Delta)$ acts discretely. Then there exists an elementary triangle Δ_{el} such that $W(\Delta) = W(\Delta_{\mathrm{el}})$. Moreover, $\overline{\Delta}$ is a finite union of copies of $\overline{\Delta_{\mathrm{el}}}$ under elements of W.*

Proof. First of all note that the vertex angles must be either 0 or rational multiples of π, otherwise the corresponding dihedral group is not discrete.

We shall show that if Δ is not elementary, then there exists a geodesic triangle Δ' such that $W(\Delta) = W(\Delta')$ and $\mathrm{Vol}(\Delta') \leq \mathrm{Vol}(\Delta)/2$. If Δ' is not elementary we repeat the process and so on. However, there is a limit to these processes since, by discreteness, there is a positive lower bound to $\mathrm{Vol}(\Delta'')$ for any Δ'' satisfying $W(\Delta) = W(\Delta'')$. Hence we must hit upon an elementary triangle Δ_{el} such that $W(\Delta) = W(\Delta_{\mathrm{el}})$.

Let α, β, γ be the edges of Δ and $r_\alpha, r_\beta, r_\gamma$ the corresponding reflections. Suppose that the vertex angle between α and β is of the form $m\pi/n$ with $\gcd(m, n) = 1$, but $m > 1$. Let δ be the geodesic between α and β whose angle with α is π/n. Let r_δ be the reflection in δ. Then the dihedral group generated by r_α and r_β is the same as the one generated by r_α and r_δ. Let Δ' be the triangle with edges α, δ, γ. Then, clearly, $W(\Delta) = W(\Delta')$. If the volume of Δ' is larger than half the volume of Δ we simply perform the above construction with α and β interchanged. $\square$

Below we give a list of non-elementary triangles $\Delta = (\lambda, \mu, \nu)$ with vertex angles $\lambda\pi, \mu\pi, \nu\pi$ which allow a dissection with elementary triangles $\Delta_{\rm el}$ such that $W(\Delta) = W(\Delta_{\rm el})$. In the spherical case discreteness of $W(\Delta)$ implies finiteness. The list of spherical cases was already found by H.A.Schwarz and F.Klein (see [Kl]). In the following table N denotes the number of congruent elementary triangles needed to cover Δ.

λ	μ	ν	N		elementary
$2/n$	$1/m$	$1/m$	2	$\times$	$(1/2, 1/n, 1/m)$ n odd
$1/2$	$2/n$	$1/n$	3	$\times$	$(1/2, 1/3, 1/n)$ n odd
$1/3$	$3/n$	$1/n$	4	$\times$	$(1/2, 1/3, 1/n)$ $n \not\equiv 0 \bmod 3$
$2/n$	$2/n$	$2/n$	6	$\times$	$(1/2, 1/3, 1/n)$ n odd
$4/n$	$1/n$	$1/n$	6	$\times$	$(1/2, 1/3, 1/n)$ n odd
$2/3$	$1/3$	$1/5$	6	$\times$	$(1/2, 1/3, 1/5)$
$1/2$	$2/3$	$1/5$	7	$\times$	$(1/2, 1/3, 1/5)$
$3/5$	$2/5$	$1/3$	10	$\times$	$(1/2, 1/3, 1/5)$
$1/3$	$2/7$	$1/7$	10	$\times$	$(1/2, 1/3, 1/7)$

As an application we construct a hypergeometric function which is algebraic over $\mathbf{C}(z)$. Take the triangle $(4/5, 1/5, 1/5)$, which is spherical. Corresponding values for a, b, c can be taken to be $1/10, -1/10, 1/5$. Hence the quotient of any two solutions f, g of the corresponding hypergeometric is algebraic. Its derivative $(f'g - fg')/g^2$ is algebraic and so is the Wronskian determinant $f'g - fg' = z^{-c}(1-z)^{c-a-b-1}$. Hence g and, a fortiori, f are algebraic. In particular, $F(1/10, -1/10, 1/5|z)$ is an algebraic function.

In many cases it is also possible to find elementary triangles $\Delta_{\rm el}$ which can be dissected into isometric copies of a smaller elementary triangle $\Delta'_{\rm el}$. Hence $W(\Delta_{\rm el}) \subset W(\Delta'_{\rm el})$. The most spectacular example is the dissection of the triangle $(1/7, 1/7, 1/7)$ into 24 copies of $(1/2, 1/3, 1/7)$. As a corollary of this dissection we find the remarkable identity

$$ {}_2F_1\left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\middle| z\right) = b(z)^{-1/28}\ {}_2F_1\left(\frac{1}{84}, \frac{29}{84}, \frac{6}{7}\middle| 12^3 \frac{z(z-1)(z^3 - 8z^2 + 5 * z + 1)}{b(z)^3}\right) $$

where $b(z) = 1 - 236z + 1666z^2 - 3360z^3 + 3395z^4 - 1736z^5 + 42z^6 + 228z^7 + z^8$. For a complete list of such dissections and the corresponding identities we refer to [V].

3.30 Some loose ends

In the Schwarz map we have assumed that the parameters a, b, c are such that $\lambda = |1 - c|, \mu = |c - a - b|, \nu = |a - b|$ are all less than 1. It turns out that in the irreducible case this is no restriction, since we can shift a, b, c by integers without affecting the monodromy group. In fact,

Lemma 3.31 *Assume that none of the numbers $a, b, c-a, c-b$ is integral. There exist $a' \in a(\bmod \mathbf{Z})$, $b' \in b(\bmod \mathbf{Z})$, $c' \in c(\bmod \mathbf{Z})$ such that*

$$ 0 \le \lambda, \mu, \nu < 1 \qquad \lambda + \mu + \nu < 1 + 2\min(\lambda, \mu, \nu) $$

where $\lambda = |1 - c'|, \mu = |c' - a' - b'|, \nu = |a' - b'|$. In the case $\lambda + \mu + \nu < 1$ there exists only one choice for a', b', c' and in the case $\lambda + \mu + \nu > 1$ there exist four possible choices.

Proof. First of all let us suppose that $0 \le a, b, c < 1$. Without loss of generality we can assume that $a \le b$. We consider the following cases.

Case i) $0 < a < c < b < 1$. We take $a' = a, b' = b, c' = c$. Then, $\lambda = 1 - c, \mu = a + b - c, \nu = b - a$ and the inequalities are satisfied. Moreover, $\lambda + \mu + \nu = 1 + 2b - 2c > 1$.

Case ii) $0 < a \leq b < c < 1$. We take $a' = a, b' = b, c' = c$. When $c \geq a + b$ we get $\lambda = 1-c, \mu = c-a-b, \nu = b-a$ and the inequalities hold. Moreover, $\lambda+\mu+\nu = 1-2a < 1$. When $c \leq a+b$ we get $\lambda = 1-c, \mu = a+b-c, \nu = b-a$ and the inequalities hold. Moreover, $\lambda + \mu + \nu = 1 + 2b - 2c < 1$.

Case iii) $0 \leq c < a \leq b < 1$. We take $a' = a, b' = b, c' = c+1$. Then, $\lambda = c, \mu = c+1-a-b, \nu = b-a$ and the inequalities are readily verified. Moreover, $\lambda+\mu+\nu = 1+2c-2a < 1$.

As to uniqueness we note that an integral shift in the a, b, c such that the corresponding values of λ, μ, ν stay below 1 necessarily gives the substitutions of the form $\lambda \to 1-\lambda, \mu \to 1-\mu, \nu \to \nu$ and similar ones where two of the parameters are replaced by 1 minus their value. In casethe condition $\lambda+\mu+\nu < 1+2\min(\lambda, \mu, \nu)$ is violated by such a substitution. For example, $\lambda + \mu + \nu \leq 1$ implies $1 - \lambda + 1 - \mu + \nu = 2 - (\lambda + \mu + \nu) + 2\nu \geq 1 + 2\nu$. In the spherical case the condition is not violated.

When we have obtained a geodesic Schwarz triangle in our construction we automatically have a metric which is invariant under the projective monodromy group. This closely reflects the nature of the natural hermitian form on the monodromy group itself.

Theorem 3.32 *Let $a, b, c \in \mathbf{R}$ be such that*

$$0 \leq \lambda, \mu, \nu < 1 \qquad \lambda + \mu + \nu < 1 + 2\min(\lambda, \mu, \nu)$$

where $\lambda = |1-c|, \mu = |c-a-b|, \nu = |a-b|$. Let M be the monodromy group of (2). *Then,*

$$M \textit{ is spheric} \iff \lambda + \mu + \nu > 1$$

$$M \textit{ is euclidean} \iff \lambda + \mu + \nu = 1$$

$$M \textit{ is hyperbolic} \iff \lambda + \mu + \nu < 1.$$

Proof. In the case when none of the numbers $a, b, c-a, c-b$ is integral, this statement can already be inferred from the proof of the previous lemma (we get only the hyperbolic and spheric case). It remains to show that if one of the numbers $a, b, c-a, c-b$ is integral, we have $\lambda + \mu + \nu = 1$. Let us suppose for example that $a \in \mathbf{Z}$. Notice that $|a-b| < 1$ and $|a+b| < |c| + 1 < 3$. Hence $|a| \leq |a-b|/2 + |a+b|/2 < 2$. So, $a = 0, \pm 1$. A case by case analysis using the inequalities for λ, μ, ν yields our statement. □

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