A rational approach to π

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Abstract

This article is based on notes for the lecture with the same title, which was held by the author on the occasion of the 'Pi in de Pieterskerk' event (Pi-day) on July 5, 2000. The present article expands these notes with short proofs of most of the theorems given, but not proved, during the lecture.

1 Introduction

During the weeks preceding Pi-day in Leiden, and of course on the day itself, it has once more become clear that the number π has an alluring appeal to a very broad audience. A possible explanation for this interest is that π is the only transcendental number which most people have ever seen and will ever see. The fact that such a transcendental number cannot be written down exactly is then a source of amazement and wonder.

In the past few years this fascination with π has resulted in a number of books on the subject of π. Some of these books are written for a wide audience, some others can be read only with a substantial mathematical background. In the bibliography of this article we give a short, descriptive listing of such books and some websites as well. Through these publications devoted to π, a body of facts and stories has developed itself around this number. We can read about Archimedes' method to compute π, Ludolf van Ceulen's record computation, Machin's formula, the arithmetic-geometric mean, Ramanujan's miraculous formulas, the impossibility of circle quadrature, computation of digits of π without knowing the previous ones. All these topics form part of what I would like to call π-folklore.

It is not the purpose of this article to provide another introduction to π-folklore. I refer the reader interested in this folklore to the bibliography. The purpose of the present article is to advertise a rather recent result around π, and thus help it find its way into π-folklore. The result deals with the explicit construction of good rational approximations to π. Let us start with two very well-known rational numbers that approximate π:

$$\frac{22}{7} - \pi \approx 0.00126 \quad , \quad \frac{355}{113} - \pi \approx 0.000000266$$

The number $\frac{22}{7}$ is so well-known as an approximation, that many people think that it equals π. The second approximation has been discovered in the fifth century by the Chinese mathematician Tsu Chung Chih. One might wonder why we call these rational approximations good, and whether more of such good approximations exist. To quantify what 'good' means we rewrite our approximations as

$$\left|\frac{22}{7}-\pi\right| \approx \frac{1}{7^{3.429}} \quad , \quad \left|\frac{355}{113}-\pi\right| \approx \frac{1}{113^{3.201}}$$

The exponents 3.429 and 3.201 will be called the *quality* of the respective approximations. In general, the quality of an approximation $\frac{p}{q}$ with $p, q \in \mathbb{N}$ and $\gcd(p,q) = 1$ is the number M such that

$$\left|\pi - \frac{p}{q}\right| = \frac{1}{q^M}$$

Using the theory of continued fractions we know that there exist infinitely many rational approximations to π whose quality is ≥ 2. However, the continued fraction expansion of π is completely intractable. Almost nothing is known about this continued fraction and thus we have no control over it. For applications one would like to have an *explicit construction* of rational approximations to π of good quality. By that we mean quality ≥ 1. This is of course less than quality ≥ 2 as with continued fractions. But this lesser quality is counter-balanced by the greater control over the approximations, due to the explicitness of the construction. This control, which is important for several applications, is lacking in the case of continued fractions.
It turns out that the number π is surprisingly resistant against contruction of good quality approximations. Despite many efforts it was only in 1993 that the Japanese mathematician Masayoshi Hata succeeded in giving such a construction in M.Hata, *Rational approximations to π and some other numbers*, Acta Arith. 63 (1993), pp 335-349.
In this article we shall describe a simple irrationality proof of π. Then we explain the role of explicit good quality approximations in irrationality proofs and irrationality measures. Finally we describe a few attempts to contruct good quality approximations crowned by Hata's successful construction.

2 Irrationality of π

The first irrationality proof of π was given in 1773 by the Swiss mathematician J.Lambert. In the long history of π this can be called a fairly recent result. The reason for the late appearance of such a proof is that proving irrationality of π is far from trivial. Lambert made use of continued fraction of the cotangent function. Such continued fractions were relatively new in Lambert's time. Here is the formula that Lambert used:

$$\cot\frac{1}{x} = x - \cfrac{1}{3x - \cfrac{1}{5x - \cfrac{1}{7x - \ddots}}}$$

This formula means that if, for a given x, we compute consecutively the truncated fractions:

$$x - \frac{1}{3x},\quad x - \cfrac{1}{3x - \cfrac{1}{5x}},\quad x - \cfrac{1}{3x - \cfrac{1}{5x - \cfrac{1}{7x}}}$$

et cetera, we get a sequence of numbers which converges to $\cot\frac{1}{x}$. In fact, it turns out that this convergence is surprisingly fast. The truncated fractions are usually called the *convergents* of the continued fraction. Although I have seen this continued fraction many times, I still think it is a wonderful formula. One of its interesting features is that the right hand side does not contain π explicitly. As the reader may know, there are several ways to expand the cotangent function. For example, Euler's summation

$$\cot\frac{1}{x} = x + \sum_{n=1}^{\infty}\frac{2x}{1 - n^2\pi^2x^2}$$

or the product formula

$$\cot\frac{1}{x} = (-1 + \pi x/2)\prod_{n=1}^{\infty}\frac{n^2\pi^2x^2 - (-1+\pi x/2)^2}{n^2\pi^2x^2 - 1}.$$

However, most of these expansions contain π explicitly. Lambert's continued fraction does not.
If, in particular, we take $x = \frac{2}{\pi}$, we obtain the equality

$$0 = 2/\pi - \cfrac{1}{6/\pi - \cfrac{1}{10/\pi - \cfrac{1}{14/\pi - \ddots}}}$$

The right hand side is really an elaborate, but useful, way to express the number zero. Writing down the truncated fractions, we get:

$$\begin{aligned} 2/\pi - \frac{1}{6/\pi} &= \frac{12-\pi^2}{6\pi} \\ &\approx 0.113 \end{aligned}$$

$$2/\pi - \cfrac{1}{6/\pi - \cfrac{1}{10/\pi}} = \frac{120 - 12\pi^2}{60\pi - \pi^3} \approx 0.00993$$

$$2/\pi - \cfrac{1}{6/\pi - \cfrac{1}{10/\pi - \cfrac{1}{14/\pi}}} = \frac{1680 - 180\pi^2 + \pi^4}{840\pi - 20\pi^3} \approx 0.000436$$

$$\frac{30240 - 3360\pi^2 + 30\pi^4}{15120\pi - 420\pi^3 + \pi^5} \approx 0.0000115$$

In particular these convergents go to zero. Lambert argued as follows. Suppose that π were rational. Then the convergents are rational numbers. By carefully estimating the numerical value of these convergents and the size of the denominator, Lambert noticed that the convergents are eventually non-zero rational numbers whose absolute values are strictly less than one divided by their denominator. This is impossible and we get a contradiction. Hence π is irrational.

Lambert's precise estimates are rather tedious though, and the above sketch may not be very illuminating. Fortunately we have nowadays a much simpler proof given by I.Niven in 1947. This proof exploits the integral

$$I_n = \frac{1}{2}\,\frac{1}{n!}\int_0^\pi x^n(\pi - x)^n \sin x \;\; \mathrm{d}x$$

for every positive integer n. We remark that D.Hilbert used integrals of this type in his simplified transendence proof of π in 1907. However, this does not simplify the question of why one should use integrals like I_n to prove irrationality of π. This is one of the charms of irrationality proving. Most of the time the initial idea seems to come clear out of the blue. Here are some particular values of I_n:

$$\begin{aligned} I_2 &= 12 - \pi^2 \\ I_3 &= 120 - 12\pi^2 \\ I_4 &= 1680 - 180\pi^2 + \pi^4 \\ I_5 &= 30240 - 3360\pi^2 + 30\pi^4 \end{aligned}$$

Looking at these polynomials in π one may observe that they coincide with the numerators of Lambert's continued fraction. So Niven's integral is not so alien after all. The reader, who is familiar with continued fractions and with partial integration, may try to find the reason for these coinciding polynomials in π. Here are a number of facts for every positive integer n:

1. $I_n \in \mathbb{Z}[\pi]$ of degree $\leq n$.
2. $I_n > 0$
3. $I_n \leq \frac{\pi^{2n+1}}{n!}$

Fact (2) is easy, as the integrand of I_n is a positive function on the segment of integration. So, $I_n > 0$. Fact (3) is also straightforward. The factors x^n, $(\pi - x)^n$, $\sin x$ in the integrand of I_n can be estimated by $\pi^n, \pi^n, 1$ respectively. So

$$I_n = \frac{1}{2}\,\frac{1}{n!}\int_0^{\pi} x^n(\pi - x)^n \sin x \;\,\mathrm{d}x \leq \frac{1}{n!}\int_0^{\pi} \pi^{2n}\;\mathrm{d}x = \frac{\pi^{2n+1}}{n!}.$$

Fact (1) follows from a number of observations. First of all, by partial integration one can see that for any polynomial $f(x)$ we have

$$\int_0^{\pi} f(x)\sin x\;\mathrm{d}x = f(\pi) + f(0) - f''(\pi) - f''(0) + f''''(\pi) + f''''(0) - \cdots$$

The second observation is that $x(\pi - x)$ is symmetric with respect to the substition $x \to \pi - x$. Suppose that $f(x)$ is symmetric in this way. Then the same holds for the even order derivatives. So, $f^{(2k)}(\pi) = f^{(2k)}(0)$ for all $k \geq 0$. Hence

$$\int_0^{\pi} f(x)\sin x\;\mathrm{d}x = 2f(0) - 2f''(0) + 2f''''(0) - \cdots$$

Now take $f(x) = x^n(\pi - x)^n$. We then see that $f^{(k)}(0) = 0$ for all $k < n$. Furthermore, by using the binomial expansion of $(\pi - x)^n$, we find that

$$f^{(k)}(0) = k!\binom{n}{k-n}(-1)^{k-n}\pi^{2n-k}$$

for all $k \geq n$. Hence $\frac{1}{n!}f^{(k)}(0) \in \mathbb{Z}[\pi]$ for every k and thus we conclude that $I_n \in \mathbb{Z}[\pi]$. Also note that the highest power of π that can occur is π^n.
Now we can finish our irrationality proof. Suppose that $\pi = \frac{p}{q}$ is rational. Since, by fact (1), I_n is a polynomial of degree n in π with integer coefficients, it is a rational number with denominator dividing q^n. Moreover, by fact (2), $I_n > 0$. Because a positive rational number is at least one divided by its denominator, we get

$$\frac{1}{q^n} \leq I_n.$$

Combine this with our upper bound for I_n (fact (3)) to get

$$\frac{1}{q^n} < \frac{\pi^{2n+1}}{n!}$$

for every positive integer n. In other words,

$$n! < q^n \pi^{2n+1}.$$

This becomes impossible when n is taken large enough! We conclude that π cannot be rational. **qed**

3 Irrationality proofs and irrationality measures

In an American court of law the evidence for the irrationality of π, which we presented in the previous section, might be called 'circumstantial'. We constructed an increasingly complicated sequence of polynomials in π and the properties of these polynomials bore indirect evidence against the rationality of π. There is sometimes a more direct way to establishing irrationality of a number. It is based on the following observation.

Observation. Let α be a real number. Suppose we have a sequence of rational numbers

$$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \ldots, \frac{p_n}{q_n}, \ldots$$

such that

$$0 < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{\epsilon_n}{q_n},$$

where $\epsilon_n \downarrow 0$ as n goes to infinity. Then α is irrational.

The proof is quite straightforward. Suppose $\alpha = \frac{p}{q}$ were rational, where $p, q \in \mathbb{Z}$ and $q > 0$. Then the difference $\Delta_n = |\alpha - p_n/q_n|$ is a positive rational number with a denominator dividing qq_n. Hence $\Delta_n \geq \frac{1}{qq_n}$. On the other hand we have the estimate $\Delta_n < \frac{\epsilon}{q_n}$ for all n. Combining the two estimates we get

$$\frac{1}{qq_n} < \frac{\epsilon_n}{q_n}$$

and hence $\frac{1}{q} < \epsilon_n$. Since $\epsilon_n \downarrow 0$ as $n \to \infty$ we conclude that $\frac{1}{q} \leq 0$. This is clearly impossible and so α is irrational. **qed**

A famous example of this principle is the irrationality proof of e, which we give here. We know that e is the sum of the series

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{n \geq 0} \frac{1}{n!}$$

Let us truncate this series after the term $\frac{1}{n!}$ and write

$$\frac{p_n}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

Then $e - \frac{p_n}{n!} = \delta_n$ where

$$\delta_n = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots$$

We can estimate δ_n by using this series expression:

$$\begin{aligned}\delta_n &= \frac{1}{(n+1)!}\left(1+\frac{1}{n+2}+\frac{1}{(n+2)(n+3)}+\cdots\right)\\ &< \frac{1}{(n+1)!}\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots\right)\\ &= \frac{e}{(n+1)!}\end{aligned}$$

Thus we see that

$$0 < e - \frac{p_n}{n!} < \frac{e}{n+1}\frac{1}{n!}.$$

Application of the above observation with $\epsilon_n = \frac{e}{n+1}$ now shows irrationality of e. **qed**

Unfortunately, a similar irrationality proof for π is very hard to find. In fact, it was only in 1993 that Hata managed to give an explicit construction for rationals approximating π sufficiently well to establish its irrationality. In the long history of π this is a recent result indeed. But there is more. In general, irrationality proofs obtained by explicit construction of rational approximations yield more information than just an irrationality proof. They often provide so-called irrationality measures as well. It turns out that Hata's construction also yields the best irrationality measure for π yet proved.
In the remainder of this section we explain what an irrationality measure is, and how they can be estimated. In the following, α will be a fixed irrational number. Consider a rational approximation $\frac{p}{q}$ to α with $p, q \in \mathbb{Z}$, $q > 0$ and $\gcd(p,q) = 1$. Recall that we defined the quality of this approximation is the number $M > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| = \frac{1}{q^M}$$

or, if it does not exist, we take $M = 0$. As a first result we prove,

Theorem. Let α be an irrational number. Then there exist infinitely many approximations to π of quality ≥ 2.

This statement is part of the theory of continued fractions. But also without knowledge of continued fractions it is not hard to show. Fix a large positive integer Q and consider the set of numbers $\{q\alpha\}$ for $q = 0, 1, 2, \ldots, Q$, where $\{x\}$ denotes the difference between x and the largest integer $\leq x$. The set of $\{q\alpha\}$ is a set of $Q+1$ numbers in the interval $[0, 1)$. So it tends to be crowded when Q gets large. In particular, there must be two values of q, say $q_1 < q_2$, such that the difference between $\{q_1\alpha\}$ and $\{q_2\alpha\}$ is less than $\frac{1}{Q}$ in absolute value. Choose integers p_1, p_2 such that $\{q_i\alpha\} = q_i\alpha - p_i$. Then, $|(q_2 - q_1)\alpha - (p_2 - p_1)| < \frac{1}{Q}$. Since clearly $0 < q_2 - q_1 \leq Q$ we see that $\frac{p_2 - p_1}{q_2 - q_1}$ is an approximation of quality at least 2. By choosing increasingly large values for Q we can produce an infinite sequence of such approximations. **qed**

In the introduction we have seen two good rational approximations to π whose quality was larger than 3. One may wonder if an infinite number of such good quality approximations exist for π, or any other irrational we are looking at. To that end we introduce the following concept.

Definition The *irrationality measure* of an irrational number α is defined as the limsup over all qualities of all rational approximations and is denoted by $\mu(\alpha)$.

We have taken the limsup in our definition rather than the maximum since we are for example interested in the question whether π has infinitely many approximations of quality at least 3. The first two occurrences from the introduction may have been exceptional coincidences. If we assume that π behaves like most other numbers, then there is very little chance that $\mu(\pi) \geq 3$. This is shown by the following Theorem.

Theorem The set of irrational numbers with irrationality measure strictly larger than 2 has Lebesgue measure zero.

This Theorem is not hard to prove. Let us restrict ourselves to the irrational numbers in the interval $[0, 1]$. Choose $\epsilon > 0$. A number α with $\mu(\alpha) \geq 2 + 2\epsilon$ is, by definition, contained in an interval of the form

$$\left[\frac{p}{q} - \frac{1}{q^{2+\epsilon}}, \frac{p}{q} + \frac{1}{q^{2+\epsilon}}\right],$$

with $0 < p < q$ integers, infinitely many times. Let us give an upper bound for the total length of these intervals with $q > Q$, where Q is some large fixed positive integer. Such a bound can be given by

$$\sum_{q=Q+1}^{\infty} \sum_{p=1}^{q} \frac{2}{q^{2+\epsilon}}$$

The inner sum is equal to $\frac{2}{q^{1+\epsilon}}$. The sum over q can be estimated by the integral criterion,

$$\sum_{q=Q+1}^{\infty} \frac{2}{q^{1+\epsilon}} < \int_{Q}^{\infty} \frac{2}{x^{1+\epsilon}}\, \mathrm{d}x = \frac{2}{\epsilon Q^{\epsilon}}.$$

When we let $Q \to \infty$ we see that the latter bound goes to zero. Hence the Lebesgue measure of the numbers in $[0, 1]$ with irrationality measure $\geq 2 + 2\epsilon$ is zero. The set of numbers in $[0, 1]$ with irrationality measure > 2 is the union of all sets of numbers with irrationality measure at least $2+2/n$ for $n = 1, 2, 3, 4, \ldots$. Since a countable union of measure zero sets has again measure zero, our result follows. **qed**

We note that numbers with irrationality measure > 2 do exist. In fact there exist irrational numbers with irrationality measure ∞. These are the so-called *Liouville numbers*. An example of such a number is given by

$$\sum_{n \geq 0} \frac{1}{2^{n!}}.$$

The reader may wish to verify as an exercise that the truncated series form a sequence of approximations whose qualities go to ∞. On the other hand, numbers like Liouville numbers are a bit artificial. They are constructed for the purpose of having large irrationality measures. It is expected that the irrationally measure for a naturally occurring number is 2. Unfortunately, there are not many instances where this is known. The algebraic numbers are known to have measure 2. This was shown by K.F.Roth in 1955, an achievement which won him the Fields medal. Another known instance is e. The fact that $\mu(e) = 2$ can easily be shown by using the continued fraction expansion of e which, contrary to that of π, is completely known. Although it is expected that $\mu(\pi) = 2$, it is very hard to get any results on $\mu(\pi)$. It was only in 1953 that K.Mahler was able to show for the first time that $\mu(\pi)$ is finite. More precisely, he showed that $\mu(\pi) < 42$. Through subsequent work, this bound was improved to lower values as is seen from the following table:

Name	year	upper bound for $\mu(\pi)$
K.Mahler	1953	42
M.Mignotte	1974	20.6
G.Chudnovsky	1979	19.89
G.Rhin, C.Viola	1993	14.8
M.Hata	1993	8.02

Mignotte and Chudnovsky used Mahler's method by improving his estimates. Rhin and Viola used certain double integrals to show that $\mu(\pi^2) \leq 7.4$. This implies that $\mu(\pi) \leq 14.8$. The reader may wish to verify this last implication as an exercise. The record of Rhin and Viola was only short-lived, since Hata derived his bound on $\mu(\pi)$ in the same year. The method of Hata differed completely from its predecessors. It uses precisely the sequence of explicit rational approximations to π which we mentioned before. How one can derive an irrationality measure by constructing a sequence of rational approximations is explained in the following Proposition.

Proposition Let α be a real number. Suppose we have a sequence of rational approximations

$$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \ldots, \frac{p_n}{q_n}, \ldots$$

to α and suppose there exist $\epsilon > 0,\ Q > 1$ with the following properties:

i) $\frac{p_n}{q_n} \neq \frac{p_{n-1}}{q_{n-1}}$ for all n.

ii) $q_n < Q^n$ for all n.

iii) $\left|\alpha - \frac{p_n}{q_n}\right| \leq \frac{1}{Q^{(1+\epsilon)n}}$ for all n.

Then

$$\mu(\alpha) \leq 1 + \frac{1}{\epsilon}.$$

Roughly speaking this Proposition says that if we can construct a sequence of explicit rational approximations to α with qualities at least $1+\epsilon$, then this allows us to show that $\mu(\alpha) < 1 + \frac{1}{\epsilon}$. So, to get a bound for $\mu(\pi)$ one might construct a sequence of good quality approximations to π. Once more we have arrived at the problem of constructing such a sequence. In the next section we eventually find such a construction. We close this section with a proof of the Proposition.
Let $\frac{p}{q}$ be any rational number with $q > 0$. Choose n such that $Q^{\epsilon n} \geq 2q > Q^{\epsilon(n-2)}$. Note that there are two such n. We make the choice such that $\frac{p_n}{q_n} \neq \frac{p}{q}$. This is possible on the basis of assumption (i). Then

$$\begin{aligned} \frac{1}{qq_n} &\leq \left|\frac{p_n}{q_n} - \frac{p}{q}\right| \\ &\leq \left|\alpha - \frac{p_n}{q_n}\right| + \left|\alpha - \frac{p}{q}\right| \\ &\leq \frac{1}{Q^{(1+\epsilon)n}} + \Delta, \end{aligned}$$

where we denoted $\Delta = \left|\alpha - \frac{p}{q}\right|$. Using $2q \leq Q^{\epsilon n}$ our last inequality implies

$$\frac{1}{qQ^n} \leq \Delta + \frac{1}{2qQ^n}$$

and hence

$$\Delta > \frac{1}{2qQ^n}$$

Now we use $2q > Q^{\epsilon(n-2)}$ to derive $Q^n < (2q)^{n/(\epsilon(n-2))}$. Hence

$$\left|\alpha - \frac{p}{q}\right| = \Delta \geq (2q)^{-1-\frac{n}{\epsilon(n-2)}}$$

As q increases, the number n increases and $n/(n-2) \to 1$. Thus we see that there cannot be infinitely many approximations of α of quality $1 + \frac{1}{\epsilon} + \delta$ for any fixed $\delta > 0$. We conclude, $\mu(\alpha) \leq 1 + \frac{1}{\epsilon}$. **qed**

4 Constructing rational approximations of π

In this section we shall make a number of attempts to construct good quality approximations to π. In presenting several attempts, only one of which is successful, we hope to illustrate the elusiveness of π with respect to explicit rational approximation.
Our first approach begins with the consideration of integrals of the form

$$J(F) = \int_0^1 \frac{F(t)}{1+t^2}\,\mathrm{d}t$$

where $F \in \mathbb{Z}[t]$ with the constraint that $F(i) = F(-i)$. The reason is that by a partial fraction expansion we see that

$$\frac{f(t)}{1+t^2} = G(t) + \frac{F(i)-F(-i)}{2i}\frac{t}{1+t^2} + \frac{F(i)+F(-i)}{2}\frac{1}{1+t^2}$$

for some polynomial $G \in \mathbb{Z}[t]$. Assuming now that $F(i) = F(-i)$ we get, using $\int_0^1 \frac{1}{1+t^2}\,\mathrm{d}t = \frac{\pi}{4}$, that

$$J(F) = \int_0^1 G(t)\,\mathrm{d}t + \frac{F(i)}{4}\pi.$$

When $J(F)$ is small, the rational number $-\frac{4}{F(i)}\int_0^1 G(t)\,\mathrm{d}t$ can be considered as a rational approximation of π. In order to make $J(F)$ small it is a good idea to choose F is such a way that it has small values in the interval $[0,1]$. For example $F(t) = t^{4n}(1-t)^{4n}$, whose maximum value on $[0,1]$ is $\frac{1}{256^n}$. Moreover, for this choice of F we get $F(i) = F(-i) = (-4)^n$. Putting

$$J_n := \int_0^1 \frac{t^{4n}(1-t)^{4n}}{1+t^2}\,\mathrm{d}t,$$

we get

$$\begin{aligned}
J_1 &= -\pi + \frac{22}{7}\\
J_2 &= 4\pi - \frac{188684}{15015}\\
J_3 &= -16\pi + \frac{431302721}{8580495}\\
J_4 &= 64\pi - \frac{5930158704872}{29494189725}\\
&\vdots \qquad \vdots
\end{aligned}$$

Without proof we mention here that the qualities of the resulting approximations eventually tend to $\log 4/\log(2e^8) \approx 0.738$. So this is not enough to get either an irrationality proof of π, nor an irrationality measure. Notice by the way that $J_1 = -\pi + \frac{22}{7}$, which gives one of the two famous approximations of π. Since I_1 is the integral of a positive function we see that $J_1 > 0$ and thus we have a proof of the fact that $\pi \neq \frac{22}{7}$. It is not clear whether there exists a natural choice of F which produces the approximation $\frac{355}{113}$.

I have tried a number of other choices of F, but they also did not give the desired infinite sequence of quality > 1 approximations to π. The reader is hereby invited to make a number of attempts for his or herself. Programs like Maple and Mathematica can be very helpful such experiments.

There is a small extension of the previous idea, namely to consider integrals of the form

$$J(F,m) = \int_0^1 \frac{F(t)}{(1+t^2)^{m+1}}\,\mathrm{d}t$$

where $m \in \mathbb{Z}_{\geq 0}$ and $F \in \mathbb{Z}[t]$. Having higher powers of $1+t^2$ in the denominator makes the integrand even smaller. In addition, the exponent $m+1$ turns out to have a positive influence on the size of the denominators of the approximations. In general $J(F,m)$ evaluates to

$$a_{F,m} + b_{F,m}\pi + c_{F,m}\log 2,$$

where $a_{F,m}, b_{F,m}, c_{F,m}$ are rational numbers.
The following evaluations of $b_{F,m}$ and $c_{F,m}$ are an exercise for the reader who feels challenged by them.

i) $c_{F,m} = -\frac{1}{2}\text{residue}_{t=\infty}\ \left[\frac{F(t)}{(1+t^2)^{m+1}}\right]$.

ii) $b_{F,m} = \frac{i}{2}\text{residue}_{t=i}\ \left[\frac{F(t)}{(1+t^2)^{m+1}}\right]$.

iii) $c_{F,m} = 0$ if $F \in \mathbb{Z}[t^2]$ or $\text{degree}(F) \leq 2m$.

We remind the reader that the residue of a rational function G at $t = \infty$ is minus the coefficient of t^{-1} in the Laurent expansion of G in increasing powers of t^{-1} Using this remark it is hopefully clear that (iii) is a direct consequence of (i).
It is clear that for the construction of rational approximations of π we must get rid of the $\log 2$ term. So we want that $c_{F,m} = 0$. By property (iii) we see that this certainly holds for integrals of the form

$$J_n := \int_0^1 \frac{t^n(1-t)^n}{(1+t^2)^{n+1}}\,\mathrm{d}t$$

Then

$$\begin{aligned} J_1 &= -\frac{1}{8}\pi + \frac{1}{2} \\ J_2 &= \frac{1}{8}\pi - \frac{3}{8} \\ J_3 &= -\frac{1}{8}\pi + \frac{19}{48} \\ J_4 &= \frac{17}{128}\pi - \frac{5}{12} \\ J_5 &= -\frac{37}{256}\pi + \frac{109}{240} \\ &\vdots \qquad \vdots \end{aligned}$$

This gives us the approximations

$$\frac{4}{1}, \frac{3}{1}, \frac{19}{6}, \frac{160}{51}, \frac{1744}{555}, \frac{644}{205}, \ldots$$

of π. It turns out that roughly the first thirty approximations in this sequence do have quality > 1. So this looks promising. Unfortunately in the long run the asymptotics have decided otherwise and the qualities go to the value $0.9058\ldots$ as $n \to \infty$. As we see, the result is better than the previous attempt, but 0.9058 is still not larger than 1.

It is not entirely without reason that we mention the above approximations. They are closely related to continued fractions. Consider the classical continued fraction

$$\arctan x = \cfrac{x}{1 + \cfrac{x^2}{3 + \cfrac{(2x)^2}{5 + \cfrac{(3x)^2}{7 + \ddots}}}}$$

Take $x = 1$ to obtain

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \ddots}}}}$$

The first few convergents read:

$$\frac{4}{1} = 4, \quad \cfrac{4}{1 + \cfrac{1^2}{3}} = 3, \quad \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5}}} = \frac{19}{6}$$

$$\cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7}}}} = \frac{160}{51}$$

Notice that these are exactly the approximations we got with the integrals J_n. Again, the reader with some experience in continued fractions and partial integration may try to show these equalities.

Let us now point out a small side track. Take $x = \frac{1}{\sqrt{3}}$ in our continued fraction for $\arctan x$. We obtain,

$$\frac{\pi}{\sqrt{3}} = \cfrac{2}{1 + \cfrac{1^2/3}{3 + \cfrac{2^2/3}{5 + \cfrac{3^2/3}{7 + \ddots}}}}$$

The first few convergents are:

$$2, \frac{9}{5}, \frac{49}{27}, \frac{185}{102}, \frac{5387}{2970}, \ldots$$

It turns out the qualities of these approximations to $\frac{\pi}{\sqrt{3}}$ tend to $1.1368\cdots$ as $n \to \infty$. So we see that for $\frac{\pi}{\sqrt{3}}$ it is possible to get a sequence of good quality approximations. In a similar vein, R.Apéry discovered in 1978 an infinite sequence of good quality approximations to π^2. So we see that the numbers $\frac{\pi}{\sqrt{3}}$ and π^2 are better amenable to explicit rational approximations. However, what we really want is π! This is a somewhat frustrating situation and many people have tried to give good explicit approximations. As we remarked before, it was not until 1993 that M.Hata succeeded in doing so.

The approximations we present now, are a variation on Hata's ideas. They have the property that they fall more into the line of approach we have adopted in this section. The resulting approximations are weaker than the ones obtained by Hata, but they do have the desired qualities > 1.

Consider the integrals

$$J_n = \int_{-1}^{1} \frac{t^{2n}(1-t^2)^{2n}}{(1+it)^{3n+1}}\,\mathrm{d}t. \tag{1}$$

This way of writing J_n is more in line with Hata's approach. Split the interval $[-1, 1]$ into its positive and negative part. Substitute $t \to -t$ on the negative part, and we obtain an integral of the form we are looking for,

$$J_n = \int_0^1 \frac{t^{2n}(1-t^2)^{2n}\left((1+it)^{3n+1} + (1-it)^{3n+1}\right)}{(1+t^2)^{3n+1}}\,\mathrm{d}t.$$

The first few values read:

$$\begin{aligned}
J_1 &= 14\pi - 44\\
J_2 &= 968\pi - \frac{45616}{15}\\
J_3 &= 75920\pi - \frac{1669568}{7}\\
J_4 &= 6288296\pi - \frac{9778855936}{495}\\
&\vdots\\
J_{40} &= a_{40}\pi - \frac{b_{40}}{c_{40}}\\
&\vdots
\end{aligned}$$

Note that J_1 gives us the aproximation $\frac{22}{7}$ again! We have not written down the values of the integers a_{40}, b_{40}, c_{40} but we would like to mention some peculiarities:

$$\begin{aligned}
&\gcd(a_{40}, b_{40}) = 2^{23}\\
&c_{40} = 3 \cdot 5 \cdot 7^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot \cdot 29 \cdot 37 \cdot 41 \cdot 43 \cdot 47 \cdot\\
&\qquad \cdot 53 \cdot 59 \cdot 83 \cdot 89 \cdot 97 \cdot \cdot 101 \cdot 103 \cdot 107 \cdot 109 \cdot 113
\end{aligned}$$

In general, let us write $J_n = a_n\pi - \frac{b_n}{c_n}$ with a_n, b_n, c_n integers and $\gcd(b_n, c_n) = 1$. It turns out that both a_n and b_n are divisible by $2^{[n/2]}$ for all n. So this means that $J_n/2^{[n/2]}$ gives us the same approximations as J_n, but its absolute value is much smaller. Secondly, we can show that c_n divides the lowest common multiple of the numbers $2, 3, \ldots, 3n$. And so c_n will be composed of many primes $\leq 3n$, as we can see from the example c_{40}. However, observe that in the factorisation of c_{40} the primes between 60 and 80 are missing. This is no coincidence. Hata discovered that in general the primes between $3n/2$ and $2n$ do not occur in the factorisation of c_n. This absence of primes improves our estimates for the size of the denominator of the approximation for π by a small amount. But it is precisely the right amount to enable us to get approximations of quality > 1. It turns out that the integrals $J_1, J_2, \ldots$ give us the sequence of approximations

$$\frac{22}{7}, \frac{5702}{1815}, \frac{104348}{33215}, \ldots$$

of π whose qualities exceed 1.0449 in the long run!
Supposing this sequence satisfies all other assumptions of our Proposition on irrationality measures, we get an irrationality measure of $1 + \frac{1}{0.0449} = 23.271\ldots$. Of course this is worse than the measure given by Hata. The construction used by Hata is somewhat different from the type of integral we considered in this section. Readers interested in this original construction are warmly encouraged to have a look at Hata's paper. The main point of the present section was to point out that a judicious choice of $F(t)$ may after all produce a sequence of good quality π-approximations. In fact, I spent considerable effort to find other choices of $F(t)$ which would give better quality approximations than those from J_n. This would have been a nice result on the occasion of Pi-day. Unfortunately I was not clever enough to find such $F(t)$. However, given the fact that the possibility of such a choice was proved only around 1993, one should not give up hoping for an improvement. It would actually be a very nice surprise if a reader of this article is more successful in finding such a sequence of better approximations.

5 Bibliography

Here we collect a number of books and URL's concerning π. First a number of books on π-folklore.

David Blatner, *The joy of π*, Walker Publishing 1997
This small book is notable for its appearance (a square) and a very rich layout. The main theme of the book, as suggested by the title, is the joy which one can have working with π. So we find lots of anecdotes, history and citations. The underlying mathematics is hardly exposed. A Dutch high school education would suffice as a suitable background.

Peter Beckmann, *A history of π*, St.Martin's Press, 1971.

Because of its publication date, the important developments from 1971 and later are missing. Nevertheless it is an interesting book for anyone concerned with the history of π. In particular the author's opinions about the scientific contributions of the Romans and medieval Christianity are merciless.

Jean-Paul Delahaye, *Le fascinant nombre* π, Pour la Science, Belin, 1997.
This book, in French, is a carefully composed book and a very rich source of π-information. The author is experienced in writing about math for a larger audience. We recognize this in the choice of subjects and the presentation of this book. On the other hand the author does not lose sight of the underlying mathematics. A real winner for anyone who likes to become acquainted with π-knowledge. A first year university background suffices.

Pierre Eymard, Jean-Pierre Lafon, *Autour du nombre* π, Hermann, 1999
This book contains an enormous amount of information but is more encyclopedic than the other books. The emphasis is on the mathematical background of anything related to π, including the more difficult points such as Ramanujan's formulae and the Gauss-Salamin algorithm. Both of these require knowledge of elliptic functions, their periods and quasi-periods, something which is avoided in most other books. A university education in mathematics is probably the required background for reading.

Lennart Berggren, Jonathan Borwein, Peter Borwein, *Pi: a source book*, Springer Verlag 1997 (2nd edition 1999).
Two of the authors, the brothers Borwein, have made considerable contributions to the mathematics behind π in the last fifteen years. This book is a collection of reprinted original publications about relevant developments around π in the last three millenia. Although a number of articles can be understood without too much mathematical background, the book is primarily of interest to readers with a university background. For π-experts this book is a 'must'. The second edition contains a reprint of Hata's article which was the inspiration for the present article.

J.M.Borwein, P.Borwein, *Pi and the AGM, A study in analytic number theory and computational complexity*, Wiley-Interscience, 1998.
This book explains the background of Ramanujan's formulas and the AGM-algorithm to compute π. As you probably know, Ramanujan himself provided hardly any details as to his method. The authors provide those details and along the way they also find new algorithms for the computation of π which are orders of magnitude faster than the already fast AGM-algorithm. A background in university mathematics is required.

F.Beukers, *Pi*, Epsilon, fall 2000. This booklet, written in Dutch, will appear in the fall of 2000 in the ZEBRA series. This series is published by Epsilon publishers in cooperation with the Dutch society of math teachers (NVvW). The books in ZEBRA are aiming at high school students who need to occupy about 40 hours of their math program with a subject of their own choosing. Consequently, this book is a light introduction to the first properties of π and its uses. Needless to say, it is suitable for anyone with a beginning knowledge of calculus in one variable.

It is very easy to find π-information on the world wide web. Starting points can be the websites of the Borwein brothers:

`http://www.cecm.sfu.ca/personal/jborwein/pi_cover.html`

`http://www.cecm.sfu.ca/personal/pborwein.`

On the latter one can find the slides that Peter Borwein used for his talks on Pi-day. Then there is the less serious

`http://www.go2net.com/useless/useless/pi/pi_pages.html.`

Of course there are many more sites, but the present ones may serve as good points of departure.