Consequences of Apéry's work on $\zeta(3)$

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1 Introduction

At the end of the 1970's it seemed that my fate as a young beginning research mathematician was closely linked with work of Roger Apéry. My first acquaintance with his work was not through $\zeta(3)$, but through the diophantine equation $x^2 + D = p^n$ in the unknown integers x, n, where D, p are given integers with p prime. This happened to be my thesis subject and the papers [A1,A2] turned out to be two very short but relevant papers on the subject. It was therefore a nice surprise for me to see Apéry "live" during the Journées Arithmetiques in 1978 in Luminy. This surprise became excitement with Apéry's announcement of his proof of $\zeta(3) \notin \mathbb{Q}$ and ended in utter confusion after hearing his famous lecture.
The ensuing history has been told in several other places in a much better way than I would be able to do. See [P],[MF]. Let me only say that it has been my good fortune to find a very simple version of the proof a few months after Apéry's announcement, see [B1]. As is well known, many people, including myself, have tried to generalise this simplified proof to obtain irrationality of $\zeta(5)$ or some other numbers of interest, like Catalan's constant. Ironically all generalisations tried so far did not give any new interesting results. Only through a combination of miracles such generalisations seem to work, which in practice means that we fall back to $\zeta(2)$ or $\zeta(3)$ again. In the early 1980's several papers of mine have dealt with such generalisations and with properties of the numbers $\sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2$ which occur in Apéry's proof. I would like to take the opportunity to give a short overview of these results in this paper.

2 Irrationality

Let us recall the plan of Apéry's irrationality proof. Consider the recurrence relation

$$(n+1)^3 u_{n+1} = (34n^3 + 51n^2 + 27n + 5)u_n - n^3 u_{n-1}$$

The solution with starting values $u_0 = 1,\ u_1 = 5$ has the peculiar property that it has integral terms, despite the fact that at every recursion step we divide by $(n+1)^3$. The n-th term is given by

$$\sum_{k=0}^{n} \binom{n+k}{k}^2 \binom{n}{k}^2.$$

We call these the *Apéry numbers* and denote them by a_n. The terms of the solution with starting values $0, 6$ are denoted by c_n. We can show that $l_n^3 c_n \in \mathbb{Z}$, where l_n denotes the lowest common multiple of $1, 2, 3, \ldots, n$. Using the prime number theorem we can show that $l_n = O_\epsilon(e^{(1+\epsilon)n})$ for any $\epsilon > 0$. Genaral theory of recurrences tells us that $\lim_{n\to\infty} c_n/a_n$ exists and Apéry showed that the limit is $\zeta(3)$. In fact, a more detailed analysis shows that

$$0 < c_n - a_n\zeta(3) < (\sqrt{2}-1)^{4n}.$$

Suppose $\zeta(3)$ were rational. Then we would have $c_n - a_n\zeta(3) >> l_n^{-3}$. However, this contradicts the upper bound since $(\sqrt{2}-1)^{4n} < e^{-3}$. Hence $\zeta(3)$ is irrational.
Now consider the generating function of the Apéry numbers $A(t) = \sum_{n=0}^{\infty} a_n t^n$. Because of the recurrence relation for the a_n, the function $A(t)$ satisfies the third order differential equation

$$(t^4 - 34t^3 + t^2)y''' + (6t^3 - 153t^2 + 3t)y'' + (7t^2 - 112t + 1)y' + (t-5)y = 0.$$

This differential equation turns out to be the symmetric square of a second order differential equation with coefficients in $\mathbb{C}(t)$. This basically means that the solution space of the third order equation is spanned, over the constants, by the squares of the second order equation. According to B.Dwork this observation had already been made by Apéry and the second order equation reads

$$(t^3 - 34t^2 + t)y'' + (2t^2 - 51t + 1)y' + \frac{1}{4}(t-10)y = 0$$

In particular, function $\sqrt{(A(t)}$ is a solution of the second order equation. Let us write $\sqrt{(A(t)} = \sum_{n=0}^{\infty} v_n t^n$. Then it is not hard to see that $4^n v_n \in \mathbb{Z}$ for all n and

$$(n+1)^2 v_{n+1} = (34n^2 + 17n + 5/2)v_n - (n-1/2)^2 v_{n-1}$$

for $n \geq 1$. Denote the solution of the recurrence with starting values $0, 1$ by w_n. General theory of recurrences tells us that $\lim_{n\to\infty} w_n/v_n$ exists. Call it α. For people who are into the business of irrationality proving the observation that $(\sqrt{2}-1)^4 < 1/4e^2$ suffices to recognize that our recurrence actually gives us an irrationality proof for α along the lines sketched above. However, this time it is not so clear what α is. We can compute it numerically up to arbitrary precision and the first thing I did was to compare its value with the values of "interesting" numbers like Euler's constant or Catalan's constant. But to no avail. Later, in [B2], I was able to give some sort of series expansion for α. Consider the infinite product

$$q\prod_{n=1}^{\infty}(1-q^n)^3(1-q^{3n})^3\frac{(1+q^{3n})^{9/2}}{(1+q^n)^{3/2}}$$

Denote its power series expansion by $\sum_{n\geq 1}\alpha_n q^n$. Then we have $\alpha = \sum_{\geq 1}\alpha_n/n^2$. Let me be the first to admit that this is not an interesting number!
Let us return to our second order differential equation. It has the four singularities $t = 0, (1\pm\sqrt{2})^4, \infty$. It is a Fuchsian equation with the following table of local exponents.

point	exponents
0	$0, 0$
$(1\pm\sqrt{2})^4$	$0, 1/2$
∞	$1/2, 1/2$

Consider the rational function $x(1-9x)/(1-x)$. It has degree 2 and ramifies above the points $(1\pm\sqrt{2})^4$ at $x=(3\pm 2\sqrt{2})/3$. Replace t in our second order equation by $x(1-9x)/(1-x)$. Then the points $x=(3\pm 2\sqrt{2})/3$ will become regular points of the new differential equation and the values of x above $t=0,\infty$ give us four singular points of the new equation. It has Riemann scheme,

point	exponents
0	$0,0$
1	$1/2,1/2$
$1/9$	$0,0$
∞	$1/2,1/2$

Replacing y by $(1-x)^{1/2}y$ gives us a second order equation with Riemann scheme

point	exponents
0	$0,0$
1	$0,0$
$1/9$	$0,0$
∞	$1,1$

which reads

$$x(x-1)(9x-1)y''+(27x^2-20x+1)y'+(9x+.......)y=0.$$

This is precisely the Picard-Fuchs equation of the modular family of elliptic curves accociated to $\Gamma_1(6)$. It is at this point that the relation of Apéry's irrationality proof with modular forms becomes clear. By rephrasing everything in terms of modular forms one obtains a new principle where one derives irrationality results for periods of modular forms. This is the subject matter of the article [B2]. Unfortunately it turns out that in only a few cases all requirements are met to indeed produce irrationality results.
We can also reverse the above procedure. Start with the recurrence relation

$$(n+1)^2u_{n+1}=(11n^2+11n+3)u_n+n^2u_{n-1}$$

which Apéry used to prove irrationality of $\zeta(2)$. In particular this recurrence has the solution $b_n=\sum_{k=0}^n\binom{n}{k}^2\binom{n+k}{k}$. Its generating function $B(x)=\sum_{n=0}^\infty b_nt^n$ satisfies the second order differential equation

$$x(x^2+11x-1)y''+(3x^2+22x-1)y'+(x+3)y=0.$$

In [B3] it is shown that this is the Picard-Fuchs equation of the modular family of elliptic curves corresponding to $\Gamma_1(5)$. Let $\lambda=-11/2-5\sqrt{5}$. Then $(1-x/\lambda)B(x)^2$ is solution of a 3rd order differential equation. Then use the independent variable $t=x(\lambda x+1)/(x-\lambda)$ instead of x to obtain the equation

$$\begin{aligned} t^2(t^2-2(124+55\sqrt{5})t+1)y'''+(6t^3-(1116+495\sqrt{5})t^2+3t)y''+ \\ +(7t^2-(812+360\sqrt{5})t+1)y'+(t-34-15\sqrt{5})y &= 0 \end{aligned}$$

There is also a corresponding recurrence relation which we prefer not to write down. However, it can be used (see [B2]) to prove that $8\zeta(3) - 5\sqrt{5}L(3,\chi) \notin \mathbb{Q}(\sqrt{5})$, where $L(s,\chi) = \sum_{n\geq 1}\left(\frac{n}{5}\right)/n^s$.
Finally, without giving any details, the Chudnovsky's [C] point out that there several other recurrences which may lead to an irrationality proof, the problem being the exact meaning of the irrational number. Such recurrences arise from Gauss-Manin systems of Shimura families of abelian surfaces having multiplication by a quaternion algebra over $\mathbb{Q}$.

3 Congruences

Another subject, which kept me busy for some time, is that of congruences for the Apéry numbers a_n, b_n defined in the previous section. Very soon after Apéry's irrationality proof it was realised by several authors, in particular Chowla, Cowles and Gessel, that the numbers a_n, b_n satisfy some interesting congruences. For example, for any prime $p \geq 5$ we have $a_{p-1} \equiv 1 \pmod{p^3}$, $b_{p-1} \equiv 1 \pmod{p^3}$. Later these congruences were generalised in [B4] to

$$\begin{aligned} \forall\, m, r \in \mathbb{N} : a_{mp^r-1} &\equiv a_{mp^{r-1}-1} \pmod{p^{3r}} \\ b_{mp^r-1} &\equiv b_{mp^{r-1}-1} \pmod{p^{3r}} \end{aligned}$$

The proof of these congruences is extremely tedious and it would be nice to have a more conceptual proof.
In [SB] the numbers b_n are related to the zeta function of certain $K3$-surfaces. As a result the following congruences were found,
For any $m, r \in \mathbb{N}$ with m odd and any odd prime we have

$$b_{(mp^r-1)/2} - \beta_p b_{(mp^{r-1}-1)/2} + p^2 b_{(mp^{r-2}-1)/2} \equiv 0 \pmod{p^r}$$

where β_p is obtained from the Taylor series expansion of $q\prod_{n=1}^{\infty}(1-q^{4n})^6 = \sum_{n\geq 1}\beta_n q^n$. This is a q-series expansion of a modular of weight 3. Moreover, the modular form is a theta series corresponding to the quadratic form $x^2 + y^2$ and we have $\beta_p = 0$ if $p \equiv 3 \pmod 4$ and $\beta_p = 4a^2 - 2p$ if $p = a^2 + b^2$, a odd. Numerical evidence suggests that the congruences hold modulo p^{2r} instead of p^r.
The phenomenon that some congruences are true modulo higher powers than expected from general theory occurs at several places. For example, consider an elliptic curve E over $\mathbb{Q}$ given in its Weierstrass normal form. Let $z = x/y$ be the local parameter at the origin of E and consider the series expansion of the holomorphic differential form,

$$\omega = \sum_{n=1}^{\infty} f_n z^{n-1} dz$$

Then we have the so-called Atkin Swinnerton-Dyer congruences. Let p be any prime for which E has good reduction.

$$\forall m, r : \; f_{mp^r} - t_p f_{mp^{r-1}} + p f_{mp^{r-2}} \equiv 0 \pmod{p^r}$$

where t_p is the trace of Frobenius on $H^1_{\rm et}(E)$. In case E has ordinary reduction at p these congruences can be rephrased as

$$\forall m, r: \ f_{mp^r} \equiv \tau_p f_{mp^{r-1}} \pmod{p^r}$$

where τ_p is the unit eigenvalue of the Frobenius map. It was shown by M.Coster and L.van Hamme that the congruences hold modulo p^{2r} instead of p^r if E has complex multiplication, see [CH] and [C] where one finds also several other references.
Motivated by the congruences for b_n I wondered if something similar holds for Apéry's numbers a_n. After a few trials the right guess was found and using the modular interpretation of the a_n it did not take long to prove the following statement (see [B5]).
Let

$$q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4 = \sum_{n\ge 1}\alpha_n q^n$$

Then, for any $m, r \in \mathbb{N}$, m odd we have

$$a_{(mp^r-1)/2} - \alpha_p a_{(mp^{r-1}-1)/2} + p^3 a_{(mp^{r-2}-1)/2} \equiv 0 \pmod{p^r}$$

Numeric experiment again seems to suggest that these congruences hold modulo p^{2r}.
Finally let me end with a statement of which I have absolutely no idea how to prove. Write n is base 5. Count the total number of 1's and 3's in this representation. Call it q. Then 5^q divides a_n. Similarly, write n in base 11. Count the total number of 5's in this representation. Then 11^q divides a_n. I would be very interested in having a proof of these statements.

4 References

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