

# NOTES ON IRRATIONALITY AND TRANSCENDENCE

Frits Beukers

September 11, 2007

## 1 Introduction

### 1.1 Irrationality

**Definition 1.1.1** Let  $\alpha \in \mathbb{C}$ . We call  $\alpha$  irrational when  $\alpha \notin \mathbb{Q}$ .

Proving irrationality and transcendence of numbers is now being considered as a branch of number theory, although the techniques that are used involve subjects like complex analysis, linear differential equations and algebraic geometry. In this chapter we shall restrict ourselves to examples in which only elementary methods are used.

The easiest numbers for which irrationality can be proved are the algebraic numbers.

**Theorem 1.1.2** Let  $\alpha$  be a zero of a polynomial  $x^m + c_1x^{m-1} + \dots + c_m \in \mathbb{Z}[x]$ . Then  $\alpha$  is either irrational or  $\alpha \in \mathbb{Z}$ . In the latter case we have  $\alpha | c_m$ .

**Proof.** Suppose  $x^m + c_1x^{m-1} + \dots + c_m$  has a rational zero  $p/q$  with  $p, q \in \mathbb{Z}$ ,  $(p, q) = 1$ ,  $q > 0$ . Then  $p^m + c_1p^{m-1}q + \dots + c_mq^m = 0$ . Hence  $q | p^m$  and since  $(p, q) = 1$  this implies  $q = 1$ . Notice that  $p^m + c_1p^{m-1} + \dots + c_m = 0$  now implies that  $p | c_m$ .  $\square$

**Corollary 1.1.3** Let  $m \in \mathbb{N}$ . If  $N \in \mathbb{Z}$  is not the  $m$ -th power of an integer then  $\alpha = \sqrt[m]{N}$  is irrational.

**Proof.** Notice that  $\alpha^m - N = 0$ . If  $\alpha \in \mathbb{Q}$  then  $\alpha \in \mathbb{Z}$  according to Theorem 1.1.2 and hence  $N$  is the  $m$ -th power of an integer. This contradicts our assumptions, hence  $\alpha \notin \mathbb{Q}$ .  $\square$

**Theorem 1.1.4** Let  $e$  be the base of the natural logarithm. Then  $e$  is irrational.

**Proof.** Suppose  $e$  is rational with denominator  $d$ . We use the series expansion

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

The number

$$\alpha_k := e - 1 - \frac{1}{1!} - \frac{1}{2!} - \dots - \frac{1}{k!}$$

is a positive rational number with a denominator dividing  $d(k!)$ . Hence, since  $\alpha_k$  is not zero,

$$\alpha_k \geq \frac{1}{k!d}.$$

On the other hand

$$\begin{aligned} \alpha_k &= \frac{1}{(k+1)!} + \frac{1}{(k+2)!} + \cdots \\ &= \frac{1}{k!} \left( \frac{1}{k+1} + \frac{1}{(k+1)(k+2)} + \cdots \right) \\ &< \frac{1}{k!} \left( \frac{1}{k+1} + \frac{1}{(k+1)^2} + \cdots \right) \\ &= \frac{1}{k!} \frac{1}{k} \end{aligned}$$

contradicting our lower bound for  $\alpha_k$  whenever  $k > d$ . □

For future use we note that the above proof depends on the derivation of a lower bound and an upper bound (for  $\alpha_k$  in this case). The lower bound is based on an arithmetic argument, which in the end comes down to the fact that a positive integer is  $\geq 1$ . The upper bound is derived using an estimate of an infinite series which is analytic in nature. It turns out that every irrationality or transcendence proof, except the proof of Theorem 1.1.2, contains these two estimates.

The irrationality proof of  $\pi$  is more complicated and we require the following lemma.

**Lemma 1.1.5** *Let  $m \in \mathbb{Z}_{\geq 0}$ . Then*

$$\pi \int_0^1 t^m \sin \pi t dt$$

*is a polynomial in  $\pi^{-2}$  with integral coefficients and degree  $[m/2]$ .*

**Proof.** By induction on  $m$ . For  $m = 0$  and  $m = 1$  we have

$$\pi \int_0^1 \sin \pi t dt = 2 \quad \pi \int_0^1 t \sin \pi t dt = 1.$$

Suppose  $m > 1$ . After a two-fold partial integration we obtain

$$\pi \int_0^1 t^m \sin \pi t dt = 1 - \frac{m(m-1)}{\pi^2} \pi \int_0^1 t^{m-2} \sin \pi t dt$$

from which our assertion follows. □

**Theorem 1.1.6** *We have  $\pi^2 \notin \mathbb{Q}$  and  $\pi \notin \mathbb{Q}$ .*

F.Beukers, Irrationality and Transcendence

**Proof.** Define  $P_m(t) = \frac{1}{m!}(\frac{d}{dt})^m t^m (1-t)^m$  and notice that  $P_m(t) \in \mathbb{Z}[t]$ . Consider the integral

$$I_n = \pi \int_0^1 (\sin \pi t) P_{2n}(t) dt.$$

After a  $2n$ -fold partial integration we find

$$I_n = \pi(-1)^n \frac{\pi^{2n}}{(2n)!} \int_0^1 (\sin \pi t) t^{2n} (1-t)^{2n} dt.$$

Hence

$$0 < |I_n| < \frac{\pi^{2n+1}}{(2n)!}.$$

On the other hand we can compute  $I_n$  term by term. Lemma 1.1.5 then implies that  $I_n = A_n(1/\pi^2)$ , where  $A_n(x) \in \mathbb{Z}[x]$  and  $\deg A_n \leq n$ .

Suppose that  $\pi^2 = a/b$  for some  $a, b \in \mathbb{N}$ . Because  $A_n$  has degree  $\leq n$  the number  $I_n = A_n(1/\pi^2) = A_n(b/a)$  is a fraction whose denominator divides  $a^n$ . Moreover,  $I_n \neq 0$ . Hence

$$|I_n| \geq \frac{1}{a^n}.$$

Together with the upper bound for  $|I_n|$  this implies

$$\frac{1}{a^{2n}} < \frac{\pi^{2n+1}}{(2n)!}$$

which becomes impossible when  $n \rightarrow \infty$ . Hence  $\pi^2 \notin \mathbb{Q}$ . This immediately implies  $\pi \notin \mathbb{Q}$ .  $\square$

Around 1740 Euler proved  $e$  to be irrational and the first proof of the irrationality of  $\pi$  was given by Lambert in 1761. This proof was based on the continued fraction expansion of  $\operatorname{arctg}(x)$ . The proof we gave can be considered as a variation of a proof given by I.Niven. On the other hand there are many 'naturally' occurring numbers for which no irrationality results are known. For example, it is not known whether Euler's constant  $\gamma = \lim_{n \rightarrow \infty} (\sum_{k=1}^n (1/k) - \log n)$  or  $e + \pi$  or  $e\pi$  is irrational. Motivated by the standard series for  $e$  P.Erdős asked the following question. Is

$$\sum_{k=1}^{\infty} \frac{1}{k! + 1}$$

irrational? Surprisingly this seems to be difficult to answer.

## 1.2 Transcendence

**Definition 1.2.1** A number  $\alpha \in \mathbb{C}$  is called algebraic if it is the zero of a non-trivial polynomial with coefficients in  $\mathbb{Q}$ . A number is called transcendental if it is not algebraic.

Obviously, proving transcendence of a number is much harder than proving irrationality. It is therefore no surprise that in the beginning of the 19th century no examples of transcendental numbers were known. In 1844 Liouville gave the first examples of transcendental numbers.

**Theorem 1.2.2 (Liouville 1844)** *The number*

$$\alpha = \sum_{k=0}^{\infty} \frac{1}{2^{k!}}$$

*is transcendental.*

**Proof** Suppose  $\alpha$  is algebraic. Let  $P(x) \in \mathbb{Z}[x]$  be the minimal polynomial of  $\alpha$  and assume its degree is  $D$ . It is a consequence of the mean value theorem there exists  $M > 0$  such that for any  $x \in [\alpha - 1, \alpha + 1]$  we have

$$|P(x)| = |P(x) - P(\alpha)| \leq M|x - \alpha|.$$

We can take  $M = \max_{\xi \in [-1, 1]} |P'(\alpha + \xi)|$ . Let now  $\alpha_n$  be the value of the truncated series

$$\alpha_n = \sum_{k=0}^n \frac{1}{2^{k!}}.$$

Notice that

$$\begin{aligned} |\alpha - \alpha_n| &= 2^{-(n+1)!} + 2^{-(n+2)!} + \dots \\ &\leq 2^{-(n+1)!}(1 + 1/2 + 1/4 + \dots) = 2 \cdot 2^{-(n+1)!} \end{aligned}$$

Once again we derive two estimates, this time for  $|P(\alpha_n)|$ . Note that  $\alpha_n$  is a rational number with denominator  $2^{n!}$ . Since  $P$  has at most finitely many zeros we have that  $P(\alpha_n) \neq 0$  for sufficiently large  $n$ . Since  $P(\alpha_n)$  is rational with denominator dividing  $2^{Dn!}$  we conclude that  $|P(\alpha_n)| \geq 2^{-Dn!}$  for sufficiently large  $n$ . On the other hand, by application of the mean value theorem and our series estimate,

$$|P(\alpha_n)| \leq M|\alpha_n - \alpha| \leq 2M2^{-(n+1)!}.$$

Putting the inequalities together we get that

$$2^{-Dn!} \leq 2M2^{-(n+1)!}$$

for sufficiently large  $n$ . Hence  $2^{(n+1-D)n!} \leq 2M$  for sufficiently large  $n$  which gives a contradiction.  $\square$

Of course this idea of proof can be applied to any number of the form

$$\sum_{n=0}^{\infty} q^{-a_n}$$

where  $q \in \mathbb{Z}_{\geq 2}$  and  $a_n$  is an increasing sequence of positive integers such that  $\lim_{n \rightarrow \infty} a_{n+1}/a_n = \infty$ . This enabled Liouville to construct infinitely many examples of transcendental numbers.

Through the pioneering work of Cantor on set theory around 1874 it also became clear that ‘almost all’ real numbers are transcendental. This follows from the following two theorems.

F.Beukers, Irrationality and Transcendence

**Theorem 1.2.3** *The set of algebraic numbers is countable.*

**Proof** It suffices to show that the set  $\mathbb{Z}[X]$  is countable. To any polynomial  $P(X) = p_n X^n + p_{n-1} X^{n-1} + \cdots + p_1 X + p_0 \in \mathbb{Z}[X]$  with  $p_n \neq 0$  we assign the number  $\mu(P) = n + |p_n| + |p_{n-1}| + \cdots + |p_0| \in \mathbb{N}$ . Clearly for any  $N \in \mathbb{N}$  the number of solutions to  $\mu(P) = N$  is finite, because both the degree and the size of the coefficients are bounded by  $N$ . Hence  $\mathbb{Z}[X]$  is countable.  $\square$

**Theorem 1.2.4 (Cantor)** *The set of real numbers is uncountable.*

**Proof** We will show that the set of real numbers in the interval  $[0, 1)$  is uncountable. Suppose that this set is countable. Choose an enumeration and denote the decimal expansion of the  $n$ -th real number by  $0.a_{n1}a_{n2}a_{n3}\cdots$ , where  $a_{nm} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$  for all  $n, m$ . Now consider the real number  $\beta$  whose decimal expansion reads  $0.b_1b_2b_3\cdots$  where the  $b_i$  are chosen such that  $b_i \neq a_{ii}$  for every  $i$ . This choice implies that  $\beta$  does not occur in our enumeration. Hence  $[0, 1)$  is uncountable.  $\square$

The principle of the proof of Theorem 1.2.4 is known as *Cantor's diagonal procedure* and it occurs in many places in mathematics.

Almost all real numbers being transcendental, it seems ironic that until the end of the 19-th century not a single 'naturally occurring' number was known to be transcendental. Only in 1873 Hermite showed that  $e$  is transcendental and in 1882 Lindemann proved  $\pi$  to be transcendental. Both these results are contained in the following Theorem.

**Theorem 1.2.5 (Lindemann-Weierstrass, 1885)** *Let  $\alpha_1, \dots, \alpha_n$  be distinct algebraic numbers contained in  $\mathbb{C}$ . Then the numbers  $e^{\alpha_1}, \dots, e^{\alpha_n}$  are linearly independent over  $\mathbb{Q}$ .*

In the 1920's C.L.Siegel extended this work widely to a much larger class of functions than the exponential function.

In his famous lecture of 1900 D.Hilbert asked whether numbers of the form  $a^b$  with  $a, b$  algebraic,  $a \neq 0, 1$  and  $b \notin \mathbb{Q}$ , are transcendental. Specific examples are  $2^{\sqrt{2}}$  and  $i^{-2i} = e^\pi$ . This problem, known as Hilbert's 23rd problem, was considered to be very difficult by Hilbert, but already in the 1930's A.O.Gel'fond and Th.Schneider independently developed techniques to solve this problem. So now we know,

**Theorem 1.2.6 (Gel'fond, Schneider, 1934)** *Let  $a, b$  be algebraic and suppose that  $a \neq 0, 1$  and  $b \notin \mathbb{Q}$ . Then  $a^b$  is transcendental.*

**Corollary 1.2.7** *Let  $\alpha, \beta$  be two positive real algebraic numbers such that  $\beta \neq 1$  and  $\log \alpha / \log \beta \notin \mathbb{Q}$ . Then  $\log \alpha / \log \beta$  is transcendental.*

**Proof** Let  $b = \log \alpha / \log \beta$  and suppose  $b$  is algebraic. Then, according to Theorem 1.2.6 the number  $\alpha = \beta^b$  is transcendental which is impossible since  $\alpha$  is algebraic.  $\square$

Nowadays the Gel'fond-Schneider theory has grown into a field of its own in which large classes of numbers, usually related to algebraic geometry, are known to be transcendental. In addition it turned out that quantification of linear independence results gave rise to applications in diophantine equations. We shall come back to this topic in later lectures.

### 1.3 Exercises

**Exercise 1.3.1** Show that  $\frac{\log 3}{\log 2}$  is irrational.

**Exercise 1.3.2** Prove, using the series expansions for  $e$  and  $e^{-1}$ , that  $e$  is not algebraic of degree 2.

**Exercise 1.3.3** Prove that  $e + \pi$  and  $e\pi$  are not both rational.

**Exercise 1.3.4** Prove that

$$\sum_{n=0}^{\infty} \left(\frac{4}{5}\right)^n \frac{1}{3^{n^2}} \notin \mathbb{Q}.$$

**Exercise 1.3.5** It is a wellknown fact that rational numbers have a decimal expansion which is periodic after some digit.

Show that the so-called Champernowne number

$$0.1234567891011121314151617\dots$$

is irrational (Actually K.Mahler proved it transcendental in the 1930's, but that is much harder).

**Exercise 1.3.6** So far we discussed irrationality and transcendence with respect to the ground field  $\mathbb{Q}$ . But we can also start with  $\mathbb{C}(z)$ , the rational functions, as ground field.

1. Show that  $e^z$  is not in  $\mathbb{C}(z)$ .
2. Let  $a_1, \dots, a_n$  be distinct real numbers. Show that  $e^{a_1 z}, \dots, e^{a_n z}$  are linearly independent over  $\mathbb{C}(z)$ .
3. Show that  $e^z$  is transcendental over  $\mathbb{C}(z)$ .
4. Same question as (2) but now with the  $a_i$  distinct numbers in  $\mathbb{C}$ .