Extra opgaven 12 november

1. Find all solutions $x, y \in \mathbb{Z}$ to $425 = x^2 + y^2$ by factorization of 425 in $\mathbb{Z}[i]$.

2. In this problem we analyse which integers n can be written as the *difference* between two squares. In other words, for which integers n is $x^2 - y^2 = n$ solvable in integers x, y. (Hint to use: $x^2 - y^2 = (x+y)(x-y)$).

   (a) Show that every odd integer can be written as the difference between two squares.

   (b) Show that every multiple of 4 is a difference of two squares.

   (c) Show that if $n \equiv 2 \pmod 4$ then n cannot be written as difference of two squares.

   (d) (Difficult) Denote the number of solutions $x, y \in \mathbb{Z}$ to $n = x^2 - y^2$ by $\delta(n)$. Prove that $\delta(n)$ is a multiplicative function.

3. Let $\epsilon(n)$ be the function defined by $\epsilon(1) = 1$, $\epsilon(n) = 0$ if n is even and $\epsilon(n) = \prod_{i=1}^{t} \left(\frac{-1}{p_i}\right)$ if $n = p_1 p_2 \cdots p_t$ with p_i an odd prime for every i. Prove that

$$\frac{r_2(n)}{4} = \sum_{d|n} \epsilon(d).$$

   You may use Theorem 7.1.2 from the course notes.