# Assignment III

Rational Points on Curves 2006

## Problem 1

Let $(p_0 : p_1 : \ldots : p_n) \in \mathbb{P}^n(\mathbb{Q})$, where $p_0, p_1, \ldots, p_n \in \mathbb{Z}$. Denote the projective height on $\mathbb{P}^n(\mathbb{Q})$ by $H_\mathbb{Q}$. Show that

$$H_\mathbb{Q}(p_0, p_1, \ldots, p_n) = \max_i |p_i| / \gcd(p_0, p_1, \ldots, p_n).$$

## Problem 2

a) Let $k$ be an algebraic number field and let $S$ be a finite set of places (norms) on $k$, including all Archimedean ones. Let $\mathcal{O}_S^*$ be the group of $S$-units. Let $a, b \in k^*$ and consider the equation

$$ax + by = 1 \quad \text{in } x, y \in \mathcal{O}_S^*. \tag{1}$$

Show, using Theorem 2.9 in the notes on Schmidt's subspace theorem, that equation (1) has at most finitely many solutions.

(b) Let $F \in \mathbb{Z}[x, y]$ be a homogeneous polynomial of degree $n$ such that the coefficient of $x^n$ is 1. Let $m \in \mathbb{Z}$ and $m \neq 0$. Consider the diophantine equation

$$F(x, y) = m \quad \text{in } x, y \in \mathbb{Z}. \qquad (2).$$

Let $k$ be the splitting field of $F(x, 1)$. In particular there exist $\alpha_1, \ldots, \alpha_n \in k$ such that $F(x, y) = (x - \alpha_1 y) \cdots (x - \alpha_n y)$. Let $S$ be the set of Archimedean norms of $k$ together with all $\mathcal{P} - adic$ norms on $k$ for which the ideal $\mathcal{P} \subset \mathcal{O}_k$ divides $m$. Let $\alpha$ be a zero of $F(x, 1)$ and $x, y$ a solution of equation (2). Show that $x - \alpha y \in \mathcal{O}_S^*$.

c) Assuming that $F(x, 1)$ has at least three distinct zeros show that (2) has at most finitely many solutions.

## Problem 3

In connection with Waring's problem it is a famous problem to show that

$$||(3/2)^k|| > (3/4)^k$$

for $k \in \mathbb{Z}_{\geq 5}$. Here $||x||$ denotes the distance of $x$ to the nearest integer. It is known that $||(3/2)^k|| > 2^{-0.9k}$. Using Schmidt's subspace theorem one can show that to any $\epsilon > 0$ there exists $k_0(\epsilon)$ such that $||(3/2)^k|| > 2^{-\epsilon k}$ for all $k > k_0$. The problem is to show this.

We proceed as follows. Let $x = ||(3/2)^k||$ and $y = 3^k - x2^k$. Now apply Schmidt's subspace theorem with three suitably chosen linear forms in two variables and $S = \{2, 3, \infty\}$ to prove the statement.
Unfortunately this proof gives us only an existence proof of $k_0$, no actual value that can be computed.