1 Home work assignment I

1.1 $y^2 = x^3 + d^2$

Problem Let $d \neq 0$ and consider the curve $E: \ y^2 = x^3 + d^2$. Let l be the straight line passingthrough a point $(x_0, y_0) \in E$ and $(0, d)$. Compute the coordinates x_1, y_1 of the third point of intersection between l and E.

Solution The line l through (x_0, y_0) and $(0, d)$ is given by

$$y - d = \frac{y_0 - d}{x_0} x.$$

Elimination of y with $y^2 = x^3 + d^2$ gives us

$$x^3 - \left(d + \frac{y_0 - d}{x_0}\right)^2 + d^2 = 0$$

The coefficient of x^2 in this cubic equation the sum of its roots $0 + x_0 + x_1$ on the one hand and on the other hand

$$x_0 + x_1 = \frac{y_0 - d}{x_0}^2.$$

Hence

$$x_1 = \frac{y_0 - d}{x_0}^2 - x_0 = \frac{y_0^2 - 2dy_0 + d^2 - x_0^3}{x_0^2} = 2d\frac{d - y_0}{x_0^2}$$

where we have used $y_0^2 = x_0^3 + d^2$. Using this equation once more we find that $\frac{d-y_0}{x_0^2} = -\frac{x_0}{y_0+d}$. hence

$$x_1 = -2d\frac{x_0}{y_0 + d}.$$

Then y_1 becomes

$$y_1 = \frac{y_0 - d}{x_0} x_1 + d = d\frac{3d - y_0}{y_0 + d}.$$

1.2 $y^2 = x^3 + k$

Problem Let $k \neq 0$ and consider the curve $E : y^2 = x^3 + k$. Choose a point $(x_0, y_0) \in E$ and let l be the tangent of E at this point. Compute the third point of intersection between l and E.

Solution The slope of l is given by $3x_0^2/2y_0$ and the line is

$$y - y_0 = \frac{3x_0^2}{2y_0}(x - x_0).$$

We determine the third point of intersection (x_1, y_1) again by looking at the coefficient of x^2 in the cubic equation

$$x^3 - \left(y_0 + \frac{3x_0^2}{2y_0}(x - x - 0)\right)^2 + k.$$

We get

$$2x_0 + x_1 = \frac{{3x_0^2}^2}{2y_0}$$

from which we deduce

$$x_1 = \frac{9x_0^4}{4y_0^2} - 2x_0 = \frac{x_0^4 - kx_0}{x_0^3 + k}.$$

Furthermore,

$$y_1 = y_0 + \frac{3x_0^2}{2y_0}\left(\frac{x_0^4 - kx_0}{x_0^3 + k} - x_0\right)$$

which equals

$$y_1 = y_0 - \frac{k}{y_0}\left(\frac{3x_0^3}{x_0^3 + k}\right) = \frac{1}{y_0}\left(\frac{x_0^6 - kx_0^3 + k^2}{x_0^3 + k}\right).$$

1.3 Hartshorne, problem 1.1

Part a) Let Y be the plane curve $y = x^2$. Show that $k[Y]$ is isomorphic to a polynomial ring in one variable over k.

Solution Notice that $A[Y] = k[x, y]/(y - x^2)$. We show that this ring is isomorphic to $k[x]$ via the k-algebra homomorphism $\phi : k[x, y] \to k[x]$ given by $x \mapsto x$, $y \mapsto x^2$. Notice that $y - x^2$ is in the kernel of ϕ. Suppose that $P(x, y)$ is in the kernel of ϕ, i.e. $P(x, x^2) \equiv 0$. We show that $P(x, y)$ is divisible by $y - x^2$ from which we can conclude that the kernel is $(y - x^2)$. Notice that for any $P \in k[x, y]$ the difference $P(x, y) - P(x, x^2)$ is divisible by $y - x^2$. In particular, if $P(x, x^2)$ vanishes we can conclude that P is divisible by $y - x^2$, as asserted.

Since the map ϕ is surjective we can apply an isomorphism theorem to conclude that $k[x, y]/(y - x^2) = k[x]$.

Part b) Let Z be the plane curve $xy = 1$. Show that $k[Z]$ is not isomorphic to a polynomial ring.

Solution Note that $k[Z] = k[x, y]/(xy - 1)$. This ring is generated by x, y which are invertible elements in $k[Z]$. Suppose there is an isomorphism of $k[Z]$ with a polynomial ring over k. Then this ring is also generated by invertible elements. However, the only invertible elements of a polynomial ring are the ground field k and such elements do bnot generate the polynomial ring. So there can be no isomorphism.

Part c) Let f be any irredicible quadratic polynomial in $k[x, y]$ and then let W be the conic defined by f. Show that $k[W]$ is isomorphic to $k[Y]$ or $k[Z]$. Which one is it when? We can assume that k is algebraically closed of characteristic zero.

Solution. Let $f = ax^2 + bxy + cy^2 + dx + ey + h$. We apply a linear change of coordinates to bring f into standard form. We distinguish several cases:

- $a = c = 0$. Then $b \neq 0$ and we can assume $b = 1$.Then $f = xy+dx+ey+f$. Substitute $x \to x - e, y \to y - d$ to get $f = xy + h - de$. Since f is irreducible we have $h - de \neq 0$. After the substitution $x \to (de - h)x$ we obtain $f = xy - 1$.
- $a \neq 0$. We assume $a = 1$. Substitute $x \to x - by/2 - d/2$ to obtain $x^2 - \Delta y^2 + e'y + h'$ where $\Delta = c - b^2/4$. We distinguish two cases,
  - $\Delta = 0$. In that case $e' \neq 0$ because f is irreducible. Substitute $y \to (y - h')/e'$ to get $f = x^2 + y$.
  - $\Delta \neq 0$. Substitute $y \to y - e'/2\Delta$ to get $f = x^2 - \Delta y^2 + h''$ with $h'' \neq 0$ because of irreducibility of f. Now replace $x - y\sqrt{\Delta}y$ by x and $x + y\sqrt{\Delta}$ by $-h''y$ to obtain $f = xy - 1$.

Thus we see that $k[W]$ is isomorphic to $k[Y]$ if and only if $\Delta = 0$. In other words $b^2 - 4ac = 0$. When $b^2 - 4ac \neq 0$ we see that $k[W]$ is isomorphic to $k[Z]$.

1.4 Hartshorne, problem 1.3

Let Y be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and their prime ideals.

Solution. Vanishing of $zx - x$ implies either $x = 0$ or $z = 1$ or both. When $x = 0$ the first equation gives $yz = 0$, which implies $y = 0$ or $z = 0$. When $z = 1$ we find from the first equation that $y = x^2$. Concluding we see that Y is a union of the zero sets of (x, y), (x, z) and $(z - 1, y - x^2)$. All three ideals are prime ideals, hence their zero sets are irreducible.