Example Answers, Rational Points on Curves

1 Feb 14

Question 1 (Feb 14, Q1). *For which values of λ does the projective cubic surface*

$$x^3 + y^3 + z^3 + u^3 = \lambda(x + y + z + u)^3 \tag{1}$$

have singularities?

Answer. If $\mathrm{char}(K) = 3$ then all points of the surface are surface are singular for all values of λ. We now assume $\mathrm{char}(K) \neq 3$, and claim that the surface is singular if and only if $\lambda = \frac{1}{16}$ or $\lambda = \frac{1}{4}$.

To prove this let $f := x^3 + y^3 + z^3 + u^3 - \lambda(x + y + z + u)^3$. The singularities are the points $P = (x, y, z, u)$ where $\nabla f(P) = 0$ and $f(P) = 0$. If $\nabla f(P) = 0$, then

$$x^2 = y^2 = z^2 = u^2 = \lambda(x + y + z + u)^2, \tag{2}$$

so that singular points can be assumed to have all 4 coordinates $\in \{\pm 1\}$. Due to symmetry there are essentially only 3 cases to analyze, namely $P = (1 : 1 : -1 : -1)$, $(1 : 1 : 1 : 1)$ or $(1 : 1 : 1 : -1)$. The case $P = (1 : 1 : -1 : -1)$ leads to $x = y = z = u = 0$ which is impossible. The case $P = (1 : 1 : 1 : 1)$ leads to $\lambda = \frac{1}{16}$ and a surface with the single isolated singularity at $P = (1 : 1 : 1 : 1)$. The case $P = (1 : 1 : 1 : -1)$ leads to $\lambda = \frac{1}{4}$ and singularities at the 4 points $(-1 : 1 : 1 : 1)$, $(1 : -1 : 1 : 1)$, $(1 : 1 : -1 : 1)$ and $(1 : 1 : 1 : -1)$. □

Question 2 (Feb 14, Q2). *Write $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$ and let A be a symmetric 4×4 matrix. Consider the intersection of the quadratic hypersurfaces in 3-dimensional projective space*

$$\boldsymbol{x}^T \boldsymbol{x}, \ \boldsymbol{x}^T A \ \boldsymbol{x}.$$

We assume that A is not a scalar multiple of the identity matrix. Give conditions on A so that the curve of intersection C is singular.

Answer. We assume $\mathrm{char}(K) \neq 2$. We claim that C is non-singular iff A has 4 distinct eigenvalues. We prove this in 2 steps.

Step 1. Assume first that A is diagonal so that the equations of the 2 surfaces are given by

$$f_1 := x_0^2 + x_1^2 + x_2^2 + x_3^2, \quad f_2 := \lambda_0 x_0^2 + \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2,$$

with $\lambda_i \in K$, not all equal. A point P of C is singular iff the rank of the matrix $((\partial f_i/\partial x_j)(P))$ is less than 2. I.e. iff there is a non-trivial linear relation between $\nabla f_1(P)$ and $\nabla f_2(P)$.

If two of the λ_i are equal ($\lambda_0 = \lambda_1 =: \lambda$, say) then $P := (1, i, 0, 0) \in C$ and

$$\lambda \nabla f_1(P) + \nabla f_2(P) = 0.$$

I.e. P is a singular point of C. Conversely suppose that the λ_i are all distinct. The equation $f_1(P) = 0$ implies that at least 2 of the coordinates of P are non-zero. Hence $\nabla f_1(P)$ and $\nabla f_2(P)$ must be linearly independent.

Also note that if exactly two of the eigenvalues are equal, e.g. $\lambda_0 = \lambda_1 = \lambda$ then the set of equations can be rewritten as $f_1 = 0,\ f_2 - \lambda f_1 = 0$. The latter equation reads $(\lambda_2 - \lambda)x_2^2 + (\lambda_3 - \lambda)x_3^2 = 0$, which is reducible. Hence C is reducible with two irreducible components.

Suppose in addition that $\lambda_2 = \lambda$(i.e three equal eigenvalues). Then $f_1 - \lambda f_2 = (\lambda_3 - \lambda)x_3^2 = 0$. Hence the ideal generated by f_1, f_2 is not reduced and C can also be given by $f_1 = 0, x_3 = 0$.

Step 2. We now drop the condition that A is diagonal, and only assume that A is symmetric. In that case there is a matrix B so that $A' := B^T AB$ is diagonal and $B^T B = I$. Using B we perform a projective coordinate change $\mathbf{x} = B\mathbf{y}$. We get the new set of equations $\mathbf{y}^T\mathbf{y} = 0$ and $\mathbf{x}^T A\mathbf{x} = 0$. Thus we are back in our previous situation. Since $B^{-1} = B^T$, A' has the same eigenvalues as A and the result for symmetric matrices follows from the result for diagonal matrices.

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