Example Answers, Rational Points on Curves

1 Feb 21

Question 1 (Feb 21, Q1). *We provide $\mathbb{P}^5$ with the 6 homogeneous co-ordinates $X_{ij}, i = 0, 1; j = 0, 1, 2$. Denote the homogeneous co-ordinates on $\mathbb{P}^1$ by x_0, x_1 and the homogeneous coordinates on $\mathbb{P}^2$ by y_0, y_1, y_2. Show that the image of the Segre embedding $\varphi : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^5$ given by $X_{ij} = x_i y_j$ is a projective variety given by the equations $X_{ij}X_{kl} = X_{kj}X_{il}$ for all $i, k =\in \{0, 1\}$ and $j, l \in \{0, 1, 2\}$.*

Answer. Clearly any $P \in \mathrm{Im}(\varphi)$ satisfies the equations. Conversely suppose $P \in \mathbb{P}^5$ satisfies the equations. At least one coordinate of $P = (X_{ij})$, X_{00} (say), is non-zero. Consider the coordinates X_{ij} as entries of a 2×3 matrix A. The equations state that the column rank of A is at most 1. Hence the row rank of A is at most 1. Since the first row is non-zero (as $X_{00} \neq 0$), we must have that the 2nd row is λ times the 1st row for some $\lambda \in K$. Therefore $P = \varphi((1 : \lambda), (X_{00} : X_{01} : X_{02}))$ and $P \in \mathrm{Im}(\varphi)$. □

Question 2 (Feb 21, Q2).

- *a) Show that the image Q of the Segre mapping $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is given by the equation $xv - yu = 0$, where x, y, u, v are the homogenous coordinates of $\mathbb{P}^3$.*
- *b) Show that Q is birationally equivalent to $\mathbb{P}^2$.*
- *c) Show that Q is not isomorphic to $\mathbb{P}^2$.*

Answer. (a) The same argument as used in the last question works.

(b) $Q \to \mathbb{P}^2$, $(x : y : u : v) \mapsto (x : y : u)$ is well-defined on $xyuv \neq 0$. $\mathbb{P}^2 \to Q$, $(x : y : u) \mapsto (x : y : u : yu/x)$ is well-defined on $xyu \neq 0$. They are inverses to each other on these open sets. Therefore Q is birationally equivalent to $\mathbb{P}^2$.

Another argument might be to remark that $\mathbb{A}^1 \times \mathbb{A}^1$ is isomorphic to $\mathbb{A}^2$. Since these two sets are open subsets of $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ this implies that $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are birationally isomorphic. (c) Q contains two non-intersecting copies of $\mathbb{P}^1$, namely the curve given by the equations $x = y = 0$ and the curve given by the equations $u = v = 0$. In $\mathbb{P}^2$ any 2 curves intersect. Therefore Q is not isomorphic to $\mathbb{P}^2$.

Another example of two non-intersecting line on $\mathbb{P}^1 \times \mathbb{P}^1$ are $\{0\} \times \mathbb{P}^1$ and $\{1\} \times \mathbb{P}^1$.

□

Question 3 (Feb 21, Q3, taken from [1] exercise 5.11).

- *a) Consider the rational projection map from* $\mathbb{P}^3$ *to* $\mathbb{P}^2$ *given as follows. The coordinates of* $\mathbb{P}^3$ *are denoted* $(x : y : z : w)$. *Set* $P := (0 : 0 : 0 : 1)$. *Take any point* $(x : y : z : w) \neq P$ *and take a line through this point and* P *and intersect with the hyperplane* $w = 0$. *Show that this gives the map* $(x : y : z : w) \mapsto (x : y : z)$. *We call this the projection from* P *onto the hyperplane* $w = 0$. *On which points is this map a morphism?*

- *b) Now consider the algebraic curve* C_1 *given by the equations* $x^2 - xz - yw = 0$ *and* $yz - xw - zw = 0$. *Show that the projection described in part (a) gives a morphism of this curve minus* P *onto the cubic plane curve* $C_2 : y^2z - x^3 + xz^2 = 0$ *minus the point* $(1 : 0 : -1)$.

Answer. (a) The line joining $(x : y : z : w)$ to P is given by $\theta : (\lambda : \mu) \mapsto \lambda(x : y : z : w) + \mu(0 : 0 : 0 : 1)$. This intersects with the hyperplane $w = 0$ at $\theta(1 : -w) = (x : y : z : 0)$. This shows that the map is as claimed. It is defined (and is a morphism) on $\mathbb{P}^3$ minus the point P.

(b) Suppose that $(x : y : z : w) \in C_1$. Multiplying the first equation of C_1 by $(x + z)$ and the second equation by y gives

$$x(x + z)(x - z) \quad = \quad yw(x + z) = y^2z, \tag{1}$$

so that $y^2z - x^3 + xz^2 = 0$ and the projection φ maps $C_1 - P$ into C_2. If $(x : y : z) \in C_2$ and $y(x + z) \neq 0$ we can use (1) and construct a preimage of φ. This leaves only the points $Q = (1 : 0 : 1)$, $(1 : 0 : -1)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$ as candidates for not being in the image. For each point Q we substitute its homogeneous coordinates into the equations for C_1 and check whether there is a common solution w. We find that all points except $Q = (1 : 0 : -1)$ lie in the image of φ.

□

Question 4 (Feb 21, Q4(a), [1] exercise 5.1). *Locate the singular points and sketch the following curves in* $\mathbb{A}^2$ *(assume char*$(K) \neq 2$*). Which is which in Figure 4 (loc.cit.) ?*

- *(a)* $x^2 = x^4 + y^4$;
- *(b)* $xy = x^6 + y^6$;
- *(c)* $x^3 = y^2 + x^4 + y^4$;
- *(d)* $x^2y + xy^2 = x^4 + y^4$.

Answer. (a) Suppose that P is a singular point of the curve defined by $f = x^2 - x^4 - y^4$. Then $\nabla f(P) = 0$, $f(P) = 0$. I.e.

$$\frac{\partial f}{\partial x} = 2x - 4x^3 = 0, \quad \frac{\partial f}{\partial y} = 4y^3 = 0, \quad f = x^2 - x^4 - y^4 = 0.$$

We deduce that $(0,0)$ is the unique singular point. Locally at $(0,0)$ the curve looks like 2 copies of the y-axis. The curve is the Tacnode of Figure 4.

(b) Suppose that P is a singular point of the curve defined by $f = xy - x^6 - y^6$. Then $\nabla f(P) = 0$, $f(P) = 0$. I.e.

$$y = 6x^5, \quad x = 6y^5, \quad xy = x^6 + y^6.$$

We have $(0,0)$ as singular point. For $\mathrm{char}(K) \neq 3$, there are other P for which $\nabla f(P) = 0$, but these do not correspond to points with $f(P) = 0$. The curve has an isolated singularity at the origin, where the curve looks locally like two lines crossing transversally. The curve is the Node of Figure 4.

(c) Suppose that P is a singular point of the curve defined by $f = x^3 - y^2 - x^4 - y^4$. Then $\nabla f(P) = 0$, $f(P) = 0$. I.e.

$$3x^2 = 4x^3, \quad 2y = -4y^3, \quad x^3 = y^2 + x^4 + y^4$$

We have $(0,0)$ as singular point. There are other P for which $\nabla f(P) = 0$, but these do not correspond to points with $f(P) = 0$. Therefore the curve has an isolated singularity at the origin. Here the curve looks locally like two copies of the x-axis. The curve is the Cusp of Figure 4.

(d) Suppose that P is a singular point of the curve defined by $f = x^3 - y^2 - x^4 - y^4$. Then $\nabla f(P) = 0$, $f(P) = 0$. I.e.

$$2xy + y^2 = 4x^3, \quad 2xy + x^2 = 4y^3, \quad x^2y + xy^2 = x^4 + y^4$$

Adding x times the first equation to y times the second equation gives

$$3(x^2y + xy^2) = 4(x^4 + y^4)$$

Combining with the 3rd equation gives $xy(x - y) = 0$. Substituting into the equation $f(P) = 0$ leads to $P = (0,0)$. This is the only singularity. Near P the curve looks locally like 3 lines crossing transversally. The curve is the Triple Point of Figure 4. $\square$

Question 5 (Feb 21, Q4(b), [1] exercise 5.2). *Locate the singular points and describe the singularities of the following surfaces in $\mathbb{A}^2$ (assume char$(K) \neq 2$). Which is which in Figure 5?*

- *(a) $xy^2 = z^2$;*

- *(b)* $x + 2 + y^2 = z^2$*;*
- *(c)* $xy + x^3 + y^3 = 0$.

Answer. (a) Suppose that P is a singular point of the surface defined by $f = xy^2 - z^2$. Then $\nabla f(P) = 0$, $f(P) = 0$. I.e.

$$\nabla(f)(P) = (y^2, 2xy, -2z) = (0,0,0), \quad xy^2 = z^2.$$

This leads us to conclude that the whole of the line $y = z = 0$ is singular. At $(0,0,0)$ the surface looks locally like two copies of the hyperplane $z = 0$. At the other singular points $(x,0,0)$, $x \neq 0$ it looks like two hyperplanes intersecting transversally. The surface is the Pinch Point of Figure 5.

(b) Suppose that P is a singular point of the surface defined by $f = x + 2 + y^2 - z^2$. Then $\nabla f(P) = 0$, $f(P) = 0$. I.e.

$$\nabla(f)(P) = 2(x, y, -z) = (0,0,0), \quad x^2 + y^2 = z^2.$$

This leads us to conclude that $P = (0,0,0)$ is the the only singularity. At $(0,0,0)$ the surface looks locally like a cylinder that has degenerated to a point. The surface is the Conical Double Point of Figure 5.

(c) Suppose that P is a singular point of the surface defined by $f = xy + x^3 + y^3$. Then $\nabla f(P) = 0$, $f(P) = 0$. I.e.

$$\nabla(f)(P) = (y + 3x^2, x + 3y^2, 0) = (0,0,0), \quad xy + x^3 + y^3 = 0.$$

Taking the inner product of $\nabla(f_(P)$ with $(x,y,0)$ leads to $2xy + 3(x^3 + y^3) = 0$. Subtract the equation $f(P) = 0$ and we get $xy = 0$. Substitute into $f = 0$ and we conclude $P = (0,0,z)$ for some $z \in K$. The whole of the line $x = y = 0$ is singular. The surface looks locally along the singularity like 2 hyperplanes crossing transversally. The surface is the Double Line of Figure 5.

□

References

[1] Robin Hartshorne , *Algebraic Geometry* Springer-Verlag .

[2] Mark Hindry, Joeseph Silverman , *Diophantine Geometry* Springer-Verlag .