Example Answers, Rational Points on Curves

1 Feb 21

Question 1 (Feb 28, Q1). *Let C be a plane affine curve in $\mathbb{A}^2$ (with coordinates x, y). Suppose that $P = (0,0)$ is a smooth point of C and suppose that the tangent line of C at P is not the line $y = 0$.*

a) Show that the equation of C can be written in the form $x + xQ(x,y) + q(y) = 0$ where $Q \in k[x,y]$ and $Q(0,0) = 0$ and $q \in k[y]$ and $q(0) = 0$.

b) Let $\mathcal{O}_P$ be the laocal ring of all $f \in k(C)$, regular in P and let $|calM_P$ be the ideal in $\mathcal{O}_P$ consisting of all f with $f(P) = 0$. Show that $\mathcal{M}_P$ is generated by y (i.e. y is a local parameter of C at P).

c) Show that to any non-trivial rational function $f \in k(C)$ there exists an integer n and a rational function $g \in \mathcal{O}_P$, invertible in $\mathcal{O}_P$, such that $f = y^n g$.

Answer. Part a) Suppose the equation of C is give by $F(x,y) = 0$. Since $F(P) = 0$ and P is non-singular we see that F has the local form $F = ax + by +$higher order terms in x, y. Since the tangent is not $y = 0$ we see that $a \neq 0$. By taking F/a instead of F we get that the equation looks locally like $0 = x + by + \ldots$. Collect the monomials in y into the polynomial $q(y)$, then the remaining terms are all divisible by x. Hence $F = x + xQ(x,y) + q(y)$. For future use we write $q(y) = y^r q^*(y)$ where $q^*(0) \neq 0$.

Part b) First of all note that $y \in \mathcal{M}_P$. Secondly, let $f \in \mathcal{M}_P$. By definition of a regular functions there exist polynomials $A(x,y), B(x,y)$ such that $f = A(x,y)/B(x,y)$ in an open neighbourhood of P and $B(0,0) \neq 0$. Moreover, $A(0,0) = 0$. Hence $A(x,y) = xa(x,y) + yb(y)$ for some polynomials $a(x,y) \in k[x,y], b(y) \in k[y]$. From part a) we see that $x = -y^r q^*(y)/(1+Q)$. Hence

$$f = \frac{A(x,y)}{B(x,y)} = \frac{y}{B(x,y)} \left(b(y) - y^{r-1} q^*(y)/(1+Q) \right).$$

Since $1 + Q(x,y), B(x,y) \in \mathcal{O}_P^*$ we conclude that $f \in (y)$. Hence $|calM_P = (y)$.

Part c) Since any $f \in k(C)$ can be written as quotient of polynomials in $k[x,y]$ it suffices to prove our statement for $f \in k[x,y]$. We say that y^k divides f if $f = y^k h$ where $h \in \mathcal{O}_P$. Now f is algebraic over y, denote the minimal polynomial by $R(X)$ and suppose R has coefficients in $k[y]$. Let us say,

$$R_m(y) f^m + \cdots R_1(y) f + R_0(y) = 0$$

with $R_i(y) \in k[y]$. We now see that if y^k divides f then y^k divides $R_0(y)$. Hence there is a maximal integer n such that y^n divides f. In particular, $f = y^n f^*$ with $f^* \in \mathcal{O}_P$. If $f^*(P) = 0$ this implies by part b) that f^* is divisible by y, contradicting the maximality of n. Thus $f^*(P) \neq 0$ and hence $f^* \in \mathcal{O}_P^*$. □

Question 2 (Feb 28, Q2). *Consider the plane projective curve C given by $x^3y + y^3z + z^3x = 0$. Let f be the rational function on C given by x/z.*

a) Determine the degree of f.

b) Determine the zeros and poles of f with their multiplicities.

c) Determine all points $P \in C$ where f has index of ramification $e_P > 1$.

Answer. Part a) Affinely written (by setting $z = 1$) the curve C has the equation $x^3y + y^3 + x = 0$ and the function f reads $f = x$. Since $F = y^3 + x^3y + x$ is a cubic polynomial, irreducble over $k(x)$, we get that $k(C) = k(x)[y]/(F)$ is an extension of degree 3 over $k(f) = k(x)$. Hence the degree of f is 3.

Part b) First we determine the zeros of x/z. Setting $x = 0$ we get from the equation of C that $y = 0$ or $z = 0$. Let us first consider the point $x = y = 0$. Write the equation affinely (by setting $z = 1$) as $y^3 + x^3y + x = 0$. We see that y is a local parameter and that the function x, hence f has a zero of order 3.

Now consider the point $x = z = 0$. Written affinely (by setting $y = 1$) we get $x^3 + z + z^3x = 0$ and $f = x/z$. We see that x is a local parameter at our point and that z has a zero of order 3 at our point. Hence x/z has a pole of order 2.

To determine the poles we set $z = 0$. From the equation of C we get $x = 0$ or $y = 0$. We just saw that f has a second order pole at $x = z = 0$. So we can expect that f has a first order pole at $z = y = 0$. let us check. Write the equation affinely by setting $x = 1$. I.e. $y + y^3z + z^3 = 0$ and $f = 1/z$. Hence z is a local parameter, and thus $f = 1/z$ has a first order pole at our point.

Part c) We solve $f(x, y, z) = b$ for every b and check when there are multiple solutions. We already dealt with $b = 0, \infty$ in the previous part. Eliminate x from $x/z = b$ and the equation of C. We get $b^3z^3y + y^3z + bz^4 = 0$. We can assume $z = 1$, since $z = 0$, hence $x = 0$, is known to be a third order zero. We are left with $y^3 + b^3y + b = 0$. This has multiple solutions if and only if the discriminant of this polynomial in y vanishes. I.e. $4b^9 + 27b^2 = 0$. Since we had already dealt with $b = 0$ we are left with $4b^7 + 27 = 0$. To each such value of b our equation has a double solution, namely $y = -3/(2b^2)$. So we find seven points P with $e_P = 2$.

Additional remark: Here is a nice illustration of the Riemann-Hurwitz formula. Above we found seven points with e_P. There is also a pole of order 2 and a zero of order 3. So $\sum_P (e_P - 1) = 10$ and we get

$$2g - 2 = -2n + \sum_P (e_P - 1) = -6 + 10 = 4.$$

We conclude that $g = 3$ as should be the case with a smooth quartic curve in ¶^{2}. $\square$