Huiswerkopgave 23 maart 2018

1. Let K be a field and L the splitting field of a separable polynomial $f \in K[x]$ of degree n. Denote the zeros of f in L by $\alpha_1, \alpha_2, \ldots, \alpha_n$.

   (a) (1 pt) Prove that for $i = 1, 2, \ldots, n$:

   $$[K(\alpha_1, \ldots, \alpha_i) : K] \leq n(n-1)\cdots(n-i+1).$$

   (hint: consider the chain of extensions $K \subset K(\alpha_1) \subset K(\alpha_1, \alpha_2) \subset \cdots$ and their degrees).

   (b) (1 pt) Prove that $[L : K]$ divides $n!$ (hint: recall that a Galoisgroup permutes zeros of polynomials).

   (c) (1/2 pt) Give an example in which the equality $[L : K] = n!$ holds for some $n > 2$.

   (d) (1 pt) Suppose that $[L : K] = n!$. Show that we have equality sign in part 1a.

2. Let $f = X^4 + 1 \in \mathbb{Q}[X]$.

   (a) (1 pt) Prove that f is irreducible over $\mathbb{Q}$.

   (b) (1/2 pt) Let α be a zero of f in $\mathbb{C}$ (no need to determine it). Show that the full set of zeros is given by $\{\alpha, -\alpha, i\alpha, -i\alpha\}$.

   (c) (1/2 pt) Show that $\alpha^2 = \pm i$. Show that $L = \mathbb{Q}(\alpha)$ is the splitting field of f over $\mathbb{Q}$.

   (d) (1/2 pt) What is the degree of L over $\mathbb{Q}$? And the order of $G := \mathrm{Gal}(L/\mathbb{Q})$?

   (e) (1 pt) Why does $\sigma(\alpha) = -\alpha$ define an element of G? Same question for $\tau(\alpha) = \alpha^3$.

   (f) (1 pt) Show that $\sigma^2 = \tau^2 = \mathrm{id}$ and $\sigma\tau = \tau\sigma$. Determine G.

   (g) (1 pt) Determine all subgroups of G and the corresponding intermediate fields in $L/\mathbb{Q}$. Write each of these intermediate fields as simple extension of $\mathbb{Q}$.

   (h) (1 pt) Show that $L = \mathbb{Q}(i, \sqrt{2})$ and express α as $\mathbb{Q}$-linear combination of $\{1, i, \sqrt{2}, i\sqrt{2}\}$.