Huiswerkopgave 9 maart 2018

The Eisenstein criterion for polynomials with coefficients in $\mathbb{Z}$ reads as follows.

Theorem 1 (Eisenstein)
Let $f(x) \in \mathbb{Z}[x]$ be primitive and let p be a prime number which divides all coefficients of $f(x)$, except the leading coefficient. Suppose also that p^2 does not divide the constant term $f(0)$ of $f(x)$. Then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

Problem 1 (2 pts): Prove this theorem using the following steps. Suppose that $f(x)$ is reducible. Then there is a factorization $f(x) = g(x)h(x)$ with $g(x), h(x) \in \mathbb{Z}[x]$ and degree of $g(x), h(x)$ is less than the degree of $f(x)$.

(i) Consider $f(x)$ modulo p and use unique factorization in $(\mathbb{Z}/p\mathbb{Z})[x]$ to show that both $g(0)$ and $h(0)$ are divisible by p.

(ii) Derive a contradiction.

Here is a generalization.

Theorem 2
Let $f(x) \in \mathbb{Z}[x]$ be primitive and let p be a prime number. Write $f(x) = f_n x^n + \cdots + f_m x^m + \cdots + f_1 x + f_0$ with $m \leq n$. Suppose that p does not divide $f_n f_m$, and $p | f_i$ for all $i < m$, and p^2 does not divide $f_0 = f(0)$. Then $f(x)$ has an irreducible factor in $\mathbb{Q}[x]$ of degree $\geq m$.

Problem 2 (3 pts): Prove this theorem (hint: show that there is at most one irreducible factor whose constant term is divisible by p).

Here is an application.

Problem 3 (2 pts): Prove that $x^n + x^{n-1} + 2$ is irreducible in $\mathbb{Q}[x]$ for all $n \geq 2$.

Here are two problems about prime ideals.

Problem 4 (2 pts): Let R be a ring and $I \subset R$ an ideal. Let $\phi : R \to R/I$ be the natural residue class homomorphism. Let $I \subset J$ be another ideal. Prove that J is a prime ideal in R if and only if $\phi(J)$ is a prime ideal in R/I.

Problem 5 (1 pt): Let n be an integer > 1. Determine all prime ideals in $\mathbb{Z}/n\mathbb{Z}$.