# The equation $x + y = 1$ in finitely generated groups

F.Beukers and H.P.Schlickewei

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## 1 Introduction

Let $H$ be a finitely generated subgroup of rank $r$ in $(\mathbf{C}^*)^2$. Denote by $G$ the $\mathbf{Q}$-closure of $H$, i.e. the subgroup of $(\mathbf{C}^*)^2$ consisting of all pairs $\mathbf{a} = (a_1, a_2) \in (\mathbf{C}^*)^2$ such that $\mathbf{a}^N = (a_1^N, a_2^N) \in H$ for some $N \in \mathbf{N}$. We are interested in an upper bound for the number of solutions $(x, y) \in G$ of the equation

$$x + y = 1 \tag{1}$$

A special case of (1) is obtained if we restrict $x$ and $y$ to the group of so-called $S$-units in an algebraic number field $K$. Here $S$ is assumed to be a finite set of places of $K$ including all infinite ones. Supposing that $d = [K : \mathbf{Q}]$, $s = \#S$ and letting $a, b \in K^*$ be fixed, J.H.Evertse [3, Theorem 1] showed that

$$ax + by = 1 \tag{2}$$

has not more than $3 \times 7^{d+2s}$ solutions. Since $s \geq d/2$ this implies that (2) has at most $3 \times 7^{4s}$ solutions. We can apply this result to equation (1). However, the estimate will depend on the degree of the field containing $H$, and on $s$, the number of places for which the elements of $H$ have non-trivial valuation. Note that for fixed $r$ the number $s$ may have arbitrarily large values.

We shall be interested in bounds which depend only on $r$. The first such uniform result for a general subgroup $G$ of $(\mathbf{C}^*)^2$ was given in [5]. There the bound

$$2^{2^{26}+36r^2}$$

was derived for the number of solutions of equation (1). This was improved in [6] to

$$2^{13r+63} r^r.$$

In this paper we obtain

**Theorem 1.1** *Let $G$ be the $\mathbf{Q}$-closure of a finitely generated subgroup of $(\mathbf{C}^*)^2$ of rank $r$. Then the equation*

$$x + y = 1, \qquad (x, y) \in G$$

*has not more than $2^{8r+8}$ solutions.*

Note that this bound, apart from the numerical constants, has the same shape as Evertse's upper bound.

It is well known that a particular application of Theorem 1.1 deals with the multiplicity of binary recurrences. Let $\{u_m\}_{m\in\mathbf{Z}}$ be a sequence of complex numbers satisfying the recurrence relation

$$u_{m+2} = \nu_1 u_{m+1} + \nu_0 u_m$$

with $\nu_0, \nu_1 \in \mathbf{C}$, $\nu_0 \neq 0$. Suppose that we have initial values $(u_0, u_1) \neq (0, 0)$. Write $f(z) = z^2 - \nu_1 z - \nu_0$. Let $\alpha, \beta$ be its zeros. Note that $\nu_0 \neq 0$ implies $\alpha, \beta \neq 0$. Let us assume that $\alpha \neq \beta$. Then there exist $a, b \in \mathbf{C}$ such that

$$u_m = a\alpha^m + b\beta^m$$

Given $c \in \mathbf{C}$ we are interested in the number of solutions $m \in \mathbf{Z}$ of $u_m = c$. Note that the cases $a, b$ or $c$ equal to zero are uninteresting since they have either at most one solution or infinitely many trivial ones. So we assume they are non-zero. Divide on both sides by $c$, and from now on we shall be interested in the equation

$$\lambda\alpha^x + \mu\beta^x = 1 \text{ in } x \in \mathbf{Z}, \tag{3}$$

where $\lambda\mu\alpha\beta \neq 0$. We shall also assume that $\alpha, \beta$ are not both roots of unity.

As a fine point we add that if $\alpha, \beta$ are roots of unity, then the set

$$\{(\alpha^x, \beta^x) |\ x \text{ solution of } (3)\}$$

consists of at most two elements. This is a consequence of the fact that there exist precisely two triangles in the complex plane two of whose sides have lengths $|\lambda|, |\mu|$, whose third side is the segment $[0, 1]$ and such that the side of length $|\lambda|$ ends in 0.

Straightforward application of Theorem 1.1 with the group $H$ generated by $(\lambda, \mu)$ and $(\alpha, \beta)$ shows that (3) has not more than $2^{24}$ solutions. However, one can do much better,

**Theorem 1.2** *Under the assumptions just mentioned the equation*

$$\lambda\alpha^x + \mu\beta^x = 1 \quad \text{in} \quad x \in \mathbf{Z}$$

*has at most* 61 *solutions.*

As a curiosity we mention that the equation with the largest number of solutions known is

$$\frac{\theta_2 - \theta_3}{\theta_2 - \theta_1}\left(\frac{\theta_1}{\theta_3}\right)^x + \frac{\theta_1 - \theta_3}{\theta_1 - \theta_2}\left(\frac{\theta_2}{\theta_3}\right)^x = 1$$

where the $\theta_i$ are the zeros of $X^3 - 2X^2 + 4X - 4$. The solutions are $x = 0, 1, 4, 6, 13, 52$. It would be interesting to have examples with more than 6 solutions, if they exist.

The first result in the situation of Theorem 1.2 with a universal bound was derived in [4] with the bound $2^{2^{23}}$.

The improvements we give in the current paper in comparison with [4],[5] and [6] depend upon two ingredients. First we use an explicit version of Thue's method via hypergeometric polynomials as given in [1], whereas the previous papers are based on a quantitative version of Roth's Theorem. To get bounds that do not depend upon degrees of number fields involved, previously a result from [7] was used on lower bounds for heights of solutions of equations. Here we apply the strongly improved bound given in Corollary 2.4 of [2].

## 2 Lemmas on algebraic numbers

First we fix our notations concerning heights. Let $K$ be an algebraic number field of degree $d$ over $\mathbf{Q}$. For any valuation $v$ we write $d_v = [K_v : \mathbf{Q}_v]$, where $K_v, \mathbf{Q}_v$ are the completions of $K, \mathbf{Q}$ with respect to $v$. For archimedean $v$ we normalise the valuation by $|x|_v = |x|^{d_v/d}$ where $|.|$ is the ordinary complex absolute value. When $v$ is non-archimedean we take $|p|_v = p^{-d_v/d}$ where $p$ is the unique rational prime such that $|p|_v < 1$. The height of an algebraic number $\alpha \in K^*$ is defined by

$$H(\alpha) = \prod_v \max(1, |x|_v)$$

Because of our normalisation $H(\alpha)$ does not depend on the choice of the field $K$ in which $\alpha$ is contained. More generally, for any $n+1$-tuple $(x_0, x_1, \ldots, x_n) \in K^n$, not all $x_i$ zero we define

$$H(x_0, x_1, \ldots, x_n) = \prod_v \max(|x_0|_v, \ldots, |x_n|_v).$$

Note that by the product formula we have $H(\lambda x_0, \ldots, \lambda x_n) = H(x_0, \ldots, x_n)$ for any $\lambda \in K^*$, so we can view this height as a height on the $K$-rational points of the projective space $\mathbf{P}^n$. In particular we have $H(\alpha) = H(1, \alpha)$.

We start with an easy Lemma.

**Lemma 2.1** *Let $a, a', b, b', A, B \in \overline{\mathbf{Q}}^*$ and $c, c' \in \overline{\mathbf{Q}}$ be such that $ab' \neq a'b$ and*

$$aA + bB = c, \qquad a'A + b'B = c'$$

*Then, $H(A, B, 1) \leq 2H(a, b, c)H(a', b', c')$.*

**Proof.** Fix a number field $K$ in which all numbers involved are contained. For each infinite valuation $v$ let $r_v = 2^{d_v/d}$ and let $r_v = 1$ if $v$ is finite. Notice that $\prod_v r_v = 2$.

One easily finds that

$$A = \frac{bc' - b'c}{\Delta}, \qquad B = \frac{a'c - ac'}{\Delta}$$

where $\Delta = a'b - ab'$. Hence

$$\begin{aligned} H(A, B, 1) &= H(bc' - b'c, a'c - ac', ba' - ab') \\ &= \prod_v \max(|bc' - b'c|_v, |a'c - ac'|_v, |ba' - ab'|_v) \\ &\leq \prod_v r_v \max(|a|_v, |b|_v, |c|_v) \max(|a'|_v, |b'|_v, |c'|_v) \\ &= 2H(a, b, c)H(a', b', c') \end{aligned}$$

□

As a corollary we get

**Corollary 2.2** *Let $a, b, A, B \in \overline{\mathbf{Q}}^*$ be such that $a \neq b$ and*

$$A + B = 1, \qquad aA + bB = 1$$

*Then, $H(A, B, 1) \leq 2H(a, b, 1)$.*

The next lemma follows from an explicit version of Thue's method using hypergeometric polynomials.

**Lemma 2.3** *Let* $a, b, A, B \in \overline{\mathbf{Q}}^*$ *and* $\rho \in \mathbf{N}$ *be such that*

$$A + B = 1, \qquad aA^{2\rho} + bB^{2\rho} = 1$$

*Then,* $H(A, B, 1) \leq 2^{1/\rho} c H(a, b, 1)^{1/\rho}$*, where* $c = 6\sqrt{3}$*.*

**Proof.** We infer from Lemma 6 of [1] that there exist three polynomials $P_\rho, Q_\rho, R_\rho$ of degree $\leq \rho$ such that

$$z^{2\rho} P_\rho(z) + (1 - z)^{2\rho} Q_\rho(z) = R_\rho(z), \qquad \forall z \in \mathbf{C}$$

$$bP_\rho(A) \neq aQ_\rho(A)$$

and

$$H(P_\rho(A), Q_\rho(A), R_\rho(A)) \leq (6\sqrt{3})^\rho H(A)^\rho.$$

Substitute $z = A$ in the polynomial identity. Application of the previous lemma with $A^{2\rho}, B^{2\rho}$ instead of $A, B$ and $c = 1$, $a' = P_\rho(A)$, $b' = Q_\rho(A)$, $c' = R_\rho(A)$ yields,

$$\begin{aligned} H(A, B, 1)^{2\rho} &\leq 2H(a, b, 1)H(P_\rho(A), Q_\rho(A), R_\rho(A)) \\ &\leq 2c^\rho H(a, b, 1)H(A)^\rho \leq 2c^\rho H(a, b, 1)H(A, B, 1)^\rho \end{aligned}$$

Divide on both sides by $H(A, B, 1)^\rho$ and take $\rho$-th roots to obtain our Lemma. □

The following lemma is due to an improvement of [7] by Corollary 2.4 in [2].

**Lemma 2.4** *Let* $\lambda, \mu \in \overline{\mathbf{Q}}^*$ *and suppose that* $\lambda + \mu = 1$*. Let* $(p_i, q_i)$*,* $i = 1, 2$ *be two solutions in* $\overline{\mathbf{Q}}$ *of* $\lambda p + \mu q = 1$ *such that the pairs* $(p_1, q_1), (p_2, q_2)$ *and* $(1, 1)$ *are all distinct. Then,*

$$H(p_1, q_1, 1)H(p_2, q_2, 1) \geq 1.0942711\ldots$$

By application of this Lemma with $\lambda = x_0, \mu = y_0$ and $p_i = x_i/x_0$, $q_i = y_i/y_0$ we obtain,

**Corollary 2.5** *Let* $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ *be three distinct solutions of* $x + y = 1$ *in* $x, y \in \overline{\mathbf{Q}}^*$*. Then,*

$$\max_{i=1,2}(\max(H(x_i/x_0), H(y_i/y_0)) \geq 1.022777\ldots$$

# 3 Normed vector spaces

Let $m \in \mathbf{N}$. For any subgroup $H \subset (\overline{\mathbf{Q}}^*)^m$ we let the $\mathbf{Q}$-closure of $H$ be the set of all $\mathbf{a} \in (\overline{\mathbf{Q}}^*)^m$ such that $\mathbf{a}^N \in H$ for some $N \in \mathbf{N}$. Let $G$ be the $\mathbf{Q}$-closure of a finitely generated subgroup of $(\overline{\mathbf{Q}}^*)^m$ of rank $r$. Let $T$ be the torsion subgroup of $G$. Then $G/T = G \otimes_{\mathbf{Z}} \mathbf{Q}$ has the natural structure of a $\mathbf{Q}$-vector space of dimension $r$. Consider the logarithmic height function $h(x) = \log H(x)$. The function

$$||(x_1, \ldots, x_m)|| = \max_{i=1,\ldots,m} h(x_i)$$

provides a natural norm on $G \otimes_{\mathbf{Z}} \mathbf{Q}$ as $\mathbf{Q}$-vector space. By continuity we can extend this norm to the real vector space $V_G = G \otimes_{\mathbf{Z}} \mathbf{R}$.

**Lemma 3.1** *The (semi)-norm $||.||$ is positive definite on $V_G$.*

**Proof.** Let us write down the semi-norm $||.||$ in an explicit way. Suppose the $\mathbf{Q}$-generators of $G$ are given by

$$\mathbf{a}_i = (a_{i1}, \ldots, a_{im}), \quad i = 1, \ldots, r.$$

Any element of $G$ can be written, modulo roots of unity, as $\mathbf{x} = (x_1, \ldots x_m) = \prod_{i=1}^r (a_{i1}, \ldots, a_{im})^{e_i}$ for some $e_i \in \mathbf{Q}$. Hence, using $h(a) = (1/2)\sum_v |\log(|a|_v)|$,

$$\begin{aligned} ||\mathbf{x}|| &= \max_{j=1,..,m} h(\prod_{i=1}^r a_{ij}^{e_i}) \\ &= \max_{j=1,..,m} (1/2) \sum_v \left| \sum_{i=1}^r e_i \log(|a_{ij}|_v) \right|. \end{aligned}$$

Extending $||.||$ to the reals is now straightforward, simply extend $e_i$ to $\mathbf{R}$. We also remark that if we take the $e_i$ integral, the components of $\mathbf{x}$ all lie in the same number field, hence the non-trivial elements of the group generated (over $\mathbf{Z}$) by the $\mathbf{a}_i$ have a norm uniformly bounded below by a positive constant, $\gamma$, say.

We now prove positive definiteness of $||.||$. Suppose there exists $\mathbf{y} \in V_G$, non-zero, such that $||\mathbf{y}|| = 0$. This implies that there exist $e_i \in \mathbf{R}$, not all zero, such that $|\sum_{i=1}^r e_i \log(|a_{ij}|_v)| = 0$ for all valuations $v$ and all $j$. Using Dirichlet's box principle we can then show that to any $\epsilon > 0$ there exist integers $m_i$, not all zero, such that $|\sum_{i=1}^r m_i \log(|a_{ij}|_v)| < \epsilon$ for all $v$ and $j$. This contradicts the existence of the uniform lower bound $\gamma$. Hence $||\mathbf{y}|| = 0$ implies that $e_i = 0$ for all $i$, as desired. □

From now on we suppose that $G \subset (\overline{\mathbf{Q}}^*)^2$. We want to bound the number of solutions of the equation

$$x + y = 1, \qquad (x, y) \in G \tag{M}$$

Consider the natural projection $p : G \to V_G$

**Lemma 3.2** *Let $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ be three distinct solutions of* (M). *Then their images under $p$ cannot be all equal.*

**Proof.** If all three images would be the same then $x_i/x_0$ and $y_i/y_0$ would be roots of unity for $i = 1, 2$. But this is impossible in view of Corollary 2.5. □

Let $\mathcal{M}$ be the image under $p$ of the solution set of (M). Then the number of solutions to (M) is bounded by $2(\#\mathcal{M})$.

We now restate the lemmas of the previous section in terms of the set $\mathcal{M} \subset V_G$. In the derivations we use the fact that $\max(H(a), H(b)) \le H(a, b, 1) \le \max(H(a), H(b))^2$.

Corollary 2.2 becomes,

**Lemma 3.3** *Let $\mathbf{w}_1, \mathbf{w}_2$ be distinct points of $\mathcal{M}$. Then,*

$$||\mathbf{w}_1|| \le \log 2 + 2||\mathbf{w}_2 - \mathbf{w}_1||$$

Lemma 2.3 becomes,

**Lemma 3.4** *Let $\mathbf{w}_1, \mathbf{w}_2$ be distinct points of $\mathcal{M}$ and $\rho \in \mathbf{N}$. Then,*

$$||\mathbf{w}_1|| \leq \log c + \frac{1}{\rho}(\log 2 + 2||\mathbf{w}_2 - 2\rho\mathbf{w}_1||)$$

Corollary 2.5 becomes,

**Lemma 3.5** *Let $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2$ be distinct points of $\mathcal{M}$. Then,*

$$\max(||\mathbf{w}_1 - \mathbf{w}_0||, ||\mathbf{w}_2 - \mathbf{w}_0||) \geq 0.022522\ldots$$

It will turn out that the cardinality of any set satisfying the inequalities in the above three lemmas can be bounded in terms of the dimension of $V_G$.

We need some additional lemmas on coverings of convex bodies. The first is straightforward,

**Lemma 3.6** *Let $V$ be an $m$-dimensional normed real vector space with norm $||.||$. Let $R > \delta > 0$. Consider the ball $B$ of radius $R$ around the origin and suppose it contains a set $U$ such that $||\mathbf{u}_1 - \mathbf{u}_2||| \geq \delta$ for any two distinct $\mathbf{u}_1, \mathbf{u}_2 \in U$. Then $\#U \leq (1 + 2R/\delta)^m$.*

**Proof.** Let $V_0$ be the volume of the unit ball $\{\mathbf{x}| \; ||\mathbf{x}|| < 1\}$. Around any point $\mathbf{u} \in U$ we consider the open ball $B_u = \{\mathbf{x}| \; ||\mathbf{x} - \mathbf{u}|| \leq \delta/2$. Since these balls are disjoint their union fills up a region of volume $(\#U)(\delta/2)^m V_0$ in the ball of radius $R + \delta/2$. The latter ball has volume $(R + \delta/2)^m V_0$. Hence $(\#U)(\delta/2)^m \leq (R + \delta/2)^m$ and our Lemma follows. □

**Lemma 3.7** *Let $\Psi$ be a convex symmetric body in $\mathbf{R}^r$. By $\lambda\Psi$ we denote the convex body obtained by multiplying the points of $\Psi$ by $\lambda$. Then, for any $\lambda > 1$, the set $\lambda\Psi$ can be covered by $(4 + 2\lambda)^r$ translated copies of $\Psi$.*

The proof of this Lemma can be found in [6, Lemma 7.2]. However, we really need the following corollary.

**Corollary 3.8** *Let $V$ be an $r$-dimensional normed real vector space with norm $||.||$. Let $\epsilon > 0$. Then there is a finite set $E \subset V$ of unit vectors such that every $\mathbf{v} \in V$ can be written as $\mathbf{v} = ||\mathbf{v}||\mathbf{e} + \mathbf{v}'$ with $\mathbf{e} \in E$ and $||\mathbf{v}'|| \leq \epsilon||\mathbf{v}||$. Moreover, $E$ can be chosen such that $\#E < (4 + 4/\epsilon)^r$.*

**Proof.** Let $B$ be the unit ball with respect to $||.||$. According to Lemma 3.7 the ball $B$ can be covered by $(4 + 4/\epsilon)^r$ translates of $(\epsilon/2)B$. Consider such a covering and let $\Delta$ be the subset of $(\epsilon/2)$-balls which have non-trivial intersection with the boundary of $B$. Clearly the balls in $\Delta$ give a covering of the boundary of $B$. For the set $E$ we take the unit vectors $\mathbf{c}/||\mathbf{c}||$ where $\mathbf{c}$ runs over the centers of the $(\epsilon/2)$-balls in $\Delta$.

Now let $\mathbf{v} \in \mathbf{R}^r$ be arbitrary. Let $\mathbf{c}$ be the center of the $(\epsilon/2)$-ball in $\Delta$ which contains $\mathbf{v}/||\mathbf{v}||$ and let $\mathbf{e} = \mathbf{c}/||\mathbf{c}||$. Notice that $||\mathbf{c} - \mathbf{e}|| = |1 - ||\mathbf{c}||\,| \leq \epsilon/2$. Hence,

$$||\frac{\mathbf{v}}{||\mathbf{v}||} - \mathbf{e}|| \leq ||\frac{\mathbf{v}}{||\mathbf{v}||} - \mathbf{c}|| + ||\mathbf{c} - \mathbf{e}|| \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus we find $||\mathbf{v} - ||\mathbf{v}||\mathbf{e}|| \leq \epsilon||\mathbf{v}||$, $\mathbf{e} \in E$ and our corollary follows. □

## 4 Proof of Theorem 1.1

Let $\Sigma$ be a subset of a normed vector space $V$ satisfying

1. $||\mathbf{w}_1|| \leq \log 2 + 2||\mathbf{w}_2 - \mathbf{w}_1||$ for any two distinct $\mathbf{w}_1, \mathbf{w}_2 \in \Sigma$
2. There exists $c_1$ such that $||\mathbf{w}_1|| \leq c_1 + \frac{1}{\rho}(\log 2 + 2||\mathbf{w}_2 - 2\rho\mathbf{w}_1||)$ for any two distinct $\mathbf{w}_1, \mathbf{w}_2 \in \Sigma$ and any $\rho \in \mathbf{N}$.
3. There exists $c_0 > 0$ such that $\max(||\mathbf{w}_1 - \mathbf{w}_0||, ||\mathbf{w}_2 - \mathbf{w}_0||) \geq c_0$ for any three distinct $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2 \in \Sigma$.

**Proposition 4.1** *Let $c_2 = \max(2\log 2, c_1 + \log 2/20)$. Then,*

$$\#\Sigma \leq \frac{1}{2}\left(44 + 2\frac{c_2}{c_0}\right)^{r+1}$$

*where $r$ is the dimension of $V$.*

**Proof.** Let $\epsilon$ be a real number such that $0 < \epsilon < 0.1$. Let $\mathbf{e}$ be a unit vector in $V$ and consider the cone

$$C_e = \{\mathbf{v} \in V|\ \mathbf{v} = ||\mathbf{v}||\mathbf{e} + \mathbf{v}',\ ||\mathbf{v}'|| \leq \epsilon||\mathbf{v}||\}$$

Let

$$c_3(\epsilon) = \frac{c_2}{1 - 10\epsilon}.$$

We will show that for any two $\mathbf{w}_1, \mathbf{w}_2 \in \Sigma \cap C_e$ with $c_3(\epsilon) < ||\mathbf{w}_1|| \leq ||\mathbf{w}_2||$ we have

$$(5/4)||\mathbf{w}_1|| \leq ||\mathbf{w}_2|| \leq (1 + 4/\epsilon)||\mathbf{w}_1|| \tag{4}$$

Suppose first $\mathbf{w}_1, \mathbf{w}_2 \in \Sigma \cap C_e$ and $||\mathbf{w}_1|| \leq ||\mathbf{w}_2|| < (5/4)||\mathbf{w}_1||$. Write $\mathbf{w}_i = ||\mathbf{w}_i||\mathbf{e} + \mathbf{w}_i'$. Then, from the first inequality on $\Sigma$ we infer,

$$\begin{aligned} ||\mathbf{w}_1|| &\leq \log 2 + 2||(||\mathbf{w}_2|| - ||\mathbf{w}_1||)e + \mathbf{w}_2' - \mathbf{w}_1'|| \\ &\leq \log 2 + 2(||\mathbf{w}_2|| - ||\mathbf{w}_1||) + 2\epsilon(||\mathbf{w}_2|| + ||\mathbf{w}_1||) \\ &\leq \log 2 + 2(1/4)||\mathbf{w}_1|| + 2\epsilon(9/4)||\mathbf{w}_1|| \end{aligned}$$

We obtain,

$$||\mathbf{w}_1|| \leq \frac{2\log 2}{1 - 9\epsilon} \leq c_3(\epsilon).$$

Suppose next that $\mathbf{w}_1, \mathbf{w}_2 \in \Sigma \cap C_e$ and $||\mathbf{w}_2|| > (1 + 4/\epsilon)||\mathbf{w}_1||$. Choose $\rho \in \mathbf{N}$ such that $||\mathbf{w}_2|| = (2\rho + \delta)||\mathbf{w}_1||$ with $|\delta| \leq 1$. Notice that $\rho \geq 2/\epsilon$. From the second inequality on $\Sigma$ it follows that

$$\begin{aligned} ||\mathbf{w}_1|| &\leq c_1 + \frac{1}{\rho}(\log 2 + 2||\delta||\mathbf{w}_1||e + \mathbf{w}_2' - 2\rho\mathbf{w}_1'|| \\ &\leq c_1 + (\log 2/20) + \frac{2}{\rho}(||\mathbf{w}_1|| + \epsilon(||\mathbf{w}_2|| + 2\rho||\mathbf{w}_1||)) \\ &\leq c_2 + \frac{2}{\rho}||\mathbf{w}_1|| + \epsilon(8 + 4/\rho)||\mathbf{w}_1|| \\ &\leq c_2 + \epsilon||\mathbf{w}_1|| + 9\epsilon||\mathbf{w}_1|| \end{aligned}$$

We get

$$||\mathbf{w}_1|| \leq \frac{c_2}{1-10\epsilon} \leq c_3(\epsilon)$$

We now put the above considerations together. Let $N$ be the smallest integer such that $(5/4)^{N-1} > 1+4/\epsilon$. Suppose $C_e$ contains $N$ points $\mathbf{w}_1,\ldots,\mathbf{w}_N$ larger than $c_3(\epsilon)$. Suppose they are ordered by size. Then, for each $i$, $||\mathbf{w}_{i+1}||/||\mathbf{w}_i| \geq 5/4$. This implies $||\mathbf{w}_N||/||\mathbf{w}_1|| > (5/4)^{N-1} > 1+4/\epsilon$ which is impossible by inequality (4). Hence any cone $C_e$ contains at most $N-1$ elements from $\Sigma$ of norm $\geq c_3(\epsilon)$. According to Lemma 3.8 the space $V$ can be covered by $(4+4/\epsilon)^r$ such cones and so the total number of points of $\Sigma$ larger than $c_3(\epsilon)$ can be estimated by $(N-1)(4+4/\epsilon)^r$. Since $\epsilon < 0.1$ it is not hard to see that $N-1 < 2/\epsilon$. Hence the number of large points is bounded by $(2/\epsilon)(4+4/\epsilon)^r$.

It remains to count the elements of $\Sigma$ with norm at most $c_3(\epsilon)$. By the third inequality on $\Sigma$ a ball of radius $c_0$ around a point of $\Sigma$ contains at most one other element from $\Sigma$. Consider a subset $\Sigma'$ of $\Sigma$ such that a ball of radius $c_0$ around any point of $\Sigma'$ contains no other point of $\Sigma'$. We can do this in such a way that $|\Sigma| \leq 2|\Sigma'|$. According to Lemma 3.6 the number of points in $\Sigma'$ can be bounded from above by $(1+2c_3(\epsilon)/c_0)^r$. Thus we conclude,

$$|\Sigma| \leq \frac{2}{\epsilon}\left(4+\frac{4}{\epsilon}\right)^r + 2\left(\frac{2c_3(\epsilon)}{c_0}+1\right)^r.$$

Now we choose $\epsilon$ such that $4/\epsilon = 2c_3(\epsilon)/c_0$, i.e $\epsilon = (10+0.5c_2/c_0)^{-1}$. Our proposition then follows immediately. □

**Proof of Theorem** 1.1. By a specialisation argument as in [5] we may assume that $G \subset (\overline{\mathbf{Q}}^*)^2$. We now complete the line of argument started in the Section (3). There we had the set $\mathcal{M}$. This set satisfies the conditions of Proposition 4.1 for the values $c_0 = 0.022522\ldots$ $c_1 = \log(6\sqrt{3}) = 2.3410\ldots$. Hence the cardinality of $\mathcal{M}$ is bounded by $\frac{1}{2} \times 256^{r+1}$. Since the number of solutions of (M) is bounded by $2\#\mathcal{M}$ our theorem follows. □

## 5 Proof of Theorem 1.2

We first need a lemma

**Lemma 5.1** *Consider the equation $\lambda\alpha^x + \mu\beta^x = 1$ in $x \in \mathbf{Z}$ where $\lambda,\mu,\alpha,\beta$ are as in the Introduction and assumed to be algebraic numbers. Suppose we have the solutions $x = 0, r, s, t$. Suppose that $t \geq 14s$. Then,*

$$s - 8.4r \leq \frac{9.1}{\log H(\alpha,\beta,1)}.$$

**Proof.** Application of Corollary 2.2 with $A=\lambda,\ B=\mu$ yields

$$H(\lambda,\mu,1) \leq 2H(\alpha,\beta,1)^r$$

Apply Lemma 2.3 with $A = \lambda\alpha^s,\ B = \mu\beta^s$ and $\rho$ such that $t = 2s\rho+\delta$, with $0 \leq \delta < 2s$. Note that $\rho \geq 7$. We obtain,

$$\begin{aligned} H(\lambda\alpha^s,\mu\beta^s,1) &\leq 2^{1/\rho}cH(\alpha^\delta\lambda^{1-2\rho},\beta^\delta\mu^{1-2\rho})^{1/\rho} \\ &\leq 2^{1/\rho}cH(\alpha,\beta,1)^{\delta/\rho}H(\lambda^{-1},\mu^{-1},1)^{2-1/\rho} \end{aligned}$$

Notice that

$$\begin{aligned} H(\alpha,\beta,1)^s &\le H(\lambda^{-1},\mu^{-1},1)H(\lambda\alpha^s,\mu\beta^s,1) \\ &\le 2^{1/\rho}cH(\alpha,\beta,1)^{\delta/\rho}H(\lambda^{-1},\mu^{-1},1)^{3-1/\rho} \end{aligned}$$

and use $H(\lambda^{-1},\mu^{-1},1)\le H(\lambda,\mu,1)^2\le 4H(\alpha,\beta,1)^{2r}$ to obtain

$$\begin{aligned} H(\alpha,\beta,1)^{s-\delta/\rho} &\le 2^{1/\rho}c2^{6-2/\rho}H(\alpha,\beta,1)^{6r} \\ &< 64cH(\alpha,\beta,1)^{6r} \end{aligned}$$

Taking log's and using $\log(64c)\le 6.5$ yields

$$s-\delta/\rho-6r\le 6.5/\log(H(\alpha,\beta,1)$$

from which our Lemma is immediate via $\delta/\rho\le 2s/7$. □

**Proof of Theorem** 1.2**.** By Theorem 2 of [1] we may assume that $\alpha,\beta,\lambda,\nu\in\overline{\mathbf{Q}}$. Without loss of generality we can also assume that

$$H(\alpha,\beta,1)\le H(\alpha^{-1},\beta^{-1},1).$$

Let $q$ be the length of the shortest closed interval containing three solutions. Let $n,n{+}p,n{+}q$ be three such solutions. Application of Lemma 2.4 to the equation $\lambda\alpha^{n+p}X+\mu\beta^{n+p}Y=1$ yields

$$H(\alpha,\beta,1)^{q-p}H(\alpha^{-1},\beta^{-1},1)^p\ge c_4,$$

where $c_4=1.0942711\ldots$. Hence $H(\alpha^{-1},\beta^{-1},1)^q\ge c_4$.

Define $\gamma=\log 8/\log c_4$ and note that $\gamma<23.1$.

Now let $k<l<m<n$ be any four solutions. First of all application of Corollary 2.2 with $A=\lambda\alpha^k,\ B=\mu\beta^k$ yields

$$H(\lambda\alpha^k,\mu\beta^k,1)\le 2H(\alpha,\beta,1)^{l-k} \tag{5}$$

In a similar way application of Corollary 2.2 with $A=\lambda\alpha^n,\ B=\mu\beta^n$ yields

$$H(\lambda\alpha^n,\mu\beta^n,1)\le 2H(\alpha^{-1},\beta^{-1},1)^{n-m} \tag{6}$$

Application of Lemma 2.1 with $A=\alpha^{k-n},\ B=\beta^{k-n}$ yields

$$\begin{aligned} H(\alpha^{k-n},\beta^{k-n},1) &\le 2H(\lambda\alpha^n,\mu\beta^n,1)H(\lambda\alpha^{n-k+l},\mu\beta^{n-k+l},1) \\ &\le 2H(\lambda\alpha^n,\mu\beta^n,1)^2H(\alpha^{l-k},\beta^{l-k},1) \end{aligned}$$

With (6) and $H(\alpha,\beta,1)\le H(\alpha^{-1},\beta^{-1},1)$ we get

$$H(\alpha^{-1},\beta^{-1},1)^{n-k}\le 8H(\alpha^{-1},\beta^{-1},1)^{2(n-m)+l-k}$$

Using our lower bound $H(\alpha^{-1},\beta^{-1},1)\ge c_4^{1/q}$ we find that

$$n-2m+l\ge-\gamma q \text{ hence } n-l-\gamma q\ge 2(m-l-\gamma q)$$

Denote the smallest solution by $n_0$ and the second smallest by $n_1$. Application of the inequality with $k=n_0,l=n_1$ yields

$$n-n_1-\gamma q\ge 2(m-n_1-\gamma q) \tag{7}$$

for any two solutions $m,n$ with $n_1<m<n$. We divide our solutions into three intervals,

- $I_1 = [n_0, n_1 + (0.9 + \gamma)q[$
- $I_2 = [n_1 + (0.9 + \gamma)q, n_1 + (230 + \gamma)q[$
- $I_3 = [n_1 + (230 + \gamma)q, \infty[$

Since any interval of length $< q$ contains at most two solutions, the interval $I_1$ contains at most $1 + 2([\gamma + 0.9] + 1) <= 49$ solutions. Because of (7) the interval $I_2$ contains at most 8 solutions.

We finally show that $I_3$ contains at most 4 solutions. Suppose $I_3$ contains 5 solutions, the largest being denoted by $N$, the smallest by $M$. Furthermore we let $k$ be a solution such that there exists another solution $l$ such that $k < l < k + q$. Because of (7) we find $k < n_1 + (1 + \gamma)q$. Since there exists at least one closed interval of length $q$ containing three solutions such a $k$ exists and we may moreover assume that $k \geq n_1$. From (7) it follows that $(N - n_1 - \gamma q) \geq 16(M - n_1 - \gamma q)$. Since $k \geq n_1$ this implies $(N - k - \gamma q) \geq 16(M - k - \gamma q)$ and since $N - k > M - k > 229q$ we get $N - k \geq (16 - 15\gamma/229)(M - k) > 14(M - k)$. Application of Lemma 5.1 to the equation $\lambda\alpha^k\alpha^x + \mu\beta^k\beta^x = 1$ with $r = l - k, s = M - k, t = N - k$ yields

$$M - k - 8.4(l - k) \leq \frac{9.1}{\log H(\alpha, \beta, 1)}.$$

Using the lower bound $H(\alpha, \beta, 1) \geq c_4^{1/2q}$ and $l - k < q$ we get $M - k < 211q$, contradicting $M - k > 229q$.

So we conclude that $I_3$ contains at most 4 solutions, which leaves us with a total of at most $49 + 8 + 4 = 61$ solutions. □

# 6 References


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