Assignment 2, UCU SCI111 spring 2004

Suppose somewhere in the plains of midwestern USA we have three villages which we call A, B, C. The villages agree to build a common electrical power plant. The only problem is where to locate it. Building the plant close to one of the villages might not sound too appealing to the others. Also lengths of electrical wiring should be minimised somehow. In brief, there are a large number of considerations that go into a decision and finally some agreement has to be reached. In the following two problems we explore the consequences of two such decisions.

1. Suppose the villages are located in the xy-plain and that their coordinates are $(x_A, y_A), (x_B, y_B), (x_C, y_C)$ respectively. We call the power plant P and its coordinates will be denoted by (x, y). The consensus is that the *sum of the squares* of the distances AP, BP, CP should be minimised.

   Do not ask why this choice is made. That problem belongs to the realm of politics. It is our job to find the location of the future power plant. In order to do that, write down the function that represents the sum of the squares of the three distances as a function of $P = (x, y)$. Determine the local extreme of this function by hand (show the calculation) as a function of the coordinates of A, B, C. Argue why the local extreme you found is precisely the desired minimum.

2. It might be that one of the villages does not agree with the previous decision, for whatever reason. An alternative consensus, with very surprising consequences, is to have the *sum of the distances* AP, BP, CP minimised. This problem is much harder to solve in general, although the answer is quite simple.

   As a first step we remark that this is a geometrical problem, namely given a triangle ABC, we must determine the point P so that $AP + BP + CP$ is minimised. To simplify our problem we rotate shift and scale our triangle in such a way that $A = (0, 0)$ and $B = (0, 2)$. Let us assume this.

   (a) Denote the sum of the distances AP, BP, CP by $d(x, y)$ and write down a formula for $d(x, y)$ in terms of x, y, x_C, y_C.

(b) Choose $y_C = 1$ and x_C some number larger than 1. Using Mathematica, determine the location of the local minimum of $d(x, y)$ by setting the two partial derivatives equal to 0. Do this for several values of x_C and make a table of the results. For each of your results, what is the angle between the villages A and B, as seen from the optimal position of the power plant?

(c) Again choose $y_C = 1$ and now take $x_C = 0.4$. You will find that the above approach yields no answer! In this case, make a contourplot of $d(x, y)$ with suitably chosen x- and y-ranges. By zooming in on this x- and y-range, make an estimate for the location of the local minimum. Apparently, taking partial derivatives did not work, but still there exists a solution to our problem.

(d) What could be the reason that taking partial derivatives did not work in this case?

(e) Repeat the steps from the previous question with the value $x_C = 0$. In this case, can you give a theoretical explanation for your result?

(f) Choose some other values for x_C, y_C and find the location of the minimum, either by taking partial derivatives, or by contourplots. In each case make an accurate drawing of the result. Can you venture a guess towards the solution of the problem? (It turns out that there are two very distinct cases. Above, you have seen an example of both cases)

3. This time we consider a different problem. There is a railway line, which is a straight line. On this line we want a railway station S such that the sum of the distances AS, BS, CS is minimised.

(a) Suppose the locations of the villages are $A = (0, 0), B = (5, 1), C = (2, 3)$. The railway line is given by the equation $x + y = 1$. Using Mathematica, determine the resulting location of S.

(b) Suppose now that we are in a general situation in which two of the villages, say A and B, are located on the railway line and C is off the line. Using common sense, can you say where the railway station should be located? Provide a convincing argument for your answer.