Assignment 3: A night at the casino

Roulette

One evening you decide to go gambling at a Casino. You have only 50 Euro in your pocket. Of course you must stop when you are broke. On the other hand, you promise yourself that you will also stop if your capital has reached 250 Euro.
You decide to play roulette and to put a fixed stake s on red at every turn of the wheel.
For those of you who are not familiar with roulette, at every turn of the wheel, you have a probability of $q = 19/37$ to lose your stake and a probability of $1 - q = 18/37$ to get your stake doubled. In the first case you lose the amount s, in the second case you gain an extra amount s.
You consider the following four options for the fixed stake s:

$$5, \quad 10, \quad 25, \quad \text{or} \quad 50 \text{ Euro.}$$

You hesitate to make the choice, since you want to minimize the risk of ending up broke, but on the other hand you want to enjoy taking part in the game for more than five seconds. And, intuitively you might feel that both the probability of ending up with 250 Euro and the expected duration of the session are influenced by the choice. But in what way? At this stage you are invited to check your intuition and to guess the answer.

I. Simulation and statistics

1. Use the random number generator to simulate the change in your capital at each turn of the roulette wheel. More specifically, write a Mathematica routine, named `riennevaplus` for example, which takes as input the stake s and which outputs $-s$ with probability $q = 19/37$ and s with probability $1 - q$.

2. Use the routine from the previous part to write a simulation for one game starting with 50 Euro and ending with 0 or 250 Euro. More specifically, write a function, named `playaround` for example, which takes s as its input, simulates a round, and outputs your capital at the end of the round, i.e. 0 or 250 Euro.

3. For each of the fixed stakes s play 100 rounds and count the number of times you won, i.e. ended up with 250 Euro. Perform this experiment a number of times to get an idea of your winning chances for each value of s. More precisely, compute the average winning chance and its standard deviation.

   Repeat the same experiment, but then with 500 rounds. What is your estimation of your winning chances based on these experiments?

4. Modify the function `playaround` so that it outputs the number of wheel turns for the round that has been played.

   For each value of s simulate 500 games, say, and make a histogram of the results. What is the average game length for each value of s ? and the standard deviation?

II. Theoretical considerations

We now turn to a Markov chain description. If the stake is fixed to s, the capital that you have in your possession at any one moment is one of the numbers $0, s, 2s, \ldots, Ks$, where $Ks = 250$. The state where our capital equals $i \times s$ is denoted by i. Let $p_i(n)$ denote the probability that your capital is in state i, i.e. it equals $i \times s$ Euro, after n turns of the roulette wheel. Denote by $\mathbf{p}(n)$ the column vector whose components are the probabilities $p_i(n)$.
Then $\mathbf{p}(n+1) = M\mathbf{p}(n)$ for some $(K+1) \times (K+1)$ matrix M.

1. Let us, from now on, assume that our fixed stake $s = 50$. Then the possible states are numbered by $0, 1, 2, 3, 4, 5$. Write down the matrix M in terms of q. Hint: first make a picture of the corresponding transition graph.

2. Write down the initial distribution $\mathbf{p}(0)$.

3. What are the eigenvectors of M with eigenvalue 1? Interpret these vectors in terms of absorbing states.

4. Check, either by hand or Mathematica, that the row vectors

$$\mathbf{I} = (1, 1, 1, 1, 1, 1)$$

$$\mathbf{W} = \left(1, \left(\frac{q}{1-q}\right), \left(\frac{q}{1-q}\right)^2, \left(\frac{q}{1-q}\right)^3, \left(\frac{q}{1-q}\right)^4 \left(\frac{q}{1-q}\right)^5\right)$$

   satisfy the relations

$$\mathbf{I}M = \mathbf{I} \qquad \mathbf{W}M = \mathbf{W}.$$

5. Now assume that $M^n\mathbf{p}(0)$ tends to a limit as $n \to \infty$ which is of the form

$$\begin{pmatrix} z_l \\ 0 \\ \vdots \\ 0 \\ z_w \end{pmatrix}$$

   What is the interpretation of the numbers z_l, z_w ?

6. Using the previous two problems show that the probability to end a round with 250 Euro is equal to

$$\frac{1-\left(\frac{q}{1-q}\right)}{1-\left(\frac{q}{1-q}\right)^5} \approx 0.179$$

Does this correspond with your results from the simulation? (see question I.3).

7. Now show that, starting with a capital of $50c$ where $c = 1, 2, 3$ or 4, the probability of winning the round is equal to

$$\frac{1-\left(\frac{q}{1-q}\right)^c}{1-\left(\frac{q}{1-q}\right)^5}.$$

8. Suppose now that s is any one of the other fixed stakes. What do you think is the answer to the previous question in the new situation? (No proof required here, just a wise guess). Use this answer to give a theoretical prediction for the results of your simulation in question I.3.

Now we make an estimate of the expected length of a single round. Suppose we are in the general situation with a fixed stake s. Denote by L_i the expected length of a round if we started with state i (i.e. $i \times s$ Euro in our pocket).

1. Give an explanation of why we should expect that

$$L_i = qL_{i-1} + (1-q)L_{i+1} + 1, \qquad 1 \leq i \leq K-1$$

and $L_0 = L_K = 0$.

2. The answer to the above recursive relations reads

$$L_i = \frac{i}{2q-1} - \frac{K}{2q-1}\,\frac{1-\left(\frac{q}{1-q}\right)^i}{1-\left(\frac{q}{1-q}\right)^K}.$$

Check that this indeed satisfies our recursive relations.

3. Compare this answer with your findings of the simulation from Problem I.4 for the four possible stakes.