**Computer session SCI 111, differential equations**

# Sample systems of first order equations

In this computer session we study systems of first order differential equations of the form

$$\begin{aligned} \frac{dx}{dt} &= f(x,y) \\ \frac{dy}{dt} &= g(x,y) \end{aligned}$$

Because the right hand sides of these equations do not depend explicitly on $t$ we call such a system *autonomous*. In the following problems we shall consider various autonomous systems using the applet *newphase* which can be found under `www.math.uu.nl/people/beukers/phase/newphase.html`.

## Linear equations with constant coefficients

These are systems of the form

$$\begin{aligned} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy \end{aligned}$$

where $a, b, c, d$ are constants. The shape of the vector field depends on the eigenvalues $\lambda_1, \lambda_2$ of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Here are the possibilities,

1. Both $\lambda_i$ are positive.
2. Both $\lambda_i$ are negative.
3. The $\lambda_i$ are real with different sign.
4. The $\lambda_i$ are complex with positive real part.
5. The $\lambda_i$ are complex with negative real part.

Here is a sample system,

$$\begin{aligned} \frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= cx + y \end{aligned}$$

**Problem 1.1** Determine (by hand) $\lambda_1, \lambda_2$ as functions of $c$.
Study the vectorfield and its integral curves as we let $c$ vary (i.e. make various choices for $c$). Pay particular attention to the cases $c > 1$, $0 < c < 1$ and $c < 0$. Which of the above instances apply to these possiblities for $c$ ?
**Problem 1.2** Answer the same questions for

$$\begin{aligned} \frac{dx}{dt} &= -x + y \\ \frac{dy}{dt} &= cx - y \end{aligned}$$

## First order differential equations

Consider the general first order differential equation

$$\frac{dy}{dx} = F(x, y).$$

We can rewrite it as an autonomous system by changing the $x$ on the left into $t$ and adding the trivial equation $\frac{dx}{dt} = 1$. So we get,

$$\begin{aligned} \frac{dx}{dt} &= 1 \\ \frac{dy}{dt} &= F(x, y) \end{aligned}$$

**Problem 1.3** Study the solutions of the equation $\frac{dy}{dx} = y(1 - y)$ and make a sketch which gives an idea of the complete set of solutions.
What happens to the solutions satisfying $0 < y(0) < 1$ ? And the solutions that satisfy $y(0) > 1$?

## Second order equations

A general second order differential equation without explicit time-dependence has the form

$$\frac{d^2x}{dt^2} = F(x, \frac{dx}{dt})$$

Introduce the new variable $y = \frac{dx}{dt}$ and our equation reads

$$\frac{dy}{dt} = F(x, y)$$

Together with $y = \frac{dx}{dt}$ this yields an autonomous system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= F(x,y) \end{aligned}$$

The $x, y$-plane is called the *phase plane* by the physicists. This is a terminology that we shall adopt.

In this way for example we can study second order linear equations $\frac{d^2x}{dt} + a\frac{dx}{dt} + bx = 0$ by studying the system

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -bx - ay \end{aligned}$$

which is what we did previously

The differential equation $x'' + \omega^2 x = 0$ corresponds to the so-called harmonic oscillator with angular frequency $\omega$. Its solutions are given by linear combinations of $\sin \omega t$ and $\cos \omega t$.

**Problem 1.4** Take $\omega = 1$ and study the solutions with our phase-plane plotter. The harmonic ocsillator with damping is decribed by $x'' + kx' + \omega^2 x = 0$ where $k > 0$ is a damping factor. Study the solutions for $\omega = 1$ and increasing $k$. Pay particular attention to the cases $k < 2$ and $k > 2$. What happens in the case of negative friction, i.e. $k < 0$?

## The pendulum

The differential equation of a pendulum is given by

$$x'' + (g/l)\sin x = 0$$

where $x$ is the angle of displacement, $g$ the accelaration of gravity and $l$ the length of the pendulum. We shall take $g/l = 1$. In the sample systems of our program you find this system included.

**Problem 1.5** Study its solutions in the phase plane and sketch them. Can you understand the periodicity of the picture? Furthermore, there are two kinds of orbits: closed ones and non-closed ones. What do they correspond to physically? Explain.

**Problem 1.6** Now introduce a small damping factor into the equation,

$$x'' + kx + \sin x = 0.$$

Study the corresponding system again with increasing $k > 0$. What happens to the diagrams as $k$ grows past $k = 2$ ?

## Van de Pol equation

The Van der Pol equation is a classical equation which arose in the study of spontaneously oscillating valve circuits. It reads

$$x'' + e(x^2 - 1)x' + x = 0, \quad e > 0.$$

**Problem 1.7** In the sample systems of our program we have the case $e = 1$. Study it by sketching the integral curves. Also consider what happens if we let $e$ increase from $e = 0$.
**Problem 1.8** Notice that the Van der Pol equation really looks like a harmonic oscillator with a damping which depends on $x$. When $e = 0$ we have exactly the harmonic oscillator. The damping becomes negative if $|x| < 1$ and $e > 0$. In the light of these remarks, do you get a better understanding of the plots you obtained above?
You have undoubtedly noticed the persistent periodic solution to Van der Pol's equation. This is an example of a *limit cycle.*

## Volterra's equation

This system of equations models a very simple predator-prey system in theoretical biology. In general it reads

$$\frac{dx}{dt} = x(a - by)$$
$$\frac{dy}{dt} = y(-c + dx)$$

It is a sample system in our program with $a = b = c = d = 1$.
**Problem 1.9** Study its solutions and make a sketch of them. Do the same thing for various other choices of $a, b, c, d$. Note that the general picture remains.
The integral curves turn out to be closed and this indicates that we are dealing with an *integrable system.* This simply means that there is a function $F(x, y)$ such that $F(x(t), y(t))$ is constant for every solution $x(t), y(t)$. The function $F$ is called a *first integral.* In our twodimensional case the existence of an integral $F$ implies that the solution curves of our system are the level lines of $F(x, y)$. In our case at hand with $a = b = c = d = 1$ this integral function is given by $F(x, y) = \ln|x| + \ln|y| - x - y$. Can you show this? If not, make a check by plotting the level lines of $F$ using Mathematica.