The Real Numbers

1 Axioms for the real numbers

Many of the axioms for integers also hold for the real numbers. So it suffices to point out the differences. One important property is that division is always possible in $\mathbb{R}$. So we add the following axiom on multiplication to that effect.

A.7 To every pair $a, b \in \mathbb{R}$ with $b \neq 0$ there exists $c \in \mathbb{R}$ such that $a = bc$. We write this element as $a/b$.

As a result of this axiom the well-ordering axiom (B.6) cannot hold anymore. The set of negative real numbers $-1/2, -1/3, -1/4, \ldots, -1/k, \ldots$ does not contain a biggest element. So we must drop the well-ordering axiom. We are now left with a set of axioms which does not quite describe the real numbers. Note for example that they are also satisfied by the rational numbers $\mathbb{Q}$. The most notable feature of the real numbers is that we can take limits. So we add the following axiom.

B.7 (Completeness axiom) To every non-empty $S \subset \mathbb{R}$ which is bounded from above, there exists a smallest upper bound in $\mathbb{R}$. We call this the supremum of $S$. Notation $\text{sup}(S)$.

In a way this axiom replaces B.6. For example, the supremum of the negative real numbers is 0. We can rephrase it in various ways. For example, to every nonempty subset of $\mathbb{R}$ which is bounded from below, there exists a biggest lower bound in $\mathbb{R}$. We call the latter the infimum of $S$. Notation: $\text{inf}(S)$.

Without going too much into further details we remark that the set $\mathbb{R}$ is uniquely determined by the axioms given above.

The Archimedean principle

Theorem 1.1 For any two $a, b \in \mathbb{R}$ with $a < b$ there exists $r \in \mathbb{Q}$ such that $a < r \leq b$.

We say that the rational numbers lie dense in the real numbers. Although well-known, it is not in our list of axioms and we need to prove it. We do this here.

Choose an integer $n$ bigger than $1/(b-a)$. Then $nb - na > 1$. Let $m$ be the largest integer $\leq nb$. Then $nb - 1 < m \leq nb$. Hence $na < nb - 1 < m \leq nb$. After division by $n$, we get $a < m/n \leq b$. 
Bounded increasing sequences

Theorem 1.2 Let \( a_1 \leq a_2 \leq a_3 \leq \ldots \) be an infinite sequence of real numbers which is increasing with \( n \) and suppose also that there exists \( A \in \mathbb{R} \) such that \( a_n \leq A \) for all \( n \in \mathbb{N} \). Then the limit \( \lim_{n \to \infty} a_n \) and equals \( \alpha = \sup\{a_1, a_2, \ldots\} \).

In the following proof we use Definition 1.5 of the word \textit{limit}.

\textbf{Proof.}: Choose \( \epsilon > 0 \). Since \( \alpha - \epsilon \) is not an upper bound for \( \{a_1, a_2, \ldots\} \), there exists \( N \) such that \( a_N > \alpha - \epsilon \). For all \( n \geq N \) we have that \( a_n \geq a_N \) and hence \( a_n > \alpha - \epsilon \). In addition we know that \( a_n < \alpha \), so \( |\alpha - a_n| \leq \epsilon \) for all \( n \geq N \). This proves the convergence to the limit \( \alpha \). \( \square \)

Limits of sequences

\textbf{Example 1.3} (Decimal expansion)

We prove that an infinite decimal fraction corresponds to a real number. Consider 
\[ 0.a_1a_2a_3a_4\ldots \text{ where } a_i \in \{0, 1, \ldots, 9\} \]. This represents an infinite increasing sequence \( S \) of rational numbers starting with 
\[ 0.a_1, 0.a_1a_2, 0.a_1a_2a_3, 0.a_1a_2a_3a_4, \ldots \]

By induction one can show that for all \( n \in \mathbb{N} \) we have 
\[ 0.a_1a_2\ldots a_n \leq 1 - \frac{1}{10^n} \]

So \( S \) is bounded from above by 1 and the sequence converges to a limit. This is the real number represented by the infinite decimal expansion.

\textbf{Example 1.4} (The exponential \( e^x \))

Let \( x \in \mathbb{R}_{\geq 0} \) and define the sequence of numbers 
\[ a_n = \left(1 + \frac{x}{n}\right)^n \]

for all \( n \in \mathbb{N} \). One can show that \( a_{n+1} > a_n \) and that the sequence is bounded. Hence \( \lim_{n \to \infty} a_n \) exists. We call this limit \( e^x \).

\textbf{Definition 1.5} Let \( a_1, a_2, a_3, \ldots \) be a sequence of real numbers. We say that this sequence converges to the limit \( L \) if for any \( \epsilon > 0 \) there exists an index \( n_\epsilon \) such that \( |a_n - L| < \epsilon \) for all \( n \geq n_\epsilon \).

\textbf{Notation}: \( \lim_{n \to \infty} a_n = L \).

Here are some rules for limits. Let \( a_1, a_2, a_3, \ldots \) and \( b_1, b_2, b_3, \ldots \) be two sequences which converge to a limit \( \alpha \) and \( \beta \) respectively. Then
1. \( \lim_{n \to \infty} a_n + b_n = \alpha + \beta \)

2. \( \lim_{n \to \infty} a_n b_n = \alpha \beta \)

3. if \( \beta \neq 0 \) and \( b_n \neq 0 \) for all \( n \), then \( \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\alpha}{\beta} \).

4. Let \( c_1, c_2, c_3, \ldots \) be a third sequence and suppose that \( a_n \leq c_n \leq b_n \) for all \( n \in \mathbb{N} \) and suppose that \( \alpha = \beta \). Then \( \lim_{n \to \infty} c_n = \alpha \).

Cauchy’s convergence criterion

Sometimes one knows about the existence of a limit without knowing its value. For example, increasing sequences which are bounded from above have a limit. Here is a general criterion.

**Theorem 1.6 (Cauchy criterion)** Let \( a_1, a_2, a_3, \ldots \) be a sequence of real numbers. Then this sequence converges to a limit if and only if for any \( \epsilon > 0 \) there exists an index \( n_\epsilon \) such that \( |a_n - a_m| < \epsilon \) for all \( m, n \geq n_\epsilon \).

**Proof.** Suppose first that the sequence has a limit, say \( L \). Choose \( \epsilon > 0 \). Then there exists an index \( n_\epsilon \) such that \( |a_n - L| < \epsilon/2 \) for all \( n \geq n_\epsilon \). Hence, for any \( n, n \geq n_\epsilon \) we have

\[
|a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |a_n - L| < \epsilon/2 + \epsilon/2 = \epsilon.
\]

Thus our convergence criterion is satisfied.

Suppose conversely that the convergence criterion is satisfied. There exists an index \( n_1 \) such that \( |a_n - a_m| < 1 \) (i.e. we took \( \epsilon = 1 \)) for all \( n \geq n_1 \). So the sequence is bounded from above and below. Now we need to find a candidate for a limit. Consider the sequence of numbers \( A_1, A_2, A_3, \ldots \) given by \( A_n = \inf\{a_n, a_n+1, a_n+2, \ldots\} \). This is an increasing sequence and it is bounded from above. So it has a limit, say \( L \). Now choose \( \epsilon > 0 \). There exists an index \( N \) such that \( |a_n - a_m| < \epsilon/3 \) for all \( n, m \geq N \) and \( |A_n - L| < \epsilon/3 \) for all \( n \geq N \). Since \( A_N \) is an infimum there exists an index \( k \geq N \) such that \( A_N \leq a_k \leq A_N + \epsilon/3 \). In particular \( |A_N - a_k| \leq \epsilon/3 \). Together with \( |a_k - a_n| < \epsilon/3 \) for all \( n \geq N \) and \( |A_N - L| < \epsilon/3 \) we get

\[
|a_n - L| = |a_n - a_k + a_k - A_N + A_N - L| \\
\leq |a_n - a_k| + |a_k - A_N| + |A_N - L| \\
< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon
\]

for all \( n \geq N \). This proves that \( \lim_{n \to \infty} a_n = L \). \( \square \)
Infinite series

Let \( a_1, a_2, a_3, \ldots \) be a sequence of real numbers. For every \( k \in \mathbb{N} \) we define the partial sums
\[
s_k = a_1 + a_2 + \cdots + a_k.
\]

We say that the infinite series \( \sum_{n=1}^{\infty} a_n \) converges if the sequence of partial sums converges. The limit is then called the sum of the infinite series.

In the book it is shown that the series
\[
\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots
\]
converges to 1. Using this series we can prove many other convergence results by comparison. This is based on the following theorem.

**Theorem 1.7** Let \( a_1, a_2, a_3, \ldots \) and \( b_1, b_2, b_3, \ldots \) be two sequence of real numbers such that \( 0 \leq a_n \leq b_n \) for all \( n \in \mathbb{N} \). If the infinite series \( \sum_{n=1}^{\infty} b_n \) converges then so does \( \sum_{n=1}^{\infty} a_n \).

We say that the series \( \sum_{n=1}^{\infty} a_n \) is dominated by \( \sum_{n=1}^{\infty} b_n \).

**Proof** Consider the partial sums \( s_k = a_1 + a_2 + \cdots + a_k \). This is an increasing sequence (i.e \( s_{k+1} \geq s_k \) for all \( k \in \mathbb{N} \)). Let \( B = \sum_{n=1}^{\infty} b_n \). Then \( s_k \leq B \) for all \( k \).

Hence \( s_1, s_2, s_3, \ldots \) is bounded from above and hence \( \lim_{n \to \infty} s_n \) exists.

Here is an application: the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges. To prove this it suffices to notice that \( \frac{1}{n^2} \leq \frac{2}{n(n+1)} \) for all \( n \in \mathbb{N} \) and apply the above theorem together with the result that \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \) converges. It was shown by Euler that
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
\]

Consider the following series with positive and negative terms,
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots
\]

Notice, by taking together every two consecutive terms, that this equals
\[
\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \cdots
\]

Note that the latter series is dominated by our convergent series \( \sum_{n} \frac{1}{n(n+1)} \). So our series \( \sum_{n} \frac{(-1)^{n-1}}{n} \) also converges. It turns out that the sum equals \( \ln(2) \).
It may happen that the sum of an infinite series depends on the order in which we sum the terms. Consider the re-ordered sum
\[ \frac{1}{1} - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots \]
In other words, after every positive term we take the next two negative terms. Now subtract from every positive term the negative term that follows. We get
\[ \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots \]
Note that this equals
\[ \frac{1}{2} \left( \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \right) = \frac{1}{2} \ln(2). \]
Surprisingly, we get a different sum. Even worse, if we take the next four negative terms after every positive term the new sum will be 0.
The explanation of the dependence of the sum on the order of summation resides in the fact that the infinite sum with terms \((-1)^{n-1}\) converges but the infinite sum of the absolute values \(\frac{1}{n}\) does not converge.

**Definition 1.8** We say that an infinite series \(\sum_{n=1}^{\infty} a_n\) converges absolutely if \(\sum_{n=1}^{\infty} |a_n|\) converges.

**Theorem 1.9** Suppose \(\sum_{n=1}^{\infty} a_n\) is an infinite series that converges absolutely. Then,
1. The series itself converges.
2. The sum of the series does not depend on the order in which we perform our summation.

Finally we give a general theorem on the convergence of series with terms that have an alternating sign.

**Theorem 1.10 (Dirichlet)** Let \(a_1, a_2, a_3, \ldots\) be a decreasing sequence with limit 0. Then the series \(\sum_{n=1}^{\infty} (-1)^{n-1}a_n\) is convergent.

Example,
\[ \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots \]
converges. It turns out that the limit equals \(\frac{\pi}{4}\) (Leibniz).

**Proof:** Let \(s_n\) be the \(n\)-th partial sum. Notice that for every \(k \in \mathbb{N}\) we have
\[ s_{2k+2} = s_{2k} + a_{2k+1} - a_{2k+2} \geq s_{2k} \text{ and } s_{2k+1} = s_{2k-1} - a_{2k} + a_{2k+1} \leq s_{2k-1}. \]
Furthermore, \(s_{2k} = s_{2k-1} - a_{2k} \leq s_{2k-1}. \) So we see that \(s_2, s_4, s_6, \ldots\) is an increasing sequence bounded from above by \(s_1\) and \(s_1, s_3, s_4, \ldots\) is a decreasing sequence bounded from below by \(s_2.\) So both sequences converge. Let \(A = \lim_{k \to \infty} s_{2k}\) and \(B = \lim_{k \to \infty} s_{2k-1}.\) Note that \(B - A = \lim_{k \to \infty} (s_{2k-1} - s_{2k}) = \lim_{k \to \infty} a_{2k} = 0.\) So \(A = B\) and both sequences of partial sums tend to \(A.\) We conclude that \(\lim_{n \to \infty} s_n = A.\) Our series is convergent.