# Home work problems for WISB324

The material up to chapter 14 are relevant for a number of these problems and the statement from Ch 15 that the number of non-equivalent irreducible representations equals the number of conjugacy classes of the group.

1. Let, as usual, $\mathbb{C}G$ be the group algebra of a finite group $G$.
   (a) Show that for every $\mathbb{C}G$-homomorphism $\phi : \mathbb{C}G \to \mathbb{C}G$ there exists $w \in \mathbb{C}G$ such that $\phi(r) = rw$ (hint: take $w = \phi(e)$).
   (b) Let $W \subset \mathbb{C}G$ be an irreducible $\mathbb{C}G$-submodule of $\mathbb{C}G$. Let $w \in W$ be a non-zero element. Show that $W = \{rw | r \in \mathbb{C}G\}$.
2. Define the group
$$G = \left\langle a, b | a^5 = b^4 = e, b^{-1}ab = a^{-1} \right\rangle .$$
   (a) Show that $b^2$ commutes with all elements of $G$.
   (b) Determine all conjugacy classes of $G$.
   (c) Determine all one-dimensional representations of $G$.
   (d) Determine the dimensions of all irreducible representations
   (e) Determine all higher dimensional (i.e. dim$>$ 1) representations of $G$. Give the matrix images (up to conjugation) of $a, b$ for these representations.
3. Define the vector space
$$V = \left\{ \sum_{1 \le i < j \le 4} a_{ij} x_i x_j \,\middle|\, a_{ij} \in \mathbb{C} \right\} \subset \mathbb{C}[x_1, \ldots, x_4].$$
   Define the representation $\rho$ of $S_4$ on $V$ by $\sigma : x_i x_j \mapsto x_{\sigma(i)} x_{\sigma(j)}$ for all $i, j$.
   (a) Determine the characters of $\rho$.
   (b) Determine the irreducible representations that compose $\rho$ (hint: use the character table of $S_4$, to be completed on Monday May 27, or consult p351 of the book).
   (c) Determine a basis for each of the irreducible subrepresentations of $\rho$.
4. Consider the representation $\rho$ of $S_5$ on $\mathbb{C}^5$ given by
$$\sigma \mathbf{e}_i \mapsto \mathbf{e}_{\sigma(i)}$$
   for all $i$, where $\mathbf{e}_1, \ldots, \mathbf{e}_5$ is the standaard basis of $\mathbb{C}^5$. Show that $\rho$ is a direct sum of the trivial representation and an irreducible one.