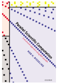
Parallel LU Decomposition
(PSC §2.3)

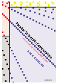
Designing a parallel algorithm

- Main question: how to distribute the data?
- What data? The matrix A and the permutation π.
- Data distribution $+$ sequential algorithm
  $\longrightarrow$ computation supersteps.
- Design backwards: insert preceding communication supersteps following the need-to-know principle.

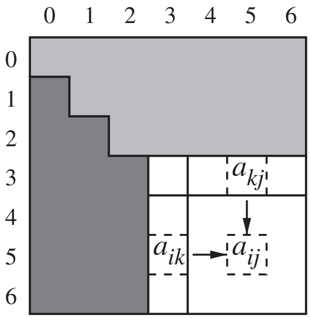


Data distribution for the matrix A

- The bulk of the work in the sequential computation is the update
$$a_{ij} := a_{ij} - a_{ik}a_{kj}$$
for matrix elements a_{ij} with $i, j \geq k+1$, taking $2(n-k-1)^2$ flops.
- The other operations take only $n-k-1$ flops. Thus, the data distribution is chosen mainly by considering the matrix update.
- Elements a_{ij}, a_{ik}, a_{kj} may not be on the same processor. Who does the update?
- Many elements a_{ij} must be updated in stage k, but only few elements a_{ik}, a_{kj} are used, all from column k or row k of the matrix. Moving those elements around causes less traffic.
- Therefore, the owner of a_{ij} computes the new value a_{ij} using communicated values of a_{ik}, a_{kj}.



Matrix update by operation $a_{ij} := a_{ij} - a_{ik}a_{kj}$



Update of row i uses only one value, a_{ik}, from column k. If we distribute row i over only N processors, then a_{ik} needs to be sent to at most $N - 1$ processors.

Matrix distribution

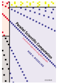
- A matrix distribution is a mapping

  $$\phi: \{(i,j) : 0 \leq i,j < n\} \rightarrow \{(s,t) : 0 \leq s < M \wedge 0 \leq t < N\}$$

  from the set of matrix index pairs to the set of processor identifiers. The mapping function ϕ has two coordinates,

  $$\phi(i,j) = (\phi_0(i,j), \phi_1(i,j)).$$

- Here, we number the processors in 2D fashion, with $p = MN$. This is just a numbering!
- Processor numberings have no physical meaning. BSPlib randomly renumbers the processors at the start.
- A processor row $P(s,*)$ is a group of N processors $P(s,t)$ with $0 \leq t < N$. A processor column $P(*,t)$ is a group of M processors $P(s,t)$ with $0 \leq s < M$.



Cartesian matrix distribution

	$t=0$	2	1	2	0	1	0
$s=0$	00	02	01	02	00	01	00
0	00	02	01	02	00	01	00
1	10	12	11	12	10	11	10
0	00	02	01	02	00	01	00
1	10	12	11	12	10	11	10
0	00	02	01	02	00	01	00
1	10	12	11	12	10	11	10

A matrix distribution is called Cartesian if $\phi_0(i,j)$ is independent of j and $\phi_1(i,j)$ is independent of i:

$$\phi(i,j) = (\phi_0(i), \phi_1(j)).$$

Parallel matrix update

(8) **if** $\phi_0(k) = s \wedge \phi_1(k) = t$ **then** put a_{kk} in $P(*, t)$;

(9) **if** $\phi_1(k) = t$ **then for all** $i : k < i < n \wedge \phi_0(i) = s$ **do**
$a_{ik} := a_{ik}/a_{kk}$;

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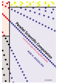
(9) **if** $\phi_1(k) = t$ **then for all** $i : k < i < n \wedge \phi_0(i) = s$ **do**
$\quad a_{ik} := a_{ik}/a_{kk}$;

(10) **if** $\phi_1(k) = t$ **then for all** $i : k < i < n \wedge \phi_0(i) = s$ **do**
$\quad$ put a_{ik} in $P(s, *)$;
if $\phi_0(k) = s$ **then for all** $j : k < j < n \wedge \phi_1(j) = t$ **do**
$\quad$ put a_{kj} in $P(*, t)$;

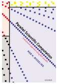
(11) **for all** $i : k < i < n \wedge \phi_0(i) = s$ **do**
$\quad$ **for all** $j : k < j < n \wedge \phi_1(j) = t$ **do**
$\qquad a_{ij} := a_{ij} - a_{ik}a_{kj}$;



Parallel pivot search

(0) **if** $\phi_1(k) = t$ **then** $r_s := \mathrm{argmax}(|a_{ik}| : k \le i < n \land \phi_0(i) = s)$;

(1) **if** $\phi_1(k) = t$ **then** put r_s and $a_{r_s,k}$ in $P(*, t)$;



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(1) **if** $\phi_1(k) = t$ **then** put r_s and $a_{r_s,k}$ in $P(*, t)$;

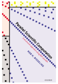
(2) **if** $\phi_1(k) = t$ **then**

$$s_{\max} := \mathrm{argmax}(|a_{r_q,k}| : 0 \leq q < M);$$
$$r := r_{s_{\max}};$$

(3) **if** $\phi_1(k) = t$ **then** put r in $P(s, *)$;

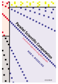
Two parallelisation methods

- The need-to-know principle: exactly those nonlocal data that are needed in a computation superstep should be fetched in preceding communication supersteps.
- Matrix update uses first parallelisation method: look at lhs (left-hand side) of assignment, owner computes.
- Pivot search uses second method: look at rhs of assignment, compute what can be done locally, reduce the number of data to be communicated.
- In pivot search: first a local search, then communication of the local winner to all processors, finally a redundant (replicated) search for the global winner.
- Broadcast of r in (3) is needed later in (4). Designing backwards, we formulate (4) first and then insert (3).



Distribution for permutation π

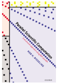
- Store π_k together with row k, somewhere in processor row $P(\phi_0(k), *)$.
- We choose $P(\phi_0(k), 0)$. This gives a true distribution.
- We could also have chosen to replicate π_k in processor row $P(\phi_0(k), *)$. This would save some **if**-statements in our programs.



Index and row swaps

(4) $\quad$ **if** $\phi_0(k) = s \wedge t = 0$ **then** put π_k as $\hat{\pi}_k$ in $P(\phi_0(r), 0)$;
$\quad\quad$ **if** $\phi_0(r) = s \wedge t = 0$ **then** put π_r as $\hat{\pi}_r$ in $P(\phi_0(k), 0)$;

(5) $\quad$ **if** $\phi_0(k) = s \wedge t = 0$ **then** $\pi_k := \hat{\pi}_r$;
$\quad\quad$ **if** $\phi_0(r) = s \wedge t = 0$ **then** $\pi_r := \hat{\pi}_k$;



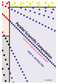
Index and row swaps

(4) **if** $\phi_0(k) = s \wedge t = 0$ **then** put π_k as $\hat{\pi}_k$ in $P(\phi_0(r), 0)$;
if $\phi_0(r) = s \wedge t = 0$ **then** put π_r as $\hat{\pi}_r$ in $P(\phi_0(k), 0)$;

(5) **if** $\phi_0(k) = s \wedge t = 0$ **then** $\pi_k := \hat{\pi}_r$;
if $\phi_0(r) = s \wedge t = 0$ **then** $\pi_r := \hat{\pi}_k$;

(6) **if** $\phi_0(k) = s$ **then for all** $j : 0 \le j < n \wedge \phi_1(j) = t$ **do**
put a_{kj} as $\hat{a}_{kj}$ in $P(\phi_0(r), t)$;
if $\phi_0(r) = s$ **then for all** $j : 0 \le j < n \wedge \phi_1(j) = t$ **do**
put a_{rj} as $\hat{a}_{rj}$ in $P(\phi_0(k), t)$;

(7) **if** $\phi_0(k) = s$ **then for all** $j : 0 \le j < n \wedge \phi_1(j) = t$ **do**
$a_{kj} := \hat{a}_{rj}$;
if $\phi_0(r) = s$ **then for all** $j : 0 \le j < n \wedge \phi_1(j) = t$ **do**
$a_{rj} := \hat{a}_{kj}$;



Optimising the matrix distribution

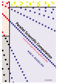
- We have chosen a Cartesian matrix distribution ϕ to limit the communication.
- We now specify ϕ further to achieve a good computational load balance and to minimise the communication.
- Maximum number of local matrix rows with index $\geq k$:

$$R_k = \max_{0 \leq s < M} |\{i : k \leq i < n \wedge \phi_0(i) = s\}|.$$

  Maximum number of local matrix columns with index $\geq k$:

$$C_k = \max_{0 \leq t < N} |\{j : k \leq j < n \wedge \phi_1(j) = t\}|.$$

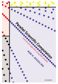
- The computation cost of the largest superstep, the matrix update (11), is then $2R_{k+1}C_{k+1}$.



Example

	$t=0$	2	1	2	0	1	0
$s=0$	00	02	01	02	00	01	00
0	00	02	01	02	00	01	00
1	10	12	11	12	10	11	10
0	00	02	01	02	00	01	00
1	10	12	11	12	10	11	10
0	00	02	01	02	00	01	00
1	10	12	11	12	10	11	10

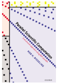
$$R_0 = 4,\ C_0 = 3 \text{ and } R_4 = 2,\ C_4 = 2$$



Bound for R_k

$$R_k \geq \left\lceil \frac{n-k}{M} \right\rceil .$$

Proof: Assume this is untrue, so that $R_k < \lceil \frac{n-k}{M} \rceil$. Because R_k is integer, we even have $R_k < \frac{n-k}{M}$. Hence all M processor rows together hold less than $M \cdot \frac{n-k}{M} = n-k$ matrix rows. But they hold all matrix rows $k \leq i < n$. Contradiction. $\square$

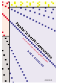


2D cyclic distribution attains bound

	$t=0$	1	2	0	1	2	0
$s=0$	00	01	02	00	01	02	00
1	10	11	12	10	11	12	10
0	00	01	02	00	01	02	00
1	10	11	12	10	11	12	10
0	00	01	02	00	01	02	00
1	10	11	12	10	11	12	10
0	00	01	02	00	01	02	00

$$\phi_0(i) = i \bmod M, \quad \phi_1(j) = j \bmod N.$$

$$R_k = \left\lceil \frac{n-k}{M} \right\rceil, \quad C_k = \left\lceil \frac{n-k}{N} \right\rceil.$$

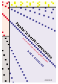


Cost of main computation superstep (matrix update)

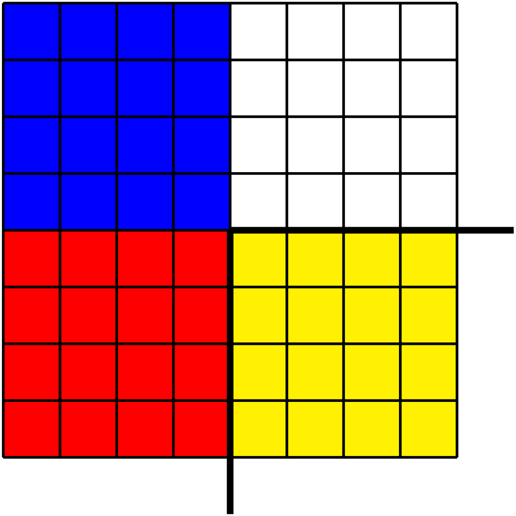
$$T_{(11),\mathrm{cyclic}} = 2 \left\lceil \frac{n-k-1}{M} \right\rceil \left\lceil \frac{n-k-1}{N} \right\rceil \geq \frac{2(n-k-1)^2}{p}.$$

$$\begin{aligned} T_{(11),\mathrm{cyclic}} &< 2\left(\frac{n-k-1}{M}+1\right)\left(\frac{n-k-1}{N}+1\right) \\ &= \frac{2(n-k-1)^2}{p} + \frac{2(n-k-1)}{p}(M+N)+2. \end{aligned}$$

The upper bound is minimal for $M = N = \sqrt{p}$. The second-order term $4(n-k-1)/\sqrt{p}$ is the additional computation cost caused by load imbalance.

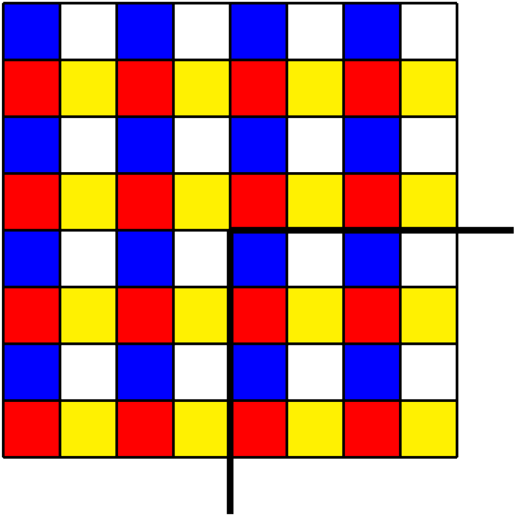


Load balance for the square block distribution



For $k \geq 4$, only the yellow processors works.

Load balance for the square cyclic distribution



For $k = 4, 5, 6$, all processors work.

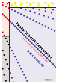
Cost of main communication superstep

The cost of the broadcast of row k and column k in (10) is

$$
\begin{aligned}
T_{(10)} &= (R_{k+1}(N-1) + C_{k+1}(M-1))g \\
&\geq \left(\left\lceil \frac{n-k-1}{M} \right\rceil (N-1) \; + \; \left\lceil \frac{n-k-1}{N} \right\rceil (M-1) \right) g \\
&= T_{(10),\mathrm{cyclic}}.
\end{aligned}
$$

$$
\begin{aligned}
T_{(10),\mathrm{cyclic}} &< \left(\left(\frac{n-k-1}{M} + 1 \right) N + \left(\frac{n-k-1}{N} + 1 \right) M \right) g \\
&= \left((n-k-1) \left(\frac{N}{M} + \frac{M}{N} \right) + M + N \right) g.
\end{aligned}
$$

The upper bound is again minimal for $M = N = \sqrt{p}$. The resulting communication cost is about $2(n-k-1)g$.



Summary

- We determined the matrix distribution, first by restricting it to be Cartesian, then by choosing the 2D cyclic distribution, based on a careful analysis of the main computation and communication supersteps, and finally by showing that a square $\sqrt{p} \times \sqrt{p}$ distribution is best.
- Developing the algorithm goes hand in hand with the cost analysis.
- We now have a correct algorithm and a good distribution, but the overall BSP cost may not be minimal yet. Wait and see . . .