Cartesian Distribution
(PSC §4.4)

Identifying 1D and 2D processor numbering

- Natural column-wise identification for $p = MN$ processors:

$$P(s,t) \equiv P(s+tM), \text{ for } 0 \le s < M \text{ and } 0 \le t < N.$$

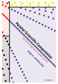
- This can also be written as

$$P(s) \equiv P(s \bmod M, s \operatorname{div} M), \text{ for } 0 \le s < p.$$

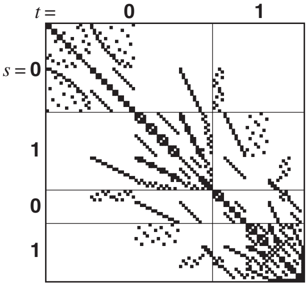
- For a Cartesian distribution (ϕ_0, ϕ_1), we map nonzeros a_{ij} to processors $P(\phi(i,j))$ by

$$\phi(i,j) = \phi_0(i) + \phi_1(j)M, \text{ for } 0 \le i,j < n \text{ and } a_{ij} \ne 0.$$

- We use 1D or 2D numbering, whichever is most convenient in the context.

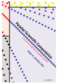


A Cartesian distribution of cage6



$n = 93$, $nz = 785$, $p = 4$, $M = N = 2$.

- The processor row of a matrix element a_{ij} is $s = \phi_0(i)$; the processor column is $t = \phi_1(j)$.
- Matrix diagonal assigned in blocks to processors $P(0) \equiv P(0,0)$, $P(1) \equiv P(1,0)$, $P(2) \equiv P(0,1)$, $P(3) \equiv P(1,1)$.



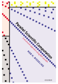
Advantages of a Cartesian distribution

Advantages:

- Main advantage for sparse matrices is the same as for dense matrices: row-wise operations require communication only within processor rows. (Similar for columns.)
- Vector component v_j has to be sent to at most M processors, and vector component u_i is computed using contributions received from at most N processors.
- Simplicity: Cartesian distributions partition the matrix orthogonally into rectangular submatrices. Non-Cartesian distributions create arbitrarily-shaped matrix parts.

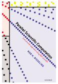
Disadvantage:

- Less general, so may not offer the optimal solution.



Matching matrix and vector distribution

- Vector component v_j is needed only by processors that possess an $a_{ij} \neq 0$, and these processors are contained in processor column $P(*, \phi_1(j))$.
- Assigning vector component v_j to one of the processors in $P(*, \phi_1(j))$ implies that v_j has to be sent to at most $M - 1$ processors, instead of M.
- If we are lucky (or clever), we may even avoid communication of v_j altogether.
- If v_j were assigned to a different processor column, it would always have to be communicated.
- Assigning u_i to a processor in processor row $P(\phi_0(i), *)$ reduces the number of contributions sent for u_i to at most $N - 1$.



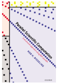
A trivial but powerful theorem

Theorem 4.4 Let A be a sparse $n \times n$ matrix and $\mathbf{u}, \mathbf{v}$ vectors of length n. Assume that:

1. distribution of A is Cartesian, $\mathrm{distr}(A) = (\phi_0, \phi_1)$;
2. distribution of $\mathbf{u}$ is such that u_i resides in $P(\phi_0(i), *)$;
3. distribution of $\mathbf{v}$ is such that v_j resides in $P(*, \phi_1(j))$.

Then: if u_i and v_j are assigned to the same processor, a_{ij} is also assigned to that processor and does not cause communication.

Proof Component u_i is assigned to $P(\phi_0(i), t)$. Component v_j to $P(s, \phi_1(j))$. Since this is the same processor, we have $(s, t) = (\phi_0(i), \phi_1(j))$, so that this processor also owns a_{ij}. $\square$



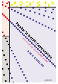
Special case $\mathrm{distr}(\mathbf{u}) = \mathrm{distr}(\mathbf{v})$

The conditions

1. distribution of A is Cartesian, $\mathrm{distr}(A) = (\phi_0, \phi_1)$;
2. distribution of $\mathbf{u}$ is such that u_i resides in $P(\phi_0(i), *)$;
3. distribution of $\mathbf{v}$ is such that v_j resides in $P(*, \phi_1(j))$;
4. $\mathrm{distr}(\mathbf{u}) = \mathrm{distr}(\mathbf{v})$;

imply that u_i and v_i are assigned to $P(\phi_0(i), \phi_1(i))$, which is the owner of the diagonal element a_{ii}.

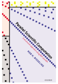
- For a fixed M and N, the choice of a Cartesian matrix distribution determines the vector distribution.
- The reverse is also true.



Example: 1D Laplacian matrix

$$A = \begin{bmatrix} -2 & 1 & & & & & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & & \ddots & & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1 \\ & & & & & 1 & -2 \end{bmatrix}.$$

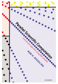
- This tridiagonal matrix represents a Laplacian operator on a 1D grid of n points.
- $a_{ij} \neq 0$ if and only if $i - j = 0, \pm 1$.



Vector distribution for tridiagonal matrix

- $a_{ij} \neq 0$ if and only if $i - j = 0, \pm 1$.
- Assume we require $\mathrm{distr}(\mathbf{u}) = \mathrm{distr}(\mathbf{v})$. Theorem 4.4 says that it is best to assign u_i and v_j (and hence u_j) to the same processor if $i = j \pm 1$.
- Therefore, a suitable vector distribution over p processors is the block distribution,

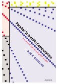
$$u_i \longmapsto P(i \text{ div } \left\lceil \frac{n}{p} \right\rceil), \text{ for } 0 \leq i < n.$$



Example: 12×12 1D Laplacian matrix

Distribution matrix for $n = 12$ and $M = N = 2$:

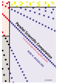
$$
\mathrm{distr}(A) = \begin{bmatrix}
0 & 0 & & & & & & & & & & \\
0 & 0 & 0 & & & & & & & & & \\
 & 0 & 0 & 0 & & & & & & & & \\
 & & 1 & 1 & 1 & & & & & & & \\
 & & & 1 & 1 & 1 & & & & & & \\
 & & & & 1 & 1 & \color{red}{3} & & & & & \\
 & & & & & \color{red}{0} & 2 & 2 & & & & \\
 & & & & & & 2 & 2 & 2 & & & \\
 & & & & & & & 2 & 2 & 2 & & \\
 & & & & & & & & 3 & 3 & 3 & \\
 & & & & & & & & & 3 & 3 & 3 \\
 & & & & & & & & & & 3 & 3
\end{bmatrix}.
$$



Example: 12×12 1D Laplacian matrix (cont'd)

Position (i, j) of $\mathrm{distr}(A)$ gives 1D identity of the processor that owns matrix element a_{ij}; $\mathrm{distr}(A)$ is obtained by:

- distributing the vectors by the 1D block distribution
- distributing the matrix diagonal in the same way as the vectors
- translating the 1D processor numbers into 2D numbers by $P(0) \equiv P(0,0)$, $P(1) \equiv P(1,0)$, $P(2) \equiv P(0,1)$, $P(3) \equiv P(1,1)$.
- determining the owners of the off-diagonal nonzeros: a_{56} is in the same processor row as a_{55}, owned by $P(1) = P(1,0)$; it is in the same processor column as a_{66}, owned by $P(2) = P(0,1)$. Thus, a_{56} is owned by $P(1,1) = P(3)$.

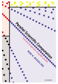


Cost analysis

Assuming a good spread of nonzeros and vector components over processors, matrix rows over processor rows, matrix columns over processor columns:

$$
\begin{aligned}
T_{(0)} &= (M-1)\frac{ng}{p} + l, \\
T_{(1)} &= \frac{2cn}{p} + l, \\
T_{(2)} &= (N-1)\frac{ng}{p} + l, \\
T_{(3)} &= \frac{Nn}{p} + l.
\end{aligned}
$$

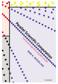
$$
T_{\mathrm{MV},\ M\times N} \le \frac{2cn}{p} + \frac{n}{M} + \frac{M+N-2}{p}ng + 4l.
$$



Efficient computation for $M = N = \sqrt{p}$

$$T_{\mathrm{MV},\ \sqrt{p}\times\sqrt{p}} \leq \frac{2cn}{p} + \frac{n}{\sqrt{p}} + 2\left(\frac{1}{\sqrt{p}} - \frac{1}{p}\right) ng + 4l.$$

- Computation is efficient if $\frac{2cn}{p} > \frac{2ng}{\sqrt{p}}$, i.e., $c > \sqrt{p}g$.
- Improvement of factor $\sqrt{p}$ compared to previous general efficiency criterion.

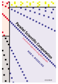


Dense matrices

- Dense matrices are the limit of sparse matrices for $c \to n$.
- Analysing the dense case is easier and it can give us insight into the sparse case as well.
- Substituting $c = n$ in previous cost formula gives

$$T_{\mathrm{MV,\ dense}} \leq \frac{2n^2}{p} + \frac{n}{\sqrt{p}} + 2\left(\frac{1}{\sqrt{p}} - \frac{1}{p}\right) ng + 4l.$$

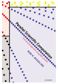
- All spreading assumptions must hold.
- Which distribution will yield this cost?



Square cyclic distribution? No!

- Previously, we have extolled the virtues of the square cyclic distribution for LU decomposition and all parallel linear algebra.
- Diagonal element a_{ii} is assigned to $P(i \bmod \sqrt{p}, i \bmod \sqrt{p})$, so that the matrix diagonal is assigned to the diagonal processors $P(s, s)$, $0 \leq s < \sqrt{p}$.
- Only $\sqrt{p}$ processors have part of the matrix diagonal and the vectors. The vector spreading assumption fails.
- The trouble is that diagonal processors must send $\sqrt{p} - 1$ copies of $\frac{n}{\sqrt{p}}$ vector components: $h_{\mathrm{s}} = n - \frac{n}{\sqrt{p}}$ in (0).
- The total cost for the square cyclic distribution is

$$T_{\mathrm{MV,\ dense,\ }\sqrt{p}\times\sqrt{p}\mathrm{\ cyclic}} = \frac{2n^2}{p} + n + 2\left(1 - \frac{1}{\sqrt{p}}\right) ng + 4l.$$

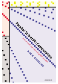


Cyclic row distribution? No!

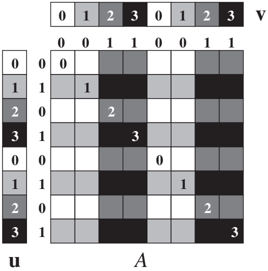
- Communication balance can be improved by choosing a distribution that spreads the matrix diagonal evenly, $\phi_{\mathbf{u}}(i) = \phi_{\mathbf{v}}(i) = i \bmod p$, and translating from 1D to 2D.
- We still have the freedom to choose M and N, where $MN = p$. For the choice $M = p$ and $N = 1$, this gives the cyclic row distribution $\phi_0(i) = i \bmod p$ and $\phi_1(j) = 0$.
- The total cost for the cyclic row distribution is

$$T_{\text{MV, dense, } p\times 1 \text{ cyclic}} = \frac{2n^2}{p} + \left(1 - \frac{1}{p}\right) ng + 2l.$$

- This distribution skips supersteps (2) and (3), since each matrix row is completely contained in one processor.
- The trouble is that the fanout is very expensive: each processor has to send $\frac{n}{p}$ vector components to all others.



Square Cartesian distribution? Yes!



$n = 8$, $p = 4$, $M = N = 2$. Square Cartesian distribution based on a cyclic distribution of the matrix diagonal.

- We take the same distribution method, $\phi_{\mathbf{u}}(i) = \phi_{\mathbf{v}}(i) = i \bmod p$, but now we choose $M = N = \sqrt{p}$ when translating from 1D to 2D.
- *Et voilà!* We achieve the optimal BSP cost.

Summary

- For Cartesian distributions, we use both 1D and 2D processor numberings to our advantage, with the identification $P(s,t) \equiv P(s+tM)$.
- We have seen the example of a tridiagonal matrix, where we obtained a 2D matrix distribution, slightly different from a 1D block row distribution. For band matrices with a wider band, this may be advantageous.
- A square Cartesian matrix distribution based on a cyclic distribution of the matrix diagonal and the input and output vectors is an optimal data distribution for dense matrices and for sparse matrices that are relatively dense.
- There exist other optimal data distributions, e.g. based on a block distribution of the matrix diagonal.