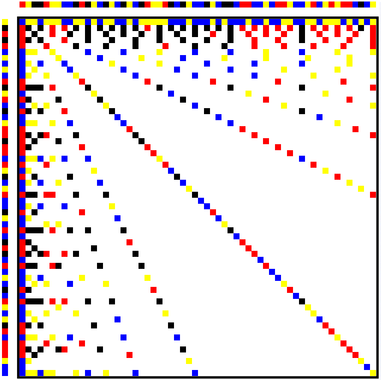
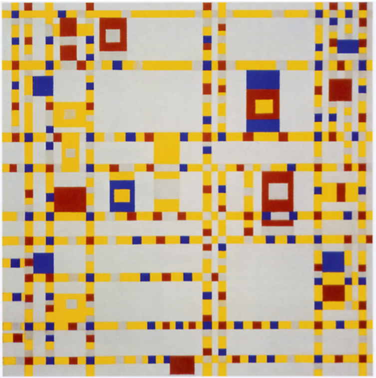
Vector Distribution
(PSC §4.6)

Vector partitioning





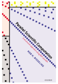
Broadway Boogie Woogie
Piet Mondriaan 1943

Balance the communication!

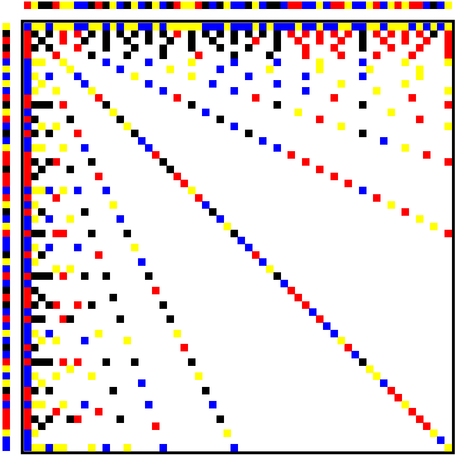
- Aim: reduce the BSP cost hg, where

$$h = \max_{0 \le s < p} h(s), \qquad h(s) = \max(h_\mathrm{s}(s), h_\mathrm{r}(s)).$$

- Thus, given a matrix distribution ϕ, we have to determine a vector distribution $\phi_\mathbf{v}$ that minimises h for the fanout and satisfies $j \in J_{\phi_\mathbf{v}(j)}$, for $0 \le j < n$.
- Constraint $j \in J_{\phi_\mathbf{v}(j)}$ means: processor $P(s) = P(\phi_\mathbf{v}(j))$ that owns v_j must own a nonzero in matrix column j, i.e., $j \in J_s$.
- We also have to find a vector distribution $\phi_\mathbf{u}$ that minimises the value h for the fanin and that satisfies the constraint $i \in I_{\phi_\mathbf{u}(i)}$, for $0 \le i < n$.

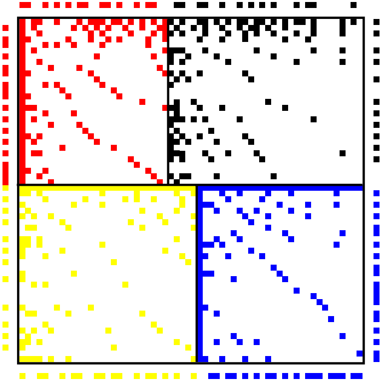


Vector partitioning for `prime60`



Global view. Both constraints are satisfied.

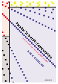
Vector partitioning for `prime60`



Local view. The local components of the vector $\mathbf{u}$ are placed to the left of the local submatrix for $P(0)$ and $P(2)$.

The two vector distribution problems are similar

- Nonzero pattern of row i of A equals the nonzero pattern of column i of A^T:
  u_{is} is sent from $P(s)$ to $P(t)$ in the multiplication by A
  $\Leftrightarrow$ v_i is sent from $P(t)$ to $P(s)$ in the multiplication by A^T.
- We can find a good distribution $\phi_{\mathbf{u}}$ given $\phi = \phi_A$ by finding a good distribution $\phi_{\mathbf{v}}$ given $\phi = \phi_{A^T}$.
- Hence, we only solve one problem, namely for **v**. We can apply this method also for **u**, with A^T instead of A.



General case: arbitrary q_j values

- Columns with $q_j = 0$ or $q_j = 1$ do not cause communication and are omitted from the problem. Hence, we assume $q_j \geq 2$, for all j.
- For processor $P(s)$:

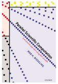
$$h_{\mathrm{s}}(s) = \sum_{0 \leq j < n,\ \phi_{\mathbf{v}}(j) = s} (q_j - 1),$$

  and

$$h_{\mathrm{r}}(s) = |\{j : j \in J_s \wedge \phi_{\mathbf{v}}(j) \neq s\}|.$$

- Aim: for given matrix distribution and hence given communication volume V, minimise

$$h = \max_{0 \leq s < p} \max\left(h_{\mathrm{s}}(s), h_{\mathrm{r}}(s)\right).$$



Egoistic local bound

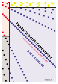
- An egoistic processor tries to minimise its own $h(s) = \max(h_{\mathrm{r}}(s), h_{\mathrm{s}}(s))$ without consideration for others.
- To minimise $h_{\mathrm{r}}(s)$, it just has to maximise the number of components v_j with $j \in J_s$ that it owns.
- To minimise $h_{\mathrm{s}}(s)$, it has to minimise the total weight of these components, where the weight of v_j is $q_j - 1$.
- A locally optimal strategy is to start with $h_{\mathrm{s}}(s) = 0$ and $h_{\mathrm{r}}(s) = |J_s|$ and grab the components in order of increasing weight, each time adjusting $h_{\mathrm{s}}(s)$ and $h_{\mathrm{r}}(s)$, as long as $h_{\mathrm{s}}(s) \leq h_{\mathrm{r}}(s)$.

Optimal values

- Denote the resulting optimal value of $h_{\mathrm{r}}(s)$ by $\hat{h}_{\mathrm{r}}(s)$, that of $h_{\mathrm{s}}(s)$ by $\hat{h}_{\mathrm{s}}(s)$, and that of $h(s)$ by $\hat{h}(s)$. We have

$$\hat{h}_{\mathrm{s}}(s) \leq \hat{h}_{\mathrm{r}}(s) = \hat{h}(s), \text{ for } 0 \leq s < p.$$

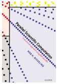
- The value $\hat{h}(s)$ is a local lower bound on the actual value that can be achieved: $\hat{h}(s) \leq h(s)$, for all s.



Example vector distribution problem

$s = 0$	1	·	1	·	1	1	1	1
1	1	1	·	1	1	1	1	·
2	·	1	·	·	·	1	1	1
3	·	·	1	1	1	·	·	1
$q_j =$	2	2	2	2	3	3	3	3
$j =$	0	1	2	3	4	5	6	7

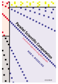
- A 1 in the table denotes that $P(s)$ owns a nonzero in column j and hence needs v_j.
- Columns are ordered by increasing q_j.
- Processor $P(0)$ wants v_0 and v_2, but nothing more, so that $\hat{h}_{\rm s}(0) = 2$, $\hat{h}_{\rm r}(0) = 4$, and $\hat{h}(0) = 4$.
- The fanout will cost at least $4g$.



Algorithm based on local bound

(R. H. Bisseling, W. Meesen, *Electronic Transactions on Numerical Analysis* **21** (2005) pp. 47–65.)

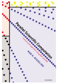
- Define the generalised lower bound $\hat{h}(J, ns_0, nr_0)$ for a given index set $J \subset J_s$ and a given initial number of sends ns_0 and receives nr_0.
- Initial communications are due to columns outside J.
- Bound is computed by the same method, but starting with $h_{\mathrm{s}}(s) = ns_0$ and $h_{\mathrm{r}}(s) = nr_0 + |J|$.
- Note that $\hat{h}(s) = \hat{h}(J_s, 0, 0)$.
- Our algorithm gives preference to the processor that faces the toughest future, i.e., the processor with the highest current value $\hat{h}(s)$.



Initialisation of algorithm

$$
\begin{array}{l}
\textbf{for } s := 0 \textbf{ to } p-1 \textbf{ do} \\
\qquad L_s := J_s; \\
\qquad h_{\mathrm{s}}(s) := 0; \\
\qquad h_{\mathrm{r}}(s) := 0;
\end{array}
$$

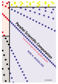
- L_s is the index set of components that may still be assigned to $P(s)$.
- The number of sends caused by the assignments done so far is registered as $h_{\mathrm{s}}(s)$; the number of receives as $h_{\mathrm{r}}(s)$.
- The current state of $P(s)$ is represented by the triple $(L_s, h_{\mathrm{s}}(s), h_{\mathrm{r}}(s))$.



Termination of algorithm

for $s := 0$ **to** $p-1$ **do**
  if $h_{\rm s}(s) < \hat{h}_{\rm s}(L_s, h_{\rm s}(s), h_{\rm r}(s))$ **then**
    $\mathrm{active}(s) := \mathit{true}$;
  else $\mathrm{active}(s) := \mathit{false}$;

- Note that $ns_0 \leq \hat{h}_{\rm s}(J, ns_0, nr_0)$, so that trivially $h_{\rm s}(s) \leq \hat{h}_{\rm s}(L_s, h_{\rm s}(s), h_{\rm r}(s))$.
- A processor will not accept more components once it has achieved its optimum, when $h_{\rm s}(s) = \hat{h}_{\rm s}(L_s, h_{\rm s}(s), h_{\rm r}(s))$.



Main loop of algorithm

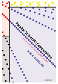
while $(\exists s : 0 \leq s < p \wedge \mathrm{active}(s))$ **do**

$s_{\mathsf{max}} := \mathrm{argmax}(\hat{h}_{\mathrm{r}}(L_s, h_{\mathrm{s}}(s), h_{\mathrm{r}}(s)) : 0 \leq s < p \wedge \mathrm{active}(s));$

$j := \mathsf{min}(L_{s_{\mathsf{max}}});$ $\{j$ has minimal $q_j\}$

$\phi_{\mathbf{v}}(j) := s_{\mathsf{max}};$

$h_{\mathrm{s}}(s_{\mathsf{max}}) := h_{\mathrm{s}}(s_{\mathsf{max}}) + q_j - 1;$



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  $\phi_{\mathbf{v}}(j) := s_{\mathsf{max}};$

  $h_{\mathrm{s}}(s_{\mathsf{max}}) := h_{\mathrm{s}}(s_{\mathsf{max}}) + q_j - 1;$

  for all $s : 0 \leq s < p \wedge s \neq s_{\mathsf{max}} \wedge j \in J_s$ **do**

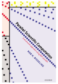
    $h_{\mathrm{r}}(s) := h_{\mathrm{r}}(s) + 1;$

  for all $s : 0 \leq s < p \wedge j \in J_s$ **do**

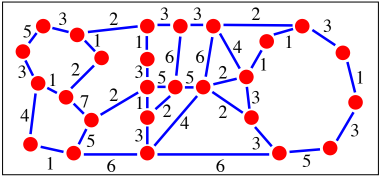
    $L_s := L_s \backslash \{j\};$

    if $h_{\mathrm{s}}(s) = \hat{h}_{\mathrm{s}}(L_s, h_{\mathrm{s}}(s), h_{\mathrm{r}}(s))$ **then**

      $\mathrm{active}(s) := \mathit{false};$

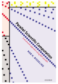


Special case: $q_j \leq 2$

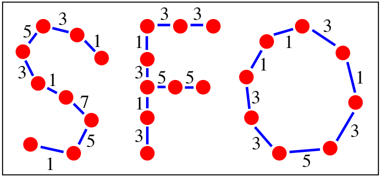


- Vertex s = processor s, $0 \leq s < p$
- Edge (s, t) = processor pair sharing matrix columns
- Edge weight $w(s, t)$ = number of matrix columns shared

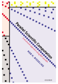
Problem: assign each matrix column/vector component to a processor, balancing the number of data words sent and received



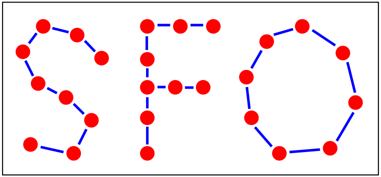
Transform into unweighted undirected graph



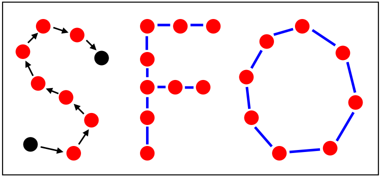
- Assign two shared columns: one to processor s, one to t.
  $w(s,t) := w(s,t) - 2$.
- Repeat until all edge weights $= 0$ or 1.



Unweighted undirected graph

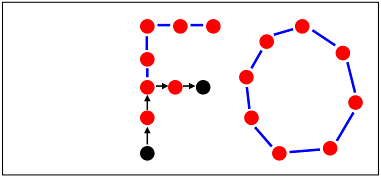


Transform into directed graph

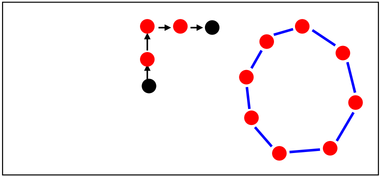


- Walk path starting at odd-degree vertex
- Remove walked edges from undirected graph
- Edge $s \to t$: processor s sends, t receives
- Even-degree vertices remain even-degree
- Repeat until all degrees in undirected graph are even.

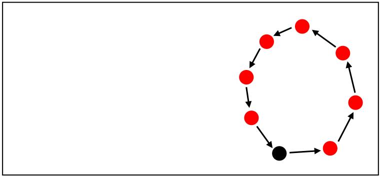
Transform into directed graph



Transform into directed graph



Transform into directed graph



- Walk path starting at even-degree vertex
- Repeat until undirected graph empty
- Solution is provably optimal (see Bisseling & Meesen 2005)

Summary

- BSP cost is a natural metric that encourages communication balancing.
- For the general vector distribution problem, we have developed a heuristic method, which works well in practice.
- The heuristic method is based on assigning vector components to the processor with the toughest future, as predicted by an egoistic local bound.
- For the special case with at most 2 processors per matrix column, we have obtained an optimal method based on walking paths in an associated graph, starting first at odd-degree vertices.