Parallel LU Decomposition

Section 2.3 of Parallel Scientific Computation, 2nd edition

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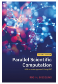
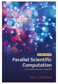


Designing a parallel algorithm

- The main question is: how to distribute the data?
- What data? The matrix A and the permutation π.
- Data distribution + sequential algorithm $\longrightarrow$ computation supersteps.
- Design the parallel algorithm backwards: insert communication supersteps where needed, following the need-to-know principle.

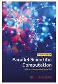


Data distribution for the matrix A

- The bulk of the work in the sequential case is the update

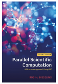
$$a_{ij} := a_{ij} - a_{ik}a_{kj}$$

  for elements a_{ij} with $i, j \geq k+1$, taking $2(n-k-1)^2$ flops.
- The other operations take only $n-k-1$ flops. Thus, the data distribution is chosen mainly by considering the matrix update.
- Elements a_{ij}, a_{ik}, a_{kj} may not be on the same processor.
- Who does the update?

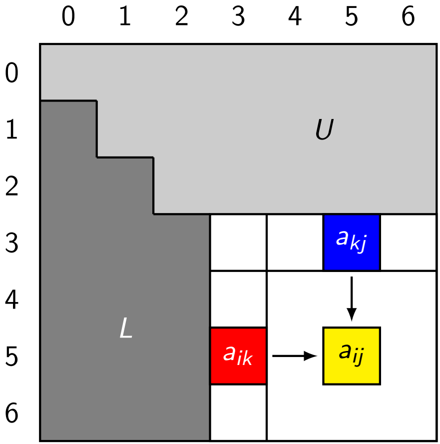


The owner computes

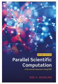
- Many elements a_{ij} must be updated in stage k, but using only few elements a_{ik}, a_{kj}, all from column k or row k. Moving those elements around causes less traffic.
- Therefore, the owner of a_{ij} computes the new value a_{ij} using communicated values of a_{ik}, a_{kj}.



Matrix update by operation $a_{ij} := a_{ij} - a_{ik}a_{kj}$



- The update of row i uses only one value, a_{ik}, from column k.
- If we distribute row i over N processors, then a_{ik} needs to be sent to $\leq N-1$ processors.



2D matrix distribution

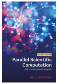
- A matrix distribution is a mapping

  $$\phi : \{(i,j) : 0 \leq i,j < n\} \rightarrow \{(s,t) : 0 \leq s < M \wedge 0 \leq t < N\}$$

  from the set of matrix index pairs to the set of processor identifiers.
- The mapping function ϕ has two coordinates,

  $$\phi(i,j) = (\phi_0(i,j), \phi_1(i,j)).$$

- Here, we number the processors in 2D fashion, where $p = MN$. This is just a numbering, without physical meaning!
- BSP newcomers should think that BSPlib randomly renumbers the processors at the start.
- A processor row $P(s,*)$ is a group of N processors $P(s,t)$ with $0 \leq t < N$.
- A processor column $P(*,t)$ is a group of M processors $P(s,t)$ with $0 \leq s < M$.

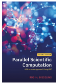


Cartesian matrix distribution

	$t =$ 0	2	1	2	0	1	0
$s =$ 0	00	02	01	02	00	01	00
0	00	02	01	02	00	01	00
1	10	12	11	12	10	11	10
0	00	02	01	02	00	01	00
1	10	12	11	12	10	11	10
0	00	02	01	02	00	01	00
1	10	12	11	12	10	11	10

- A matrix distribution is called Cartesian if

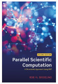
$$\phi(i,j) = (\phi_0(i), \phi_1(j)).$$



Parallel algorithm for Cartesian distribution: divisions

if $\phi_0(k) = s \wedge \phi_1(k) = t$ **then** ▷ Superstep (8)
    put a_{kk} in $P(*, t)$;

if $\phi_1(k) = t$ **then** ▷ Superstep (9)
    for all $i : k < i < n \wedge \phi_0(i) = s$ **do**
        $a_{ik} := \frac{a_{ik}}{a_{kk}}$;

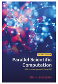


Parallel algorithm: matrix update

if $\phi_1(k) = t$ **then** ▷ Superstep (10)
    for all $i : k < i < n \land \phi_0(i) = s$ **do**
        put a_{ik} in $P(s, *)$;

if $\phi_0(k) = s$ **then**
    for all $j : k < j < n \land \phi_1(j) = t$ **do**
        put a_{kj} in $P(*, t)$;

for all $i : k < i < n \land \phi_0(i) = s$ **do** ▷ Superstep (11)
    for all $j : k < j < n \land \phi_1(j) = t$ **do**
        $a_{ij} := a_{ij} - a_{ik}a_{kj}$;



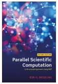
Parallel pivot search

if $\phi_1(k) = t$ **then** ▷ Superstep (0)

$r_s := \mathrm{argmax}(|a_{ik}| : k \leq i < n \wedge \phi_0(i) = s)$;

if $\phi_1(k) = t$ **then** ▷ Superstep (1)

put r_s and $a_{r_s,k}$ in $P(*, t)$;



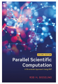
Parallel pivot search

if $\phi_1(k) = t$ **then** $\triangleright$ Superstep (0)
  $r_s := \mathrm{argmax}(|a_{ik}| : k \leq i < n \wedge \phi_0(i) = s);$

if $\phi_1(k) = t$ **then** $\triangleright$ Superstep (1)
  put r_s and $a_{r_s,k}$ in $P(*, t)$;

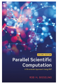
if $\phi_1(k) = t$ **then** $\triangleright$ Superstep (2)
  $s_{\max} := \mathrm{argmax}(|a_{r_q,k}| : 0 \leq q < M);$
  $r := r_{s_{\max}};$

if $\phi_1(k) = t$ **then** $\triangleright$ Superstep (3)
  put r in $P(s, *)$;



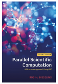
Two parallelization methods

- The need-to-know principle: exactly those nonlocal data that are needed in a computation superstep should be fetched in preceding communication supersteps.
- Matrix update uses first parallelization method: look at lhs (left-hand side) of assignment; the owner computes.
- Pivot search uses second method: look at rhs of assignment; compute what can be done locally, which reduces the number of data to be communicated.
- In pivot search: first a local search, then communication of the local winner to all processors, finally a redundant search for the global winner.
- Broadcast of r in superstep (3) is needed later in (4). Designing backwards, we formulate (4) first and then insert (3).



Distribution for permutation π

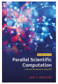
- We should store π_k together with row k, somewhere in processor row $P(\phi_0(k), *)$.
- We could choose a single location such as $P(\phi_0(k), 0)$. This gives a true distribution.
- We choose, however, to replicate π_k in processor row $P(\phi_0(k), *)$. This saves some **if**-statements in our algorithm and removes clutter.



Index swaps

if $\phi_0(k) = s$ **then** ▷ Superstep (4)
    put π_k as $\hat{\pi}_k$ in $P(\phi_0(r), t)$;
if $\phi_0(r) = s$ **then**
    put π_r as $\hat{\pi}_r$ in $P(\phi_0(k), t)$;

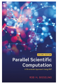
if $\phi_0(k) = s$ **then** $\pi_k := \hat{\pi}_r$; ▷ Superstep (5)
if $\phi_0(r) = s$ **then** $\pi_r := \hat{\pi}_k$;



Row swaps

if $\phi_0(k) = s$ **then** ▷ Superstep (6)
  for all $j : 0 \leq j < n \wedge \phi_1(j) = t$ **do**
    put a_{kj} as $\hat{a}_{kj}$ in $P(\phi_0(r), t)$;
if $\phi_0(r) = s$ **then**
  for all $j : 0 \leq j < n \wedge \phi_1(j) = t$ **do**
    put a_{rj} as $\hat{a}_{rj}$ in $P(\phi_0(k), t)$;

if $\phi_0(k) = s$ **then** ▷ Superstep (7)
  for all $j : 0 \leq j < n \wedge \phi_1(j) = t$ **do**
    $a_{kj} := \hat{a}_{rj}$;
if $\phi_0(r) = s$ **then**
  for all $j : 0 \leq j < n \wedge \phi_1(j) = t$ **do**
    $a_{rj} := \hat{a}_{kj}$;



Optimizing the matrix distribution

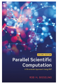
- We have chosen a Cartesian matrix distribution ϕ to limit the communication.
- We now specify ϕ further to achieve a good computational load balance and to minimize the communication.
- Maximum number of local matrix rows with index $\geq k$:

  $$R_k = \max_{0 \leq s < M} |\{i : k \leq i < n \wedge \phi_0(i) = s\}|.$$

  Maximum number of local matrix columns with index $\geq k$:

  $$C_k = \max_{0 \leq t < N} |\{j : k \leq j < n \wedge \phi_1(j) = t\}|.$$

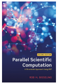
- The computation cost of the largest superstep, the matrix update (11), is then $2R_{k+1}C_{k+1}$.



Example

	$t =$ 0	2	1	2	0	1	0
$s =$ 0	00	02	01	02	00	01	00
0	00	02	01	02	00	01	00
1	10	12	11	12	10	11	10
0	00	02	01	02	00	01	00
1	10	12	11	12	10	11	10
0	00	02	01	02	00	01	00
1	10	12	11	12	10	11	10

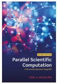
$$R_0 = 4, C_0 = 3$$



Lower bound on R_k

$$R_k \geq \left\lceil \frac{n-k}{M} \right\rceil .$$

Proof: Assume this is false, so that $R_k < \lceil \frac{n-k}{M} \rceil$. Because R_k is integer, we even have $R_k < \frac{n-k}{M}$. Hence all M processor rows together hold fewer than $M \cdot \frac{n-k}{M} = n - k$ matrix rows. But they hold all matrix rows $k \leq i < n$, which are $n - k$ rows. Contradiction. □

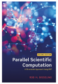


2D cyclic distribution attains the lower bound

	$t =$ 0	1	2	0	1	2	0
$s =$ 0	00	01	02	00	01	02	00
1	10	11	12	10	11	12	10
0	00	01	02	00	01	02	00
1	10	11	12	10	11	12	10
0	00	01	02	00	01	02	00
1	10	11	12	10	11	12	10
0	00	01	02	00	01	02	00

$$\phi_0(i) = i \bmod M, \quad \phi_1(j) = j \bmod N.$$

$$R_k = \left\lceil \frac{n-k}{M} \right\rceil, \quad C_k = \left\lceil \frac{n-k}{N} \right\rceil.$$

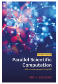


Cost of main computation superstep (the matrix update)

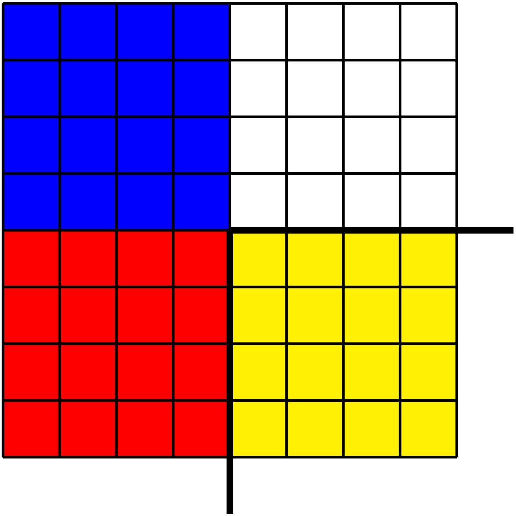
$$T_{(11),\mathrm{cyclic}} = 2 \left\lceil \frac{n-k-1}{M} \right\rceil \left\lceil \frac{n-k-1}{N} \right\rceil \geq \frac{2(n-k-1)^2}{p}.$$

$$\begin{aligned} T_{(11),\mathrm{cyclic}} &< 2\left(\frac{n-k-1}{M}+1\right)\left(\frac{n-k-1}{N}+1\right) \\ &= \frac{2(n-k-1)^2}{p} + \frac{2(n-k-1)}{p}(M+N)+2. \end{aligned}$$

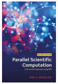
- The upper bound is minimal for a square distribution, $M = N = \sqrt{p}$.
- The second-order term $\frac{4(n-k-1)}{\sqrt{p}}$ is the additional computation cost caused by load imbalance.



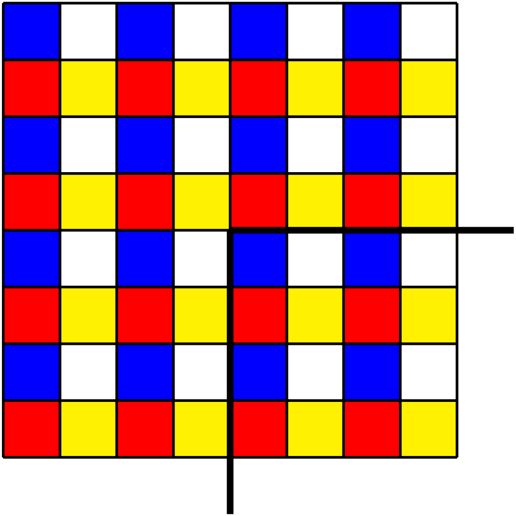
Bad load balance for the square block distribution



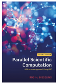
For $k = 4, 5, 6, 7$, only the yellow processor works.



Better load balance for the square cyclic distribution



For $k = 4, 5, 6$, all processors work.



Cost of main communication superstep (the broadcast)

- The cost of the broadcast of row k and column k in (10) for a Cartesian distribution is

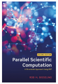
$$
\begin{aligned}
T_{(10)} &= (R_{k+1}(N-1) + C_{k+1}(M-1))g \\
&\geq \left(\left\lceil \frac{n-k-1}{M} \right\rceil (N-1) \; + \; \left\lceil \frac{n-k-1}{N} \right\rceil (M-1) \right) g \\
&= T_{(10),\mathrm{cyclic}},
\end{aligned}
$$

  so the $M \times N$ cyclic distribution is the best.

- The broadcast cost for the 2D cyclic distribution has an upper bound

$$
\begin{aligned}
T_{(10),\mathrm{cyclic}} &< \left(\left(\frac{n-k-1}{M} + 1 \right) N + \left(\frac{n-k-1}{N} + 1 \right) M \right) g \\
&= \left((n-k-1) \left(\frac{N}{M} + \frac{M}{N} \right) + M + N \right) g.
\end{aligned}
$$

- This upper bound is minimal for $M = N = \sqrt{p}$. The resulting communication cost is about $2(n-k-1)g$.



Summary

- We determined the matrix distribution, first by restricting it to be Cartesian, then by choosing it to be 2D cyclic.
- We did this based on a careful analysis of the main computation and communication supersteps.
- We then showed that a square $\sqrt{p} \times \sqrt{p}$ distribution is best.
- Cliffhanger: we now have a correct algorithm and a good distribution, but the overall BSP cost might be improved. Wait and see . . .