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DENSITY RESULTS FOR AUTOMORPHIC FORMS ON HILBERT MODULAR GROUPS

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1 Introduction

Let F be a totally real number field of dimension d, and let $\mathcal{O}_F$ be its ring of integers. If $\mathfrak{q}$ is an ideal in $\mathcal{O}_F$ let $\Gamma = \Gamma_0(\mathfrak{q})$ denote the congruence subgroup of Hecke type of the Hilbert modular group. In the present paper, we derive density results for cuspidal automorphic representations of $G = \mathrm{SL}_2(\mathbb{R})^d$ in $L^2(\Gamma(\mathfrak{q})\backslash G)$. The main result, Theorem 3.3, implies that there are infinitely many cuspidal automorphic representations $\varpi = \otimes_{j=1}^{d} \varpi_j$, even if we restrict some components ϖ_j. In particular, let Λ be the set of eigenvalue vectors $\lambda_\varpi = (\lambda_{\varpi_1}, \dots, \lambda_{\varpi_d})$ in $\mathbb{R}^d$. For each $j = 1, \dots, d$ let $\Lambda^{(j)}$ be the projection in $\mathbb{R}^{d-1}$ by omitting the j-th coordinate. A consequence of Theorem 3.3 is that $\Lambda^{(j)} \cap [1/4, \infty)^{d-1}$ is dense in $[1/4, \infty)^{d-1}$.

Proposition 3.8 makes this statement more precise. Let $\mathcal{P}_X$ be the set of cuspidal representations for which the λ_{ϖ_j} are in fixed intervals $I_j \subset \mathbb{R}_{\geq 0}$ at all places $j \neq l$, and such that $1/4 < \lambda_{\varpi_l} < X$. In the case when $I_j \subset (1/4, \infty)$ for each j, i.e. if all components are of principal series type, we will show that $X^{-1}\mu_r(\mathcal{P}_X)$ tends to a positive constant, where μ_r denotes a suitable measure in the unitary dual of G. That constant depends on the measure of the intervals and on F, but not on r, nor on $\mathfrak{q}$.

We consider also the set $\mathcal{D}_X$ of those automorphic representations that have a prescribed discrete series eigenvalue at each place $j \neq l$, and also a discrete series eigenvalue at $j = l$, with $-X \leq \lambda_{\varpi_l} \leq 0$. We shall show that this set has also positive density. (See Proposition 3.7.)

Our results imply that there are infinitely many automorphic representations that have a non-zero Fourier coefficient of order r and components with a prescribed type; i.e. for each j the type of ϖ_j can be prescribed to be either unitary principal series or discrete series.

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On the other hand, we will see in Proposition 3.6, that if at least one of the intervals I_j is contained in $[0,1/4]$, then the limit $X^{-1}\mu_r(\mathcal{P}_X)$ is zero. This implies that the automorphic representations which are of complementary series type at least at one place, are rare, i.e. they have density zero with respect to the measure μ_r. We note that this is expected, given the Ramanujan–Petersson conjecture which predicts no components of complementary series type in cuspidal automorphic representations.

As another main application, if we consider the set of all automorphic representations $\mathcal{A}_X$ with eigenvalue components λ_{ϖ_j} satisfying the condition $\sum_{j=1}^{d}|\lambda_{\varpi_j}| < X$, then we have that $X^{-d}\mu_r(\mathcal{A}_X)$ tends to a positive limit as $X \to \infty$ (see Corollary 3.4). This result can be seen as similar to a Weyl law, weighted by Fourier coefficients of automorphic representations.

A Weyl law for spherical automorphic representations for congruence subgroups of the Hilbert modular group was proved by Efrat [E1], by using the Selberg trace formula on $\Gamma\backslash G/K$ ([E2]). This result implies the existence of infinitely many K-spherical automorphic representations in this context. This existence result follows from Theorem 3.3. We observe that Efrat counts vectors of eigenvalues by their l^2-norm, whereas our result uses the l^1-norm. Moreover, our distribution results have squares of Fourier coefficients as weights.

Results related to those in this paper, in the case when $d = 1$, were obtained by Bruggeman [B, §4] and by Deshouillers–Iwaniec (see [DI, Theorem 2]). In the case of the Lie group $\mathrm{SU}(2,1)$ and the trivial K-type, a similar result was given by Reznikov in [R].

To prove these density results, we use a sum formula of Kuznetsov type for discrete cofinite subgroups $\Gamma \subset G$, in which all weights contribute. We obtained this sum formula in [BMP], and applied it there to give estimates for averages of Kloosterman sums for F.

The sum formula has the following type:

$$\int_Y \prod_{j=1}^{d} k_j(\nu_j)\, d\sigma_{r,r}(\nu) = \int_Y \prod_{j=1}^{d} k_j(\nu_j)\, d\delta(\nu) + K(\mathsf{B}k)\,.$$

The index j runs over the infinite places of F, the test functions k_j are even and holomorphic on a strip in $\mathbb{C}$ and the set Y of spectral parameters is a subset of $\mathbb{C}^d$. The measure $d\delta$ on Y has an elementary description (see (34) and (3)). The measure $d\sigma_{r,r}$ is supported on the set of spectral parameters of automorphic representations, and has weights that are essentially products of Fourier coefficients of automorphic forms for Γ (see (44), (78), (2)).

The term $K(\mathsf{B}k)$ is a sum of Kloosterman sums, depending on a Bessel transform $\mathsf{B}k$ of $k = \times_j k_j$.

In this paper we take a proper subset $E \subset \{1, \dots, d\}$, choosing k_j conveniently for $j \in Q := \{1, \dots, d\} \setminus E$ and leaving k_j free for $j \in E$. This choice will lead to a partial sum formula involving test functions of product type at the places in E (Theorem 3.1). To prove this, we apply the sum formula in [BMP] to suitable test functions depending on a parameter, and estimate the contributions of the different terms in the formula. The main task will be to show that the contributions of the so called Kloosterman and Eisenstein terms are of lower order of magnitude than that of the delta term (see sections 4 and 5). For the estimation of the Eisenstein term, we will need to give an estimate on the vertical line $\mathrm{Re}\,\nu = 0$, for Fourier coefficients of Eisenstein series at each cusp of $\Gamma_0(\mathfrak{q})$, making explicit the dependence on the order r of the Fourier term. In the estimate of the Fourier coefficient, we will use a logarithmic lower bound for ray class L-functions on the critical line. This will be obtained by an argument similar to one given by Landau in the case of the Dedekind zeta function [L].

The sum formula holds for any discrete subgroup $\Gamma \subset G$ of finite covolume. To arrive at the density results, we have critically used that Γ is a congruence subgroup to bound the Fourier coefficients of Eisenstein series. Without a good grasp on the contribution of the Eisenstein series, density results for the cuspidal spectrum are out of reach. We have employed a trivial estimate of Kloosterman sums. Using the Weil bound can improve some intermediate bounds. However, this does not influence the final density results.

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2 Preliminaries

As in [BMP], let F be a totally real number field, and let $\mathcal{O}$ be its ring of integers. We consider the algebraic group $\mathbf{G} = R_{F/\mathbb{Q}}(\mathrm{SL}_2)$ over $\mathbb{Q}$ obtained by restriction of scalars applied to SL_2 over F.

Let $\sigma_1, \dots, \sigma_d$ be the embeddings $F \to \mathbb{R}$. We have

$$G := \mathbf{G}_\mathbb{R} \cong \mathrm{SL}_2(\mathbb{R})^d, \qquad \mathbf{G}_\mathbb{Q} \cong \left\{ (x^{\sigma_1}, \dots, x^{\sigma_d}) : x \in \mathrm{SL}_2(F) \right\}. \qquad (1)$$

G contains $K := \prod_{j=1}^d \mathrm{SO}_2(\mathbb{R})$ as a maximal compact subgroup.

The image of $\mathrm{SL}_2(\mathcal{O}) \subset \mathrm{SL}_2(F)$ corresponds to $\mathbf{G}_{\mathbb{Z}}$. This is a discrete subgroup of $\mathbf{G}_{\mathbb{R}}$ with finite covolume. It is called the *Hilbert modular group*, see [F, §3]. We choose a non-zero ideal $\mathfrak{q}$ in $\mathcal{O}$ and form the congruence subgroup of Hecke type $\Gamma = \Gamma_0(\mathfrak{q}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}) : c \in \mathfrak{q} \right\}$, which has finite index in $\mathbf{G}_{\mathbb{Z}}$. Actually, the results in this paper can be easily extended to any Γ satisfying $\Gamma_0(\mathfrak{q}) \supset \Gamma \supset \Gamma(\mathfrak{q})$.

We are concerned with functions on $\Gamma\backslash G$. For simplicity, we restrict ourselves to functions that satisfy $f(\pm g_1, \pm g_2, \ldots, \pm g_d) = f(g_1, g_2, \ldots, g_d)$ for all choices of the $\pm$. That will imply that we will consider only automorphic forms with even weight. There should be no great difficulty in extending the results in this paper to functions on $\Gamma\backslash G$ that are odd at some real places.

By $L^2(\Gamma\backslash G)^+$ we mean the Hilbert space of (classes of) functions that are invariant by the center of G, and square integrable on $\Gamma\backslash G$ for the measure induced by the Haar measure. This Hilbert space contains the closed subspace $L^2_c(\Gamma\backslash G)^+$ generated by integrals of Eisenstein series. The orthogonal complement $L^2_d(\Gamma\backslash G)^+$ of $L^2_c(\Gamma\backslash G)^+$ is the closure of $\sum_{\varpi} V_{\varpi}$, where V_{ϖ} runs through an orthogonal family of closed irreducible subspaces for the G-action in $L^2(\Gamma\backslash G)$ by right translation.

The constant functions constitute one of these irreducible spaces. All the others have infinite dimension, and consist of cusp forms. Non-cuspidal, infinite dimensional V_{ϖ} would have to come from singularities of Eisenstein series. These do not occur in the present situation. The Fourier coefficients of Eisenstein series can be expressed in terms of number theoretical functions. This enables us to spot the singularities. The essential point is that the denominators $L(1+2\nu, \bar{\lambda}_{\mu}, \chi)$ in (58) have no zeros for $\mathrm{Re}\,\nu > 0$.

Irreducible unitary representations. Each representation ϖ has the form $\varpi = \otimes_j \varpi_j$, with ϖ_j an even unitary irreducible representation of $\mathrm{SL}_2(\mathbb{R})$. Table 1 lists the possible isomorphism classes for each ϖ_j. For each ϖ we define a spectral parameter $\nu_{\varpi} = (\nu_{\varpi,1}, \ldots, \nu_{\varpi,d})$, with $\nu_{\varpi,j}$ as in the last column of the table. There are Casimir operators C_j acting on each coordinate for $1 \le j \le d$. The eigenvalue $\lambda_{\varpi} \in \mathbb{R}^d$ is given by $\lambda_{\varpi,j} = \frac{1}{4} - \nu^2_{\varpi,j}$. We note that if ϖ_j lies in the complementary series, $\lambda_{\varpi,j} \in (0, 1/4)$ and if ϖ_j is isomorphic to a discrete series representation $D^{\pm}_b$, $b \in 2\mathbb{Z}$, $b \ge 2$, then $\lambda_{\varpi,j} = \frac{b}{2}\left(1 - \frac{b}{2}\right) \in \mathbb{Z}_{\le 0}$. If Q is an arbitrary subset of $\{1, \ldots, d\}$, then we shall denote by $\|\lambda_{\varpi,Q}\|_1 = \sum_{j \in Q} |\lambda_{\varpi,j}|$, the 1-norm of the projection $\lambda_{\varpi,Q}$, of λ_{ϖ} onto the subspace of $\mathbb{R}^d$ corresponding to Q.

notation		name	weights	ν
1		trivial representation	0	$\frac{1}{2}$
$H(s)$	$s \in i[0,\infty)$	unitary principal series	$q \in 2\mathbb{Z}$	$\frac{s}{2}$
$H(s)$	$s \in (0,1)$	complementary series	$q \in 2\mathbb{Z}$	$\frac{s}{2}$
D_b^+	$b \geq 2,\, b \in 2\mathbb{Z}$	holomorphic discrete series	$q \geq b,\, q \in 2\mathbb{Z}$	$\frac{b-1}{2}$
D_b^-	$b \geq 2,\, b \in 2\mathbb{Z}$	antiholomorphic discrete series	$q \leq -b,\, q \in 2\mathbb{Z}$	$\frac{b-1}{2}$

Table 1: *Irreducible unitary even representations of the Lie group* $\mathrm{SL}_2(\mathbb{R})$. All characters of $SO_2(\mathbb{R})$ occur at most once; the characters that occur are listed under *weights*. The last column gives a spectral parameter ν, with $\mathrm{Re}\,\nu \geq 0$, such that $\lambda(\nu) = \frac{1}{4} - \nu^2$ is the eigenvalue of the Casimir operator. See [La2, Chap. VI, §6].

The constant functions give rise to $\varpi = \mathbf{1} := \otimes_j 1$. It occurs with multiplicity one. If V_ϖ does not consist of the constant functions, then $\varpi_j \neq 1$ for all j.

If $\nu_j \in \left(0, \frac{1}{2}\right)$ for some $j = 1, \ldots, d$, then we call $\lambda(\nu) := \frac{1}{4} - \nu^2 := \left(\frac{1}{4} - \nu_j^2\right)_j$ an *exceptional eigenvalue*. We call such a ν_j an *exceptional coordinate*. If $d = 1$, it is known that only finitely many exceptional eigenvalues can occur for a given Γ. For $d > 1$ such a result has not been proved. In principle, there might be infinitely many exceptional eigenvalues, since one coordinate can stay small, while others tend to ∞. On the other hand, the Ramanujan–Petersson conjecture predicts that there are none, that is, $\nu_j \in i\mathbb{R} \cup \left(\frac{1}{2} + \mathbb{Z}_{\geq 0}\right)$, for all j. We note that the results of Efrat [E1] imply that there exist only finitely many eigenvalues such that *all* coordinates are exceptional.

In the case $F = \mathbb{Q}$, Selberg showed that $\nu_j \notin (1/4, 1/2]$ for such exceptional coordinates, that is $\lambda(\nu_j) \geq 3/16$ ([S]). This estimate has been improved and extended to arbitrary number fields by several authors, see [LuRS], [KS1], and [KS2]. Kim and Shahidi give the bound $\lambda(\nu_j) > \frac{1}{4} - \frac{1}{9^2}$, see [KS1].

Automorphic forms and Fourier coefficients. The elements of V_ϖ that transform on the right according to a character of the maximal compact subgroup K are square integrable automorphic forms, cusp forms if $\varpi \neq \mathbf{1}$. The Fourier coefficients of these cusp forms are independent of the actual choice within V_ϖ, and are determined by the number $c^r(\varpi) = c^r_\infty(\varpi)$

in equation (17) of [BMP]. As we consider only the cusp ∞, we omit the parameter κ from the notation. The number $r \in \mathcal{O}' = \{\, x \in F : \mathrm{Tr}_{F/\mathbb{Q}}(xy) \in \mathbb{Z} \text{ for all } y \in \mathcal{O}\,\}$ determines the order of the Fourier coefficient.

We take $r \neq 0$. That implies that $\varpi \neq \mathbf{1}$ if $c^r(\varpi) \neq 0$ for some $r \neq 0$. If $q = (q_1, \cdots, q_d) \in 2\mathbb{Z}$ is a weight occurring in ϖ, then there exists $\psi \in \varpi$, normalized as in [BMP, (15)], with the following Fourier term of order r at the point $g = \left(\begin{pmatrix} \sqrt{y_j} & x_j/\sqrt{y_j} \\ 0 & 1/\sqrt{y_j} \end{pmatrix} \begin{pmatrix} \cos\vartheta_j & \sin\vartheta_j \\ -\sin\vartheta_j & \cos\vartheta_j \end{pmatrix} \right) \in G$:

$$\frac{c^r(\varpi)}{\mathrm{vol}((\Gamma\cap N)\backslash N)} \cdot \prod_{j=1}^{d} e^{2\pi i r_j x_j} \frac{(-1)^{q_j/2}(2\pi|r_j|)^{-1/2}}{\Gamma\left(\frac12+\nu_j+\frac12 q_j\,\mathrm{sign}(r_j)\right)} W_{\mathrm{sign}(r_j)q_j/2,\nu_j}\big(4\pi|r_j|y_j\big)e^{iq_j\vartheta_j}, \quad (2)$$

where $N = \left\{ \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \right\} \subset G$, and $W_{\cdot,\cdot}$ is the exponentially decreasing Whittaker function.

Definition 2.1 (Test functions). Fix $\tau \in (1/2, 3/4)$. Let $\mathcal{L}$ be the space of even functions on the set

$$\{\nu \in \mathbb{C} : |\,\mathrm{Re}\;\nu| \le \tau\} \cup \left(\tfrac12 + \mathbb{Z}\right)$$

that are holomorphic on $|\,\mathrm{Re}\,\nu| \le \tau$ and satisfy the estimates

$$k(\nu) \ll \big(1 + |\,\mathrm{Im}\,\nu|\big)^{-a} \text{ for } |\,\mathrm{Re}\,\nu| \le \tau \text{ for some } a > 2, \text{ and}$$

$$\sum_{b\in 2\mathbb{Z},\, b\ge 2} \frac{b-1}{2}\left| k\left(\frac{b-1}{2}\right)\right| < \infty\,.$$

For each such test function, we define the following quantity:

$$\mathsf{H}(k) := \frac{i}{2}\int_{\mathrm{Re}\,\nu=0} k(\nu)\,\nu\,\tan\pi\nu\,d\nu + \sum_{b\ge 2,\, b\in 2\mathbb{Z}} \frac{b-1}{2} k\left(\frac{b-1}{2}\right). \quad (3)$$

We set

$$\mathcal{A}_d := \left\{\, \tfrac{b}{2}\left(1-\tfrac{b}{2}\right) : b \text{ even}, b \ge 2 \,\right\} \subset \mathbb{Z}_{\le 0}\,, \quad \mathcal{Y} := (0,\infty) \cup \mathcal{A}_d \subset \mathbb{R}\,. \quad (4)$$

The eigencoordinates $\lambda_{\varpi,j}$ are elements of $\mathcal{Y}$. The test functions $k \in \mathcal{L}$ give rise to functions of the form

$$g\left(\tfrac14 - \nu^2\right) = k(\nu)\,. \quad (5)$$

The domain of these functions consists of the $\frac14 - \nu^2 \in \mathbb{C}$ with $|\,\mathrm{Re}\,\nu| \le \tau$, together with the numbers $\frac{b}{2}\big(1-\frac{b}{2}\big)$, $b \in 2\mathbb{Z}$. We denote by $\tilde{\mathcal{L}}$ the class of functions g obtained in this way. If $g \in \tilde{\mathcal{L}}$ we set

$$\tilde{\mathsf{H}}(g) = \frac12\int_{1/4}^{\infty} g(y)\tanh\left(\pi\sqrt{y-1/4}\right)dy + \sum_{y\in\mathcal{A}_d} \sqrt{1/4-y}\,g(y)\,. \quad (6)$$

The functional $\tilde{\mathsf{H}}$ on $\tilde{\mathcal{L}}$ is described by a measure $d\eta$ on $\mathcal{Y}$, which is given by $d\eta(y) = \frac{1}{2}\tanh\bigl(\pi\sqrt{y-1/4}\bigr)dy$ on $(1/4,\infty)$, and by the sum of $\sqrt{1/4-y}$ times the delta measure at the points $y \in \mathcal{A}_d$.

3 Statement of Main Results

Theorem 3.1 and Theorem 3.3 state the main results of this paper. We give some consequences in Propositions 3.6–3.8 and Remark 3.9. We keep the notation from section 2.

All statements in this section depend on the partition of $\{1,\dots,d\}$ into three disjoint subsets E, Q_+ and Q_-, with $Q := Q_+ \cup Q_- \neq \emptyset$.

To each such partition we attach a set of irreducible representations of G,

$$\mathcal{R}(E,Q_+,Q_-) = \{\varpi \neq \mathbf{1} : \lambda_{\varpi,j} \geq 0 \text{ if } j \in Q_+, \lambda_{\varpi,j} < 0, \text{ if } j \in Q_-\}. \quad (7)$$

We shall often write $\mathcal{R} = \mathcal{R}(E,Q_+,Q_-)$. Our main results will follow from the next theorem.

Theorem 3.1. *Let $r \in \mathcal{O}' \smallsetminus \{0\}$. Choose a partition E, Q_+, Q_- of the set $\{1,\dots,d\}$ with $Q = Q_+ \cup Q_- \neq \emptyset$ and let $\mathcal{R}$ be the corresponding set as in* (7). *If $g = \times_{j\in E}\, g_j$, with $g_j \in \tilde{\mathcal{L}}$ for each $j \in E$, then the series*

$$Z_s(g) := \sum_{\varpi\in\mathcal{R}} \bigl|c^r(\varpi)\bigr|^2 e^{-s\|\lambda_{\varpi,Q}\|_1} \prod_{j\in E} g_j(\lambda_{\varpi,j}) \quad (8)$$

converges absolutely for each $s > 0$, and

$$\lim_{s\downarrow 0} s^{d-|E|} Z_s(g) = \frac{2^{1+|E|}}{(2\pi)^d}\sqrt{|D_F|}\prod_{j\in E}\tilde{\mathsf{H}}(g_j) \quad (9)$$

Here $\tilde{\mathsf{H}}(g)$ is as given in (6). *If $E = \emptyset$ then the products over $j \in E$ on the right-hand side of* (8) *and* (9) *are interpreted as* 1.

The theorem will be proved in section 6, as an application of the sum formula of Kuznetsov type given in [BMP]. In this section we shall use Theorem 3.1 to obtain density results for automorphic representations. We first prove the following proposition:

Proposition 3.2. *Let $r \in \mathcal{O}' \smallsetminus \{0\}$. Let E, $Q_\pm$ and $\mathcal{R}$ be as above. If $g_j \in \tilde{\mathcal{L}}$ for $j \in E$, then*

$$\lim_{X\to\infty} X^{|E|-d} \sum_{\substack{\varpi\in\mathcal{R}\\ \|\lambda_{\varpi,Q}\|_1 \leq X}} |c^r(\varpi)|^2 \prod_{j\in E} g_j(\lambda_{\varpi,j}) = \frac{2^{1+|E|}\sqrt{|D_F|}}{(d-|E|)!\,(2\pi)^d}\prod_{j\in E}\tilde{\mathsf{H}}(g_j)\,, \quad (10)$$

with $\tilde{\mathsf{H}}(g_j)$ *as given in* (6). *If* $E=\emptyset$ *then the products over* $j\in E$ *in* (10) *equal* 1.

Proof. For each $X>0$, and each $g=\times_{j\in E}\, g_j\in\tilde{\mathcal{L}}^{|E|}$, we define

$$\mu_g(X)=\sum_{\substack{\varpi\in\mathcal{R}\\ \|\lambda_{\varpi,Q}\|_1\le X}}\left|c^r(\varpi)\right|^2\prod_{j\in E} g_j(\lambda_{\varpi,j})\,. \tag{11}$$

The absolute convergence follows from Theorem 3.1.

If $Z_s(g)$ is as in (8), we have

$$Z_s(g)=\sum_{\varpi\in\mathcal{R}}\left|c^r(\varpi)\right|^2 e^{-s\|\lambda_{\varpi,Q}\|_1}\prod_{j\in E} g_j(\lambda_{\varpi,j})=\int_0^\infty e^{-sX}\,d\mu_g(X)\,. \tag{12}$$

By applying Theorem 3.1, we find that

$$\lim_{s\to 0} s^{d-|E|}\int_0^\infty e^{-sX}\,d\mu_g(X)=\frac{2^{1+|E|}}{(2\pi)^d}\sqrt{|D_F|}\prod_{j\in E}\tilde{\mathsf{H}}(g_j)\,. \tag{13}$$

In the case when the functions g_j are non-negative on $\mathcal{Y}=(0,\infty)\cup\mathcal{A}_d$, then the function $X\mapsto\mu_g(X)$ is non-decreasing and we may apply a Tauberian theorem (see Theorem 4.3 in Chap. V of [W]) to obtain

$$\lim_{X\to\infty} X^{|E|-d}\mu_g(X)=\frac{2^{1+|E|}\sqrt{|D_F|}}{(d-|E|)!\,(2\pi)^d}\prod_{j\in E}\tilde{\mathsf{H}}(g_j)\,. \tag{14}$$

For general $g_j\in\tilde{\mathcal{L}}$, the bounds in Definition 2.1 allow us to construct $\tilde{g}_j\in\tilde{\mathcal{L}}$ such that $\tilde{g}_j\ge 0$ on $\mathcal{Y}$ and $|g_j|\le\tilde{g}_j$.

Let $\gamma(X)$ be equal to $\operatorname{Re}\mu_g(X)$ or $\operatorname{Im}\mu_g(X)$. For $X_1>X$ we have

$$\gamma(X_1)-\gamma(X)=\sum_{\substack{\varpi\in\mathcal{R}\\ X<\|\lambda_{\varpi,Q}\|_1\le X_1}}\left|c^r(\varpi)\right|^2(\text{Re or Im})\prod_{j\in E} g_j(\lambda_{\varpi,j})$$

hence $|\gamma(X_1)-\gamma(X)|\le\mu_{\tilde{g}}(X_1)-\mu_{\tilde{g}}(X)$. So $\beta(X):=\gamma(X)+\mu_{\tilde{g}}(X)$ is non-decreasing and we have

$$\lim_{X\mapsto\infty} X^{|E|-d}\beta(X)=\frac{2^{1+|E|}}{(d-|E|)!(2\pi)^d}\sqrt{|D_F|}\Big((\text{Re or Im})\prod_{j\in E}\tilde{\mathsf{H}}(g_j)+\prod_{j\in E}\tilde{\mathsf{H}}(\tilde{g}_j)\Big).$$

Since (14) holds for $\tilde{g}=\prod_{j\in E}\tilde{g}_j$, this implies that (14) holds also for arbitrary $g_j\in\tilde{\mathcal{L}}$, and the proposition follows. □

We now state the main result in this paper. In the proof we will extend the validity of (10) to a larger class of functions g than those considered so far.

Theorem 3.3. *Let $r \in \mathcal{O}' \smallsetminus \{0\}$. Let $\{1, \dots, d\}$ be the disjoint union of the subsets E, Q_+ and Q_-, with $E \neq \{1, \dots, d\}$. Let H be a hypercube $H = \prod_{j \in E}[a_j, b_j] \subset \mathbb{R}^{|E|}$, such that $a_j, b_j \notin \left\{\frac{b}{2}\left(1 - \frac{b}{2}\right) : b \geq 2,\ b \text{ even}\right\}$. Then, if $\mathcal{R} = \mathcal{R}(E, Q_+, Q_-)$ is as in (7), we have*

$$\lim_{X \to \infty} X^{|E|-d} \sum_{\substack{\varpi \in \mathcal{R},\, a_j \leq \lambda_{\varpi,j} \leq b_j,\, j \in E \\ \|\lambda_{\varpi,Q}\|_1 \leq X}} |c^r(\varpi)|^2$$

$$= \frac{2\sqrt{|D_F|}}{(d - |E|)!\,(2\pi)^d} \prod_{j \in E} \Bigg(\int_{[a_j,b_j] \cap [1/4,\infty)} \tanh\left(\pi\sqrt{y - 1/4}\right) dy + \sum_{\substack{b \geq 2,\, b \text{ even}, \\ a_j < \frac{b}{2}(1 - \frac{b}{2}) < b_j}} (b-1) \Bigg). \quad (15)$$

Remark. Note that the factors in the product over $j \in E$ on the right-hand side of (15) are twice the volume of $[a_j, b_j]$ for the measure $d\eta$ discussed after (6).

Proof. As a first step in the proof we will show that (10) is also valid for any function $k(\lambda) = \times_{j \in E} k_j(\lambda)$ with $k_j(\lambda)$ an arbitrary continuous, compactly supported and real valued function on $\mathcal{Y}$. For this purpose, we carry out an approximation argument based on the fact that for each $\varepsilon > 0$ there exists $h_\varepsilon = \times_{j \in E} h_j \in \tilde{\mathcal{L}}^{|E|}$ such that

$$\left|k(\lambda) - h_\varepsilon(\lambda)\right| \leq \varepsilon b(\lambda) \quad \text{for all } \lambda \in \mathcal{Y}^{|E|}, \quad (16)$$

where $b = \times_{j \in E} b_j$ is also an element of $\tilde{\mathcal{L}}^{|E|}$, not depending on ε with $b_j > 0$, for each j.

We first show how (16) leads to the assertion. After that, we construct b and h_ε satisfying (16). In the remainder of the proof we shall write $\mu_X(g) := \mu_g(X)$ to stress the dependence on the test function g. We will use the fact that μ_X defines a positive measure on $\mathcal{Y}^{|E|}$ for each fixed X. Also, we denote by μ the non-negative measure on $\mathcal{Y}^{|E|}$ on the right-hand side of (10), given by

$$\int_{\mathcal{Y}^{|E|}} f(y)\,d\mu(y) = \frac{2^{1+|E|}\sqrt{|D_F|}}{(d - |E|)!\,(2\pi)^d} \int_{\mathcal{Y}^{|E|}} f\big((y_j)_{j \in E}\big) \prod_{j \in E} d\eta(y_j)\,.$$

We have

$$h_\varepsilon(\lambda) - \varepsilon b(\lambda) \leq k(\lambda) \leq h_\varepsilon(\lambda) + \varepsilon b(\lambda)$$

for $\lambda \in \mathcal{Y}^{|E|}$. Hence

$$\mu_X(h_\varepsilon) - \varepsilon \mu_X(b) \leq \mu_X(k) \leq \mu_X(h_\varepsilon) + \varepsilon \mu_X(b)\,, \quad (17)$$

$$\mu(h_\varepsilon) - \varepsilon\mu(b) \le \mu(k) \le \mu(h_\varepsilon) + \varepsilon\mu(b)\,. \tag{18}$$

The inequality (18) shows that

$$\big|\mu(k) - \mu(h_\varepsilon)\big| \le \varepsilon\mu(b)$$

for each $\varepsilon > 0$. Hence $\lim_{\varepsilon\downarrow 0}\mu(h_\varepsilon) = \mu(k)$.

Proposition 3.2 shows that $\lim_{X\to\infty} X^{|E|-d}\mu_X(h_\varepsilon) = \mu(h_\varepsilon)$ for each $\varepsilon > 0$, and $\lim_{X\to\infty} X^{|E|-d}\mu_X(b) = \mu(b)$. We want to prove the same for k.

We derive from (17) that

$$0 \le \limsup_{X\to\infty} X^{|E|-d}\mu_X(k) - \liminf_{X\to\infty} X^{|E|-d}\mu_X(k) \le 2\varepsilon\mu(b)\,.$$

Hence $\lim_{X\to\infty} X^{|E|-d}\mu_X(k)$ exists; by taking limits in (17) we obtain:

$$\mu(h_\varepsilon) - \varepsilon\mu(b) \le \lim_{X\to\infty} X^{|E|-d}\mu_X(k) \le \mu(h_\varepsilon) + \varepsilon\mu(b)\,.$$

The desired equality follows by taking the limit as $\varepsilon \downarrow 0$.

Now we turn to the construction of h_ε and b. First we focus our attention on one place $j \in E$. For positive u, we put

$$h_j(\lambda) := \sqrt{\tfrac{u}{\pi}} \int_{-\infty}^{\infty} e^{-uy^2} k_j(\lambda - y)dy \qquad (\lambda \in \mathbb{R})\,. \tag{19}$$

The dependence of h_j on u is not visible in the notation. This expression shows that $\nu \mapsto h_j(1/4 - \nu^2)$ has a holomorphic extension to $\mathbb{C}$, and then has an exponential decay on the strip $|\mathrm{Re}\,\nu| < \tau$ and along the real axis. This implies that $h_j \in \tilde{\mathcal{L}}$. We note that for each j

$$\|h_j\|_\infty \le \|k_j\|_\infty\,, \tag{20}$$

where $\|\cdot\|_\infty$ denotes the supremum norm on $\mathbb{R}$. The function h_j gives a holomorphic approximation of k_j. By taking u sufficiently large, we obtain, for prescribed $\varepsilon_j > 0$,

$$\|h_j - k_j\|_\infty < \varepsilon_j\,. \tag{21}$$

Let us take $A_j > 1$ so that $\mathrm{Supp}(k_j) \subset [-A_j + 1, A_j - 1]$. If $|\lambda| > A_j$ then we have

$$\begin{aligned}\big|h_j(\lambda)\big| &\le 2(A_j - 1)\|k_j\|_\infty \sqrt{u/\pi}\; e^{-u(|\lambda|+1-A_j)^2}\\ &\le 2(A_j - 1)\|k_j\|_\infty \sqrt{u/\pi}\; e^{-u(|\lambda|+1-A_j)}\,.\end{aligned}$$

Further enlarging u, if necessary, we get $e^{-u}\sqrt{u/\pi}\,2(A_j - 1)\|k_j\|_\infty \le \varepsilon_j$, and

$$\big|h_j(\lambda)\big| \le \varepsilon_j b_j(\lambda) \quad \text{for } |\lambda| \ge A_j\,, \tag{22}$$

where

$$b_j(\lambda) := \begin{cases} \left(\frac{1+\lambda}{1+A_j}\right)^{-2} & \text{for } \lambda \ge -\frac{1}{2}\,,\\ \left(\frac{1-\lambda}{1+A_j}\right)^{-2} & \text{for } \lambda < -\frac{1}{2}\,.\end{cases}$$

We extend b_j as a holomorphic function on a neighborhood of $[0,\infty)$, such that $\nu \mapsto b_j(1/4 - k\nu^2)$ is holomorphic on $|\mathrm{Re}\,\nu| \le \tau$, with τ slightly larger than $1/2$. The estimate $b_j(1/4-\nu^2) = O((1+|\nu|)^{-4})$ is sufficient to conclude that $b_j \in \tilde{\mathcal{L}}$.

Note that the h_j depend on $\varepsilon_j > 0$ but the b_j do not. If $|\lambda| \le A_j$, then we have that $b_j(\lambda) \ge 1$ and we conclude that

$$\bigl|h_j(\lambda)\bigr| \le \beta_j b_j(\lambda) \tag{23}$$

where $\beta_j = 2(A_j - 1)\|k_j\|_\infty \sqrt{u/\pi}$.

We now take $h_\varepsilon = \times_{j\in E} h_j$ and $b = \times_{j\in E} b_j$, and assume that all $\varepsilon_j \in (0,1)$. For a given $\lambda \in \mathcal{Y}^{|E|}$, let $F = \{\, j \in E : |\lambda_j| \le A_j \,\}$. If $F = E$, then we have, by (20), (21) and the fact that $b_j(\lambda_j) \ge 1$ if $|\lambda_j| \le A_j$,

$$\begin{aligned}\bigl|k(\lambda) - h_\varepsilon(\lambda)\bigr| &\le \sum_{j\in E} \bigl|k_j(\lambda_j) - h_j(\lambda_j)\bigr| \prod_{i\in E,\, i\neq j} \|k_i\|_\infty \\ &\le \Bigl(\sum_{j\in E} \varepsilon_j\Bigr) \prod_{i\in E} \max\bigl(1, \|k_i\|_\infty\bigr) b(\lambda)\,.\end{aligned}$$

The product over $i \neq j$ is taken outside the sum over j by estimating it by

$$\prod_{i\in E} \max\bigl(1, \|k_i\|_\infty\bigr) b(\lambda)\,.$$

If $F \neq E$ then $k(\lambda) = 0$, as at least one λ_j is outside the support of k_j. We use (23) and (22), to obtain in the case $F \neq E$:

$$\bigl|k(\lambda) - h_\varepsilon(\lambda)\bigr| = \bigl|h_\varepsilon(\lambda)\bigr| \le b(\lambda) \prod_{j\in F} \beta_j \prod_{i\in E\smallsetminus F} \varepsilon_i\,.$$

Thus we see that we can adjust the ε_i in such a way that condition (16) is satisfied.

As the final step in the proof of the theorem we extend (10) to the characteristic function of a hypercube $\prod_{j\in E}[a_j, b_j] \subset \mathbb{R}^{|E|}$.

Let us denote by χ_j the characteristic function of $[a_j, b_j]$. Now, for any $\varepsilon > 0$, it is easy to construct functions $u_j, U_j \in C_c(\mathbb{R})$ such that $0 \le u_j \le \chi_j \le U_j$, $\|u_j\|_\infty \le 1$, $\|U_j\|_\infty \le 1$, and $\int_{-\infty}^{\infty}(U_j - u_j)\,d\eta \le \varepsilon$. The measure $d\eta$ has point masses at the elements of $\mathcal{A}_d$. So we need the assumption $a_j, b_j \notin \mathcal{A}_d$ to attain the last inequality for all $\varepsilon > 0$.

If we let $u = \times_{j\in E} u_j$ and $U = \times_{j\in E} U_j$, we have

$$\mu(U) - \mu(u) \le C_1\,|E|\varepsilon \quad \text{and} \quad u \le \chi \le U\,, \tag{24}$$

with C_1 a constant depending on the supports of the U_j and on the measure μ.

Now since $\mu_X(u) \le \mu_X(\chi) \le \mu_X(U)$ and (10) is valid for u and U we see that

$$\mu(u) \le \liminf_{X\to\infty} X^{|E|-d}\mu_X(\chi) \le \limsup_{X\to\infty} X^{|E|-d}\mu_X(\chi) \le \mu(U)\,. \tag{25}$$

The existence of $\lim_{X\to\infty} X^{|E|-d}\mu_X(\chi)$ follows from (24) and (25). Since $\mu(u) \le \mu(\chi) \le \mu(U)$, we have that

$$\lim_{X\to\infty} X^{|E|-d}\mu_X(\chi) = \mu(\chi)\,.$$

This is the statement in the theorem. □

If we let $E = \emptyset$ in Theorem 3.3, we get the following result that can be seen as a Weyl-type law, weighted by Fourier coefficients.

Corollary 3.4. *Let $r \in \mathcal{O}' \smallsetminus \{0\}$ and let $\{1,\dots,d\} = Q_+ \sqcup Q_-$. Then we have*

$$\lim_{X\to\infty} X^{-d} \sum_{\substack{\|\lambda_\varpi\|_1<X\\ \lambda_{\varpi,j}>0,\, j\in Q_+\\ \lambda_{\varpi,j}<0,\, j\in Q_-}} \left|c^r(\varpi)\right|^2 = \frac{2\sqrt{|D_F|}}{d!\,(2\pi)^d}\,. \tag{26}$$

The possibility to prescribe Q_+ and Q_- allows us to count representations having discrete series type factors at some places, and factors of principal or complementary series type at the other places. The following result ignores this distinction.

Corollary 3.5. *Let $r \in \mathcal{O}' \smallsetminus \{0\}$. Let E be a proper subset of $\{1,\dots,d\}$, and put $Q = \{1,\dots,d\} \smallsetminus E$. Take $[a_j,b_j]$, for $j \in E$, as in the theorem. Then*

$$\lim_{X\to\infty} X^{|E|-d} \sum_{\substack{a_j\le\lambda_{\varpi,j}\le b_j,\, j\in E\\ \|\lambda_{\varpi,Q}\|_1\le X}} \left|c^r(\varpi)\right|^2$$

$$= \frac{2\sqrt{|D_F|}}{(d-|E|)!\,\pi^d 2^{|E|}} \prod_{j\in E}\Bigg(\int_{[a_j,b_j]\cap[1/4,\infty)} \tanh\big(\pi\sqrt{y-1/4}\big)\,dy + \sum_{\substack{b\ge 2,\, b\,\text{even}\\ a_j<\frac{b}{2}(1-\frac{b}{2})<b_j}} (b-1)\Bigg). \tag{27}$$

Proof. This is obtained from the theorem by adding the contributions over all possible choices of $Q_+ \sqcup Q_- = \{1,\dots,d\} \smallsetminus E$. □

We now show that Theorem 3.3 can be used to derive density results for automorphic representations subject to restrictions at some places.

The first result (Proposition 3.6) confirms that the representations of complementary series type are rare. We fix one place $l \in \{1,\dots,d\}$, and apply Corollary 3.5 with $E = \{l\}$, and $[a_l, b_l] \subset (0, 1/4]$. We catch all complementary series eigenvalues at the place l if we take $0 < a_l \leq 77/324$, since, according to [KS1], all complementary series factors correspond to eigenvalues in $(77/324, 1/4)$. The right-hand side in (27) vanishes for this choice, and leads to case (i) in the following proposition.

To obtain part (ii), we take $E = \{1, \ldots, d\} \setminus \{l\}$, and fix $k \neq l$. We take $[a_k, b_k] \subset (0, 1/4]$, and for $j \neq k$, we let $[a_j, b_j]$ be any interval in $\mathbb{R}$, as in the theorem.

Proposition 3.6. *For each $r \in \mathcal{O}' \setminus \{0\}$, we have*

$$\text{(i)} \qquad \lim_{X\to\infty} X^{1-d} \sum_{\substack{\sum_{j\neq l} |\lambda_{\varpi,j}| \leq X \\ 0 \leq \lambda_{\varpi,l} \leq 1/4}} \left|c^r(\varpi)\right|^2 = 0\,,$$

$$\text{(ii)} \qquad \lim_{X\to\infty} X^{-1} \sum_{\substack{|\lambda_{\varpi,l}| \leq X \\ 0 \leq |\lambda_{\varpi,k}| \leq 1/4 \\ a_j \leq |\lambda_{\varpi,j}| \leq b_j,\ j\neq k,l}} \left|c^r(\varpi)\right|^2 = 0\,.$$

Replacing $\lambda_{\varpi,l} \leq 1/4$ by $\lambda_{\varpi,l} < 1/4$, we obtain a density zero result for exceptional eigenvalues.

In the next application, we restrict our attention to discrete series type eigenvalues, and, moreover, prescribe the eigenvalue at all places but one. So we choose $E = \{1, \ldots, d\} \setminus \{l\}$ and $Q = Q_- = \{l\}$. For each $j \in E$, we pick $\lambda_j \in \mathcal{A}_d$, and choose $[a_j, b_j]$ such that $[a_j, b_j] \cap \mathcal{Y} = \{\lambda_j\} \subset (a_j, b_j)$, and $b_j < 77/324$ if $\lambda_j = 0$. Application of Theorem 3.3 gives

Proposition 3.7. *Let $r \in \mathcal{O}' \setminus \{0\}$. Let $1 \leq l \leq d$, and take $\lambda_j \in \mathcal{A}_d$ for $j \neq l$. Then*

$$\lim_{X\to\infty} X^{-1} \sum_{\substack{-X \leq \lambda_{\varpi,l} < 0 \\ \lambda_{\varpi,j} = \lambda_j,\, j\neq l}} \left|c^r(\varpi)\right|^2 = \frac{\sqrt{|D_F|}}{\pi^d} \prod_{j\neq l} \sqrt{1/4 - \lambda_j}\,.$$

This shows that there are infinitely many ϖ that have discrete series type factors at all places, and a prescribed eigenvalue at all but one place. If we take r totally positive, we restrict the sum to ϖ that are generated by a holomorphic Hilbert modular cusp form (see Proposition 2.2.3 in [BMP]). The occurrence of the Fourier coefficients $c^r(\varpi)$ in our result makes it hard to find a connection to dimension formulas for spaces of holomorphic Hilbert modular forms like those in Theorem 3.5 of [F].

The number of places at which we have restricted ϖ is reflected in the exponent of X. If $d > 2$, the positive density here might in principle correspond to a lower density than the zero density in (i) of the previous proposition.

Finally, we take $E = \{1, \dots, d\} \smallsetminus \{l\}$, $Q = Q_+ = \{l\}$ and confine $\lambda_{\varpi,j}$, $j \neq l$, to a small interval $[a_j, b_j]$ of principal series eigenvalues. We obtain

Proposition 3.8. *Let $r \in \mathcal{O}' \smallsetminus \{0\}$; let $1 \le l \le d$, and take $[a_j, b_j] \subset [1/4, \infty]$ for $j \neq l$. Then*

$$\begin{aligned}\lim_{X\to\infty} X^{-1} & \sum_{\substack{0\le\lambda_{\varpi,l}\le X\\ a_j\le\lambda_{\varpi,j}\le b_j,\, j\neq l}} \left|c^r(\varpi)\right|^2 \\ &= \frac{2^{1-d}\sqrt{|D_F|}}{\pi^d} \prod_{j\neq l} \int_{a_j}^{b_j} \tanh\left(\pi\sqrt{y-1/4}\right) dy \\ &= \frac{2^{1-d}\sqrt{|D_F|}}{\pi^d} \prod_{j\neq l} \left((b_j - a_j)\left(1 + O\left(e^{-2\pi\sqrt{a_j-1/4}}\right)\right)\right).\end{aligned}$$

Remark 3.9. In the case when $d = 1$, necessarily $E = \emptyset$ in Theorem 3.3 and $Q = Q_+ = \{1\}$ or $Q = Q_- = \{1\}$. The asymptotic result obtained in each case is expressed in Corollary 3.4. We get, as $X \mapsto +\infty$:

$$\begin{aligned}\sum_{0\le\lambda_\varpi\le X} \left|c^r(\varpi)\right|^2 &\sim \tfrac{X}{\pi}, \quad \text{if } Q_- = \emptyset, \\ \sum_{-X\le\lambda_\varpi<0} \left|c^r(\varpi)\right|^2 &\sim \tfrac{X}{\pi}, \quad \text{if } Q_+ = \emptyset.\end{aligned} \tag{28}$$

We note that in [B], Corollary 4.4 gives a special case of Theorem 3.1, implying the first asymptotic formula in (28). Also, in [DI, Theorem 2], a result that applies to Fourier coefficients of general automorphic forms is given in the form of an upper bound.

In comparing the result for $d = 1$ with those in [B] and [DI], it is useful to note that a Maass form u on $\mathfrak{H}$, normalized with respect to the usual measure $y^{-2}dx\,dy$, corresponds to $\frac{1}{\sqrt{2\pi}}f_0$, where $f_0 \in L^2(\Gamma\backslash G)$ has length 1 and weight 0. The factor π arises from the normalization of the Haar measure on N that we choose in §4.1, and the factor 2 from $\Gamma\backslash G \cong (\Gamma\backslash\mathfrak{H}) \times (Z\backslash K)$, where $Z = \{I, -I\}$.

Remark 3.10. We note that density results entirely similar to those in this section could have been obtained using the 2-norm of the eigenvalues λ_ϖ in place of the 1-norm.

Indeed, if one uses a zeta function involving $e^{-s\|\lambda_\varpi\|_2}$ in place of $e^{-s\|\lambda_\varpi\|_1}$, the limit in Theorem 3.1 is essentially the same, except for a different multiplicative constant on the right-hand side. By using this limit and following the arguments in this section, we obtain entirely similar results with some changes in the multiplicative constants. The proof of the analogue of Theorem 3.1 is very similar to the one given in sections 4–6, but one has to make a different choice of the test functions, hence there are several computations and estimates that need to be worked out again.

4 Sum Formula. Delta and Kloosterman Terms

We shall use the sum formula in Theorem 2.7.1 of [BMP] in a way similar to the application in section 3 of that paper. For completeness, we will recall most of the notation. We refer the reader to [BMP] for any unexplained facts or notation.

We apply the sum formula with $r = r' \in O' \smallsetminus \{0\}$. That implies that the test functions k have the form $k = \times_{j=1}^{d} k_j : \nu \mapsto \prod_{j=1}^{d} k_j(\nu_j)$ with all $k_j \in \mathcal{L}$, see Definition 2.1.

Throughout this section, we shall take the cusps κ and κ' equal to ∞, and omit them from the notation.

The sum formula gives the following equality for each test function k:

$$\int_Y k(\nu)\, d\sigma_{r,r}(\nu) = \Delta_{r,r}(k) + K_{-r,-r}(\mathsf{B}k)\,. \tag{29}$$

The integral on the left constitutes the *spectral side* of the sum formula. The measure $d\sigma_{r,r}$ contains information on the spectral decomposition of $L^2(\Gamma\backslash G)^+$ and on the Fourier coefficients of the automorphic forms occurring in this decomposition; see sections 5 and 6. The *geometric side* consists of the delta term $\Delta_{r,r}(k)$, defined in (34), and the Kloosterman term $K_{-r,-r}(\mathsf{B}k)$, see (37). The latter depends on a Bessel transform $\mathsf{B}k$ of the test function k.

In this section, we shall fix a special test function of product type, leaving $k_j \in \mathcal{L}$ free for $j \in E$ and choosing it in a special way for $j \in Q$, depending on a parameter $s > 0$. The purpose of this section will be to investigate the behavior of the geometric side as s tends to 0. In §4.1 we consider the delta term. The study of the Kloosterman term, in §4.2, takes more work. It turns out that the delta term gives the main contribution.

In all estimates, we take into account the dependence on r and the k_j, with $j \in E$.

The spectral side has the same behavior. This we shall use in section 6 to prove Theorem 3.1.

Notation. The map $\xi \mapsto (\xi^{\sigma_1}, \ldots, \xi^{\sigma_d})$ gives an embedding of the number field F in $\mathbb{R}^d$. We will often write ξ_j instead of ξ^{σ_j}.

Accordingly, we define, for $x, y \in \mathbb{R}^d$, the product $xy \in \mathbb{R}^d$ by $(xy)_j = x_j y_j$.

For $x \in \mathbb{R}^d$, we put $S(x) := \sum_{j=1}^d x_j$, extending the trace $\mathrm{Tr}_{F/\mathbb{Q}}$. Similarly, $N(y) := \prod_{j=1}^d y_j$ extends the norm $N_{F/\mathbb{Q}}$ to $N : (\mathbb{R}^*)^d \to \mathbb{R}^*$.

Test functions. We write the set $\{1, \ldots, d\}$ of places of F as the disjoint union of three sets E, Q_+ and Q_-, where $Q_+ \cup Q_- \neq \emptyset$.

At the places $j \in E$, we keep $k_j \in \mathcal{L}$ arbitrary. At the other places, we make a special choice, depending on a parameter $s > 0$:

$$\text{If } j \in Q_+\text{:}\quad k_j(\nu) = \begin{cases} e^{-s(1/4-\nu^2)} & \text{if } |\operatorname{Re}\nu| \le \tau\,, \\ 0 & \text{if } \nu \in \tfrac12 + \mathbb{Z},\ |\nu| > \tau\,; \end{cases} \tag{30}$$

$$\text{if } j \in Q_-\text{:}\quad k_j(\nu) = \begin{cases} 0 & \text{if } |\operatorname{Re}\nu| \le \tau\,, \\ e^{-s(\nu^2-1/4)} & \text{if } \nu \in \tfrac12 + \mathbb{Z},\ |\nu| > \tau\,. \end{cases} \tag{31}$$

Norms of test functions. In Theorem 3.1, we have not stated any uniformity of the limit in terms of the k_j. In Proposition 6.1 we shall give some information on the uniformity in k. To do that, we now introduce some norms.

For $a > 2$, let $\mathcal{L}_a$ be the subspace of $k \in \mathcal{L}$ for which $k(\nu) \ll (1+|\operatorname{Im}\nu|)^{-a}$ on the strip $|\operatorname{Re}\nu| \le \tau$.

For $\alpha \in [0, \tau]$ and $b \le a$, we put

$$N_{\alpha,b}(k) := \sup_{\nu:\, \operatorname{Re}\nu = \alpha} \bigl(1 + |\operatorname{Im}\nu|\bigr)^b |k(\nu)| \text{ on } \mathcal{L}_a\,,$$

$$\mathrm{N}^{\mathrm{discr}}(k) := \sum_{b\ge 2,\, b\in 2\mathbb{Z}} \frac{b-1}{2} \left| k\left(\frac{b-1}{2}\right)\right| \quad \text{on } \mathcal{L}\,,$$

$$\mathsf{N}_{\alpha,a}(k) := N_{0,a}(k) + N_{\alpha,a}(k) + \mathrm{N}^{\mathrm{discr}}(k) \quad \text{on } \mathcal{L}_a\,. \tag{32}$$

We extend $N_{\alpha,b}$ to $\mathcal{L}$ by defining it equal to ∞ outside $\mathcal{L}_a$.

We recall the definition of the integral transformation $\mathsf{H}(k)$ from (3).

$$\mathsf{H}(k) = \frac{i}{2} \int_{\operatorname{Re}\nu = 0} k(\nu)\, \nu\, \tan \pi\nu\, d\nu + \sum_{b\ge 2,\, b\in 2\mathbb{Z}} \frac{b-1}{2} k\left(\frac{b-1}{2}\right).$$

Note that H is continuous on $\mathcal{L}_a$ with respect to $N_{0,a} + \mathrm{N}^{\mathrm{discr}} \le \mathsf{N}_{\alpha,a}$ for any $\alpha \in [0, \tau]$. We also use the following notation:

$$\|k\|_{\alpha,a,E} := \prod_{j\in E} \mathsf{N}_{\alpha,a}(k_j) \quad \text{if } \ k = \times_{j\in E}\, k_j \in \mathcal{L}_a^{|E|}. \tag{33}$$

4.1 Delta term. Section 2.6 and Definition 2.5.2 in [BMP] give the definition of the delta term:

$$\Delta_{r,r}(k) = 2\,\mathrm{vol}(\Gamma_N\backslash N)\prod_{j=1}^{d}\mathsf{H}(k_j)\,. \tag{34}$$

The group N consists of the matrices

$$n[x] = \left(\begin{pmatrix}1 & x_1\\ 0 & 1\end{pmatrix},\dots,\begin{pmatrix}1 & x_d\\ 0 & 1\end{pmatrix}\right)\in G\,,$$

with $x\in\mathbb{R}^d$. In [BMP] we have chosen the Haar measure $dn = \frac{dx_1}{\pi}\cdots\frac{dx_d}{\pi}$. The characters of N have the form

$$\chi_r(n[x]) = e^{2\pi i S(rx)}, \tag{35}$$

with $r\in\mathbb{R}^d$. The intersection $\Gamma_N = \Gamma\cap N$ consists of the $\left(\begin{smallmatrix}1&\xi\\0&1\end{smallmatrix}\right)$ with $\xi\in\mathcal{O}\subset\mathbb{R}^d$. The volume of $\Gamma_N\backslash N$ is equal to $\pi^{-d}\sqrt{|D_F|}$, where D_F is the discriminant of the number field F; see, e.g. p. 115 in [La1]. The character χ_r is trivial on Γ_N precisely if $r\in\mathcal{O}'$.

The factor $\alpha(r,r)$ in [BMP] is equal to 2 in the present context. To see this in Definition 2.6.1 in [BMP], note that the matrices $\left(\begin{smallmatrix}\varepsilon&0\\0&1/\varepsilon\end{smallmatrix}\right)$ with $\varepsilon\in\mathcal{O}^*$ form a system of representatives of $\Gamma_N\backslash\Gamma_P$. We take $g_\infty = 1$ at the cusp $\kappa=\infty$, hence a_γ is equal to $\left(\begin{smallmatrix}|\varepsilon^{\sigma_j}|&0\\0&1/|\varepsilon^{\sigma_j}|\end{smallmatrix}\right)$ at the place j. So only $\varepsilon=\pm1$ contribute to $\alpha(r,r)$, and $\chi_r(n_\infty(\gamma))=1$ for $\gamma=\pm\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$.

The factors k_j of k with $j\in E$ are general. Let us put $\mathsf{H}_E(k) := \prod_{j\in E}\mathsf{H}(k_j)$.

We are left with the factors for $j\in Q$. We have chosen the corresponding k_j in (30) and (31). For $j\in Q_+$,

$$\begin{aligned}\mathsf{H}(k_j) &= \frac{i}{2}\int_{\mathrm{Re}\,\nu=0} e^{s(\nu^2-1/4)}\nu\,\tan\pi\nu\,d\nu + \frac12\\ &= s^{-1}e^{-s/4}\int_{t=0}^{\infty} e^{-t^2}t\,dt + O\biggl(\int_0^\infty e^{-st^2-s/4}te^{-2\pi t}dt\biggr) + \frac12\\ &= \tfrac12 s^{-1} + O(1)\,,\end{aligned}$$

and for $j\in Q_-$,

$$\mathsf{H}(k_j) = \sum_{m=2}^{\infty}\left(m-\frac12\right)e^{s(m-m^2)}.$$

In order to estimate this quantity, we replace the sum by $\int_1^\infty f(x)dx = \frac12 s^{-1}$, with $f(x) = \bigl(x-\frac12\bigr)e^{-s(x^2-x)}$, and then we need to estimate the error.

The maximum of f occurs at $x_{\max} = \frac{1}{2} + \frac{1}{\sqrt{2s}}$. To estimate the error we consider two intervals: The sum over $2 \le m \le m_1 = [x_{\max}]$ is larger than the integral over $[1, m_1]$, and the difference between sum and integral is smaller than $f(x_{\max})$. The sum over $m \ge m_1 + 1$ is larger than the integral over $[m_1 + 1, \infty)$, and again the difference is smaller than $f(x_{\max})$. The missing integral over $[m_1, m_1 + 1]$ is smaller than $f(x_{\max})$ as well. So the error is $\ll f(x_{\max}) \ll 1/\sqrt{s}$ and we obtain for $0 < s \le 1$

$$\begin{aligned} \mathsf{H}(k_j) &= \tfrac{1}{2} s^{-1} + O(s^{-1/2}) \,, \\ \Delta_{r,r}(k) &= \frac{2^{1+|E|}}{(2\pi)^d} \sqrt{|D_F|}\, \mathsf{H}_E(k) s^{|E|-d} \\ &\qquad \cdot \big(1 + (\text{if } Q_- \neq \emptyset) O(s^{1/2}) + O(s)\big) \,. \end{aligned} \tag{36}$$

With the convention in (33), we can restate this as

Proposition 4.1. *Let $k = \times_{j=1}^d k_j$, with $k_j \in \mathcal{L}$ arbitrary for $j \in E$, and with k_j as defined in* (30) *and* (31)*, for $j \in Q$. Then we have for each $\alpha \in [0, \tau]$, as $s \downarrow 0$:*

$$\Delta_{r,r}(k) = \frac{2^{1+|E|}}{(2\pi)^d} \sqrt{|D_F|}\, \mathsf{H}_E(k)\, s^{|E|-d} + \|k\|_{\alpha,a,E}\, O_F(s^{|E|-d+c}) \,,$$

where $c = 1/2$ if $Q_- \neq \emptyset$ and $c = 1$ if $Q_- = \emptyset$.

4.2 Kloosterman term. In §3.3 of [BMP] it was sufficient for our purposes to estimate the Kloosterman term by a quantity that is of the same order as the delta term. Here we want to use the main term in (36) for an asymptotic result. So we have to do better on the Kloosterman term.

For our purposes, it will suffice to employ trivial bounds for the Kloosterman sums.

For any function $f : (\mathbb{R}^*)^d \to \mathbb{C}$ that decreases sufficiently fast as the y_j tend to zero and infinity, we define the following sum of Kloosterman sums:

$$K_{r,r}(f) := \sum_{c \in \mathfrak{q},\, c \neq 0} \frac{S(r,r;c)}{|N(c)|} f\left(\frac{r^2}{c^2}\right). \tag{37}$$

Here $\mathfrak{q}$ is the ideal $\mathfrak{q} \subset \mathcal{O}$ such that $\Gamma = \Gamma_0(\mathfrak{q})$. We have $(r^2/c^2)_j = (r^2/c^2)^{\sigma_j}$. The quantity $|N(c)|$ is the norm of the ideal $(c) \subset \mathcal{O}$. The Kloosterman sum is defined by

$$S(r,r;c) = \sum\nolimits^*_{d \bmod c} e^{2\pi i \operatorname{Tr}_{F/\mathbb{Q}}(r(d+a)/c)}, \tag{38}$$

where d runs over representatives of $\mathcal{O}$ mod (c) for which there exists $a \in \mathcal{O}$ such that $ad \equiv 1 \bmod (c)$.

The aim of this subsection is to prove the following estimate for the Kloosterman term.

PROPOSITION 4.2. *Let $k = \times_{j=1}^{d} k_j$, with $k_j \in \mathcal{L}$ arbitrary for $j \in E$, and with k_j as defined in* (30) *and* (31) *for $j \in Q$. For $\alpha \in (1/2, \tau]$, and $0 < \varepsilon < 2\alpha - 1$, as $s \downarrow 0$, we have*

$$K_{-r,-r}(\mathsf{B}k) \ll_{F,\alpha,\varepsilon} |N(r)|^{2\alpha} \|k\|_{\alpha,a,E}\, s^{-(1+\varepsilon-\alpha)|Q_+|-(1-\alpha)|Q_-|}.$$

The choice $\alpha = \tau$ is optimal in the s-aspect. We see that the Kloosterman term is indeed of smaller order than $s^{|E|-d} = s^{-|Q_+|-|Q_-|}$ in (36).

Bessel transform. The transform $\mathsf{B}k$ occurring in the Kloosterman term is of product type: $\mathsf{B}k(y) = \prod_{j=1}^{d} \beta_+ k_j(y_j)$ for $y \in (0,\infty)^d$, with the following Bessel transformation:

$$\beta_+ k(y) = \frac{i}{2} \int_{\mathrm{Re}\,\nu=0} k(\nu)\big(J_{-2\nu}(4\pi\sqrt{y}) - J_{2\nu}(4\pi\sqrt{y})\big)\frac{\nu\, d\nu}{\cos \pi\nu} \qquad (39)$$
$$+ 2 \sum_{b\ge 2,\, b\in 2\mathbb{Z}} (-1)^{b/2} k\left(\frac{b-1}{2}\right)\frac{b-1}{2} J_{b-1}(4\pi\sqrt{y}),$$
$$J_w(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\;\Gamma(w+n+1)}\left(\frac{t}{2}\right)^{w+2n}.$$

In (25) and (26) of [BMP], we have rewritten the integral defining $\beta_+ k$ in several ways. As in §3.3 of [BMP], we use some estimates for the Bessel function to find bounds for $\beta_+ k$. With $0 \le \alpha \le \tau$ we have

$$\text{for } \mathrm{Re}\,\nu = \alpha \qquad J_{2\nu}(y) \ll y^{2\alpha} e^{\pi|\mathrm{Im}\,\nu|}\big(1 + |\mathrm{Im}\,\nu|\big)^{-2\alpha-1/2} \text{ as } y \downarrow 0, \qquad (40)$$
$$\text{for } b \in 2\mathbb{Z},\ b \ge 2 \quad J_{b-1}(y) \ll y^{b-1}\,\Gamma(b)^{-1} \text{ as } y \downarrow 0,$$
$$\text{for } \mathrm{Re}\,\nu = 0 \qquad J_{2\nu}(y) \ll e^{\pi|\mathrm{Im}\,\nu|} \text{ for all } y > 0,$$
$$\text{for } u > 0 \qquad J_u(y) \ll u^{-1} \text{ for all } y > 0.$$

For convenience, we write $t = 4\pi\sqrt{y}$. We fix $\alpha \in (1/2, \tau]$. For $k \in \mathcal{L}_a$, with $a > 2$, we find the following estimates along the same lines as in [BMP], §3.3.

$$\beta_+ k(y) = -i \int_{\mathrm{Re}\,\nu=\alpha} k(\nu) J_{2\nu}(t)\frac{\nu\, d\nu}{\cos\pi\nu} + 2\sum_{b\ge 4}(-1)^{b/2}\frac{b-1}{2} k\left(\frac{b-1}{2}\right) J_{b-1}(t)$$
$$\ll_\alpha N_{\alpha,a}(k) y^\alpha + y^{3/2}\,\mathrm{N}^{\mathrm{discr}}(k) \ll \mathsf{N}_{\alpha,a}(k) y^\alpha \quad \text{as } y \downarrow 0,$$
$$\beta_+ k(y) = -i \int_{\mathrm{Re}\,\nu=0} k(\nu) J_{2\nu}(t)\frac{\nu\, d\nu}{\cos\pi\nu} + 2\sum_{b\ge 2}(-1)^{b/2}\frac{b-1}{2} k\left(\frac{b-1}{2}\right) J_{b-1}(t)$$

$$\ll N_{0,a}(k) + \mathrm{N}^{\mathrm{discr}}(k) \ll \mathsf{N}_{\alpha,a}(k) \qquad \text{as } y \to \infty\,,$$

thus

$$\beta_+ k(y) \ll \mathsf{N}_{\alpha,a}(k) \min(y^\alpha, 1) \qquad \text{for } y > 0\,.$$

If we specialize $k = k_j$ as in (30) and (31) for $j \in Q$, we can obtain better estimates by reconsidering the integrals. First we note that, for $\mathrm{Re}\,\nu > \alpha$,

$$J_{2\nu}(y) = \frac{1}{2\pi i} \int_{\mathrm{Re}\,w=-\alpha} \left(\frac{y}{2}\right)^{-2w} \frac{\Gamma(\nu+w)}{\Gamma(1+\nu-w)}\, dw\,. \tag{41}$$

(To derive this integral representation from the power series expansion, move the line of integration to the left.) We take $\frac{1}{2} < \alpha < \gamma < \alpha + \frac{1}{2} < \frac{3}{2}$, and write $\nu = \gamma + iq$, $w = -\alpha + it$. To estimate $J_{2\nu}$, it suffices to consider the case $q \geq 0$. We find, for all $y > 0$,

$$\begin{aligned}
J_{2\nu}(y) &\ll_{\alpha,\gamma} y^{2\alpha} \int_{-\infty}^{\infty} e^{-\frac{\pi}{2}|q+t| + \frac{\pi}{2}|q-t|} \big(1 + |q+t|\big)^{\gamma-\alpha-1/2} \\
&\qquad\qquad \cdot \big(1 + |q-t|\big)^{-\gamma-\alpha-1/2} dt \\
&= y^{2\alpha} e^{\pi q} \int_0^\infty (1+x)^{\gamma-\alpha-1/2} (1+2q+x)^{-\gamma-\alpha-1/2} dx \\
&\quad + y^{2\alpha} q \int_{-1}^{1} e^{-\pi q x} \big(1 + q(1{+}x)\big)^{\gamma-\alpha-1/2} \big(1 + q(1{-}x)\big)^{-\gamma-\alpha-1/2} dx \\
&\quad + y^{2\alpha} e^{-\pi q} \int_0^\infty (1+x)^{-\gamma-\alpha-1/2} (1+2q+x)^{\gamma-\alpha-1/2}\, dx \\
&\ll y^{2\alpha} e^{\pi q} \left(\int_1^{q+1} x^{\gamma-\alpha-1/2} (1+q)^{-\gamma-\alpha-1/2}\, dx + \int_{q+1}^\infty x^{-2\alpha-1} dx \right) \\
&\quad + y^{2\alpha} q \left(\int_0^1 e^{\pi q x} (1+q)^{-\alpha-\gamma-1/2}\, dx + (1+q)^{\gamma-\alpha-1/2} \right) \\
&\quad + y^{2\alpha} e^{-\pi q} q^\gamma \int_0^\infty (1+x)^{-2\alpha-1}\, dx \\
&\ll y^{2\alpha} e^{\pi q} (1+q)^{1/2-\alpha-\gamma}.
\end{aligned}$$

The best choice of γ seems $\alpha + \frac{1}{2} - \varepsilon$, with $\varepsilon > 0$ small. We shall apply the estimate with $1/2 < \gamma < 3/2$, hence we take $\varepsilon < 1/2$. The advantage of the present estimate above (40) is its validity for all $y > 0$.

In the case of real ν, $\nu \geq 3/2$, we find

$$\begin{aligned}
J_{2\nu}(y) &\ll_\alpha y^{2\alpha} \int_{-\infty}^{\infty} e^{-t(\arg(\nu-\alpha+it) - \arg(1+\nu+\alpha-it))} \frac{|\nu-\alpha+it|^{\nu-\alpha-1/2}}{|1+\nu+\alpha-it|^{\nu+\alpha+1/2}}\, dt \\
&\ll y^{2\alpha} \int_{-\infty}^{\infty} |\nu+it|^{-2\alpha-1}\, dt \ll y^{2\alpha} \nu^{-2\alpha}.
\end{aligned}$$

We apply these estimates to find bounds for the local Bessel transforms. We take $\alpha \in \left(\frac{1}{2}, \tau\right]$, $\gamma = \alpha + \frac{1}{2} - \varepsilon$, $0 < \varepsilon < \frac{1}{2}$. For $j \in Q$, the functions k_j are holomorphic on $|\mathrm{Re}\,\nu| < 3/2$, so $k_j(\nu)$ makes sense if $\mathrm{Re}\,\nu = \gamma$.

For $j \in Q_+$, we have, uniformly for $0 < s < 1$,

$$\begin{aligned}
\beta_+ k_j(y) &= -i \int_{\mathrm{Re}\,\nu=\gamma} k_j(\nu) J_{2\nu}(t) \frac{\nu\, d\nu}{\cos \pi\nu} \\
&\ll \int_{-\infty}^{\infty} e^{s(\gamma^2-1/4-u^2)} y^\alpha \big(1+|u|\big)^{1/2-\alpha-\gamma} |u|\, du \\
&\ll y^\alpha s^{\alpha/2+\gamma/2-5/4} \int_0^\infty e^{-u^2} (\sqrt{s}+u)^{1/2-\alpha-\gamma} u\, du \\
&\ll y^\alpha s^{\alpha-1-\varepsilon}, \\
\beta_+ k_j(y) &\ll e^{-s/4} \int_{-\infty}^{\infty} e^{-sw^2} \frac{|w| e^{\pi w}}{\cosh \pi w}\, dw + 1 \\
&\ll e^{-s/4} \int_0^\infty e^{-sw^2} w\, dw + 1 \\
&\ll s^{-1}, \\
\beta_+ k_j(y) &\ll \min(s^{\alpha-1-\varepsilon} y^\alpha, s^{-1}) \qquad \text{for } y > 0\,.
\end{aligned}$$

In the last estimate, we have the parameter s inside the minimum. This will enable us to improve the estimates in [BMP, §3.3].

For $j \in Q_-$ we find a similar, but slightly better estimate, uniformly for $0 < s < 1$,

$$\begin{aligned}
\beta_+ k_j(y) &= 2 \sum_{b \geq 4,\, b \in 2\mathbb{Z}} (-1)^{b/2} \frac{(b-1)}{2} e^{s\frac{b}{2}(1-\frac{b}{2})} J_{b-1}(t) \\
&\ll \sum_{b \geq 4,\, b \in 2\mathbb{Z}} y^\alpha (b-1)^{1-2\alpha} e^{-s((b-1)/2)^2} \\
&\ll y^\alpha \left(\left(\sqrt{\frac{2-2\alpha}{2s}} \right)^{1+1-2\alpha} 1 + \int_{x=\sqrt{\frac{2-2\alpha}{2s}}}^{\infty} x^{1-2\alpha} e^{-sx^2} dx \right) \\
&\ll_\alpha y^\alpha s^{\alpha-1}, \\
\beta_+ k_j(y) &\ll \sum_{b \geq 4,\, b \in 2\mathbb{Z}} e^{s\frac{b}{2}(1-\frac{b}{2})} \leq \int_{x=1}^{\infty} e^{sx(1-x)} dx \ll s^{-1/2}, \\
\beta_+ k_j(y) &\ll \min(y^\alpha s^{\alpha-1}, s^{-1/2}) \qquad \text{for all } y > 0\,.
\end{aligned}$$

In this way we have proved

Lemma 4.3. *Let $1/2 < \alpha \leq \tau$, $0 < \varepsilon < 1/2$. For $k = \times_{j=1}^d k_j$, with $k_j \in \mathcal{L}$ arbitrary for the places $j \in E$, and k_j as in (30) and (31) for $j \in Q$, we* *have, uniformly for* $s \in (0,1)$,

$$\beta_+ k_j(y) \ll \mathsf{N}_{\alpha,a}(k) \min(y^\alpha, 1) \quad \textit{for } j \in E\,,$$
$$\beta_+ k_j(y) \ll \min(s^{\alpha-1-\varepsilon} y^\alpha, s^{-1}) \quad \textit{for } j \in Q_+\,,$$
$$\beta_+ k_j(y) \ll \min(s^{\alpha-1} y^\alpha, s^{-1/2}) \quad \textit{for } j \in Q_-\,.$$

Sum over the units. We shall apply Lemma 8.1 in [BM2] in the following form:

LEMMA 4.4. *Let* $a, b \in \mathbb{R}$, $a+b>0$. *Let* $p_j, q_j > 0$ *for* $j = 1, \dots, d$. *There exists* $C \geq 0$ *such that for all* $f : (\mathbb{R}^*)^d \to \mathbb{C}$ *satisfying*

$$|f(y)| \leq \prod_{j=1}^{d} \min\left(p_j |y_j|^a, q_j |y_j|^{-b}\right),$$

we have

$$\sum_{\varepsilon \in \mathcal{O}^*} |f(\varepsilon y)| \ll_{a,b} \min\left(N(p)\,|N(y)|^a, N(q)\,|N(y)|^{-b}\right)$$
$$\cdot \left(1 + \left|\log|N(y)| + \frac{1}{a+b} \log \frac{N(p)}{N(q)}\right|^{d-1}\right).$$

Proof. This is a direct consequence of Lemma 8.1 in [BM2]. There we had to take into account the complex places of F. We apply the lemma with $e=0$ and all $n_j = 1$.

With $\eta_j = p_j^{1/(a+b)} q_j^{-1/(a+b)} |y_j|$, we have

$$\min\left(p_j|y_j|^a, q_j|y_j|^{-b}\right) = p_j^{b/(a+b)} q_j^{a/(a+b)} \min\left(\eta_j^a, \eta_j^{-b}\right).$$

Lemma 8.1 in [BM2] bounds the sum over the units by

$$N(p)^{b/(a+b)} N(q)^{a/(a+b)} \left(1 + |\log N(\eta)|^{d-1}\right) \min\left(N(\eta)^a, N(\eta)^{-b}\right)$$
$$= \left(1 + \left|\log\left|\frac{N(p)^{1/(a+b)}}{N(q)^{1/(a+b)}} N(y)\right|\right|^{d-1}\right) \min\left(N(p)|N(y)|^a, N(q)|N(y)|^{-b}\right). \quad \square$$

Kloosterman term. For all test functions k in the sum formula, the sum $K_{r,r}(f)$ converges absolutely. This convergence is part of the statement of Theorem 2.7.1 in [BMP]. For our choice of test functions, the absolute convergence follows from the estimates of Bessel transforms obtained above, and the next lemma.

LEMMA 4.5. *Let* $f : (\mathbb{R}^*)^d \to \mathbb{C}$ *satisfy*

$$|f(y)| \leq \prod_{j=1}^{d} \min\left(p_j(f), q_j(f)|y_j|^\alpha\right),$$

with $p_j(f), q_j(f) > 0$, for $j = 1, \dots, d$, and with $\alpha > 1/2$. Let $r \in \mathcal{O}' \smallsetminus \{0\}$. Then $K_{r,r}(f)$ converges absolutely, and

$$K_{r,r}(f) \ll_{F,\alpha,\varepsilon} N(p_f) \min\left(|N(r)|^{2\alpha} \frac{N(q_f)}{N(p_f)}, |N(r)|^{1+\varepsilon} \left(\frac{N(q_f)}{N(p_f)}\right)^{1/2\alpha+\varepsilon} \right),$$

with $N(p_f) = \prod_{j=1}^d p_j(f)$, and $N(q_f)$ similarly.

In the proof below, we use the trivial estimate $|S(r,r;c)| \leq |N(c)|$ of the Kloosterman sums given in (38). The resulting estimate suffices for our purpose. However, the Kloosterman sums satisfy a much better estimate of Weil–Salié type (see Theorem 10 in [BM1]). With this estimate and the method of Lemma 3.2.1 in [BMP], we can obtain the following bound for the Kloosterman term:

$$K_{r,r}(f) \ll_{F,\varepsilon,\alpha} N(p_f) \left(\frac{N(q_f)}{N(p_f)}\right)^{1/4\alpha+\varepsilon} N(\mathfrak{q})^{-1/2+\varepsilon} |N(r)|^{1/2+\varepsilon}. \tag{42}$$

In the present context, we prefer the less complicated reasoning below.

Proof.

$$\begin{aligned} K_{r,r}(f) &\leq \sum_c \frac{|S(r,r;c)|}{|N(c)|} \prod_{j=1}^d \min\left(p_j(f), q_j(f)|r_j|^{2\alpha}|c_j|^{-2\alpha}\right) \\ &\leq \sum_{(c)\subset\mathfrak{q},\,(c)\neq(0)} \sum_{\zeta\in\mathcal{O}^*} \prod_{j=1}^d \min\left(p_j(f), q_j(f)|r_j|^{2\alpha}|\zeta_j c_j|^{-2\alpha}\right). \end{aligned}$$

We use Lemma 4.4 to estimate the sum over $\zeta \in \mathcal{O}^*$, where we take $a = 0$, $b = 2\alpha$, $p_j = p_j(f)$, $q_j = q_j(f)$, and $y = c/r$. This gives the bound

$$\begin{aligned} \sum_\zeta * \ll_\alpha & \min\left(N(p_f), N(q_f)|N(r/c)|^{2\alpha}\right) \\ & \cdot \left(1 + \left|\log|N(c/r)| + \tfrac{1}{2\alpha}\log(N(p_f)/N(q_f))\right|^{d-1}\right). \end{aligned}$$

We use that there are $O(n^\varepsilon)$ ideals with norm n, and split up the sum at $n \approx \kappa := |N(r)||N(q_f)/N(p_f)|^{1/2\alpha}$:

$$\begin{aligned} K_{r,r}(f) &\ll_{F,\alpha,\varepsilon} \sum_{n=1}^\infty n^\varepsilon \min\left(N(p_f), N(q_f)n^{-2\alpha}|N(r)|^{2\alpha}\right) \\ &\qquad \cdot \left(1 + \left|\log n - \log|N(r)| + \frac{1}{2\alpha}\log\frac{N(p_f)}{N(q_f)}\right|^{d-1}\right) \\ &\ll_{F,\alpha,\varepsilon} \sum_{1\leq n<\kappa} n^\varepsilon N(p_f)\left(1 + (\log\kappa - \log n)^{d-1}\right) \\ &\qquad + \sum_{n\geq\max(1,\kappa)} n^{\varepsilon-2\alpha} N(p_f)\kappa^{2\alpha}\left(1 + (\log n - \log\kappa)^{d-1}\right) \end{aligned}$$

$$\ll_{F,\alpha,\varepsilon} N(p_f)\kappa^{1+2\varepsilon} \ (\text{ if } \kappa \geq 1) + N(p_f)\kappa^{2\alpha} \min\left(1, \kappa^{2\varepsilon+1-2\alpha}\right)$$
$$= N(p_f) \min(\kappa^{2\alpha}, \kappa^{1+2\varepsilon}) . \qquad \square$$

Now we are in a position to prove Proposition 4.2. Let $f = \mathsf{B}k$ and let $p_j(f) = s^{-1}$ (resp. $p_j(f) = s^{-1/2}$) if $j \in Q_+$ (resp. Q_-). Furthermore, let $q_j(f) = s^{\alpha-1-\varepsilon}$ (resp. $q_j(f) = s^{\alpha-1}$), if $j \in Q_+$ (resp. Q_-). Let $p_j(f) = q_j(f) = \mathsf{N}_{\alpha,a}(k_j)$, if $j \in E$. By Lemma 4.3 we have

$$|f(y)| \leq \prod_{j=1}^{d} \min\left(p_j(f), q_j(f)|y_j|^\alpha\right).$$

We also note that

$$N(p_f) = \|k\|_{\alpha,a,E}\, s^{-(|Q_+|+\frac{1}{2}|Q_-|)},$$
$$N(q_f) = \|k\|_{\alpha,a,E}\, s^{-((1+\varepsilon-\alpha)|Q_+|+(1-\alpha)|Q_-|)},$$
$$N(q_f)/N(p_f) = s^{(\alpha-\varepsilon)|Q_+|+(\alpha-1/2)|Q_-|}.$$

We are interested in the asymptotic behavior as $s \downarrow 0$. This corresponds to the case when $N(q_f)/N(p_f)$ is small. So we take the bound $N(p_f)|N(r)|^{2\alpha}\frac{N(q_f)}{N(p_f)}$ in Lemma 4.5,

$$K_{-r,-r}(\mathsf{B}k) \ll_{F,\alpha,\varepsilon} |N(r)|^{2\alpha}\|k\|_{\alpha,a,E}\, s^{-(1+\varepsilon-\alpha)|Q_+|-(1-\alpha)|Q_-|}. \tag{43}$$

5 Sum Formula. The Eisenstein Term

The goal of this section is to give an estimate for the Eisenstein term of the sum formula. We start by introducing some notation.

The set Y in the sum formula (29) has the form

$$Y := \left(i[0,\infty) \cup \left(0, \frac{1}{2}\right) \cup \left\{\frac{b-1}{2} : b \geq 2,\ b \in 2\mathbb{Z}\right\}\right)^d.$$

The measure $d\sigma_{r,r}(\nu)$ on Y in the left-hand side of (29) is the sum of a measure $d\sigma_{r,r}^{\text{discr}}$, to be discussed in section 6, and a measure $d\sigma_{r,r}^{\text{cont}}$, given by

$$\int_Y f(\nu)d\sigma_{r,r}^{\text{cont}}(\nu) := \sum_{\kappa\in\mathcal{P}} c_\kappa \sum_{\mu\in\mathcal{L}_\lambda} \int_{-\infty}^{\infty} f(iy+i\mu)\left|D_\kappa^{\infty,r}(iy,i\mu)\right|^2 dy \tag{44}$$

for compactly supported continuous functions on Y. All test functions $k = \times_{j=1}^d k_j$, $k_j \in \mathcal{L}$, are integrable for this measure, in particular if we take k_j as in (30) and (31).

$\mathcal{P}$ is a full set of representatives of $\Gamma_0(\mathfrak{q})$-inequivalent cusps. For each $\kappa \in \mathcal{P}$, the number c_κ is positive; $D_\kappa^{\infty,r}(\nu, i\mu)$ is a normalized Fourier coefficient at ∞ of the Eisenstein series $E_q(P^\kappa, \nu, i\mu, g)$ with weight q and spectral parameter $\nu \in \mathbb{C}$, and μ is in a lattice $\mathcal{L}_\kappa$ in the hyperplane $\sum_{j=1}^d x_j = 0$ depending on κ.

To show that the contribution of the Eisenstein term is negligible in the context of this paper, we need an estimate for $D_\kappa^{\infty,r}(\nu, i\mu)$ in terms of $\operatorname{Im}\nu$ and μ. This will take quite some work, which we carry out in this section. It is at this point that we essentially use that we are in an arithmetic situation. In general, the contribution of the Eisenstein series could be comparable to, or even larger than that of the discrete spectrum. For congruence subgroups, there are expressions for the Fourier coefficients of Eisenstein series in terms of number theoretic quantities, for which one has additional information.

Our estimate will depend on many quantities, among these the field F, the ideal $\mathfrak{q}$, and the choices we make for each cusp. In the present context, we want to show that (44) gives a smaller contribution than $\Delta_{r,r}(k)$ in Proposition 4.1. We have decided not to burden the paper with an attempt to make explicit all those dependences.

5.1 Fourier coefficients of Eisenstein series. In view of (18) in [BMP], in order to estimate the coefficient $D_\kappa^{\infty,r}(\nu, i\mu)$, it suffices to consider the weight $q = 0 \in (2\mathbb{Z})^d$. As usual, let $\Gamma(\mathfrak{q}) \subset \Gamma_0(\mathfrak{q})$ be the principal congruence subgroup of level $\mathfrak{q}$, where $\mathfrak{q}$ is an ideal in $\mathcal{O}$.

We denote

$$a[y] = \left(\begin{pmatrix} \sqrt{y_1} & 0 \\ 0 & 1/\sqrt{y_1} \end{pmatrix}, \ldots, \begin{pmatrix} \sqrt{y_d} & 0 \\ 0 & 1/\sqrt{y_d} \end{pmatrix} \right) \in G$$

for $y \in \mathbb{R}_{>0}^d$, and $a[y]^\rho = \prod_j y_j^{1/2}$, $a[y]^{i\mu} = \prod_j y_j^{i\mu_j}$. The Eisenstein series is, for $\operatorname{Re}\nu > 1/2$, given by the sum

$$E_0(P^\kappa, \nu, i\mu; g) = \sum_{\gamma \in \Gamma_0(\mathfrak{q})_\kappa \backslash \Gamma_0(\mathfrak{q})} a_\kappa(\gamma g)^{\rho + 2\nu\rho + i\mu}, \tag{45}$$

where $g_\kappa \in G$ satisfies $\kappa = g_\kappa \infty$, where $g = g_\kappa n_\kappa(g) a_\kappa(g) k_\kappa(g)$ for each $g \in G$, with $n_\kappa(g) \in N = \{n[x] : x \in \mathbb{R}^d\}$, $a_\kappa(g) \in A = \{a[y] : y \in \mathbb{R}_{>0}^d\}$, $k_\kappa(g) \in K$, and where $\Gamma_0(\mathfrak{q})_\kappa$ is the subgroup fixing the cusp κ. The lattice $\mathcal{L}_\kappa$ consists of the μ in the hyperplane $\sum_{j=1}^d x_j = 0$ that satisfy $a_\kappa(\delta)^{i\mu} = 1$ for all $\delta \in \Gamma_0(\mathfrak{q})_\kappa$.

The Eisenstein series $E_0(P^\kappa, \nu, i\mu, g)$ is a linear combination of Eisenstein series $E_0^{\Gamma(\mathfrak{q})}(P^{\kappa_j}, \nu, i\mu, g)$ for the principal congruence subgroup $\Gamma(\mathfrak{q})$, with κ_j running through the cusps of $\Gamma(\mathfrak{q})$ above κ. The coefficients in this linear combination depend on the choice of the $g_{\kappa_j} \in \mathbf{G}_\mathbb{Q}$ such that $\kappa_j = g_{\kappa_j}\infty$. Different choices introduce factors of the form $a[y]^{\rho+2\nu\rho+i\mu}$, so the size will not be influenced if $\mathrm{Re}\,\nu = 0$.

To see this, we note that if γ runs through a set of representatives of $\Gamma_0(\mathfrak{q})_\kappa \backslash \Gamma_0(\mathfrak{q})$, then γ^{-1} runs through a set $R \subset \Gamma_0(\mathfrak{q})$ such that each element $\gamma \cdot \kappa$ of the orbit $\Gamma_0(\mathfrak{q})\kappa$ occurs exactly once. This orbit consists of a finite number of $\Gamma(\mathfrak{q})$-orbits, for which we choose representatives $\gamma_j \in \Gamma_0(\mathfrak{q})$. Let $\kappa_j = \gamma_j \kappa$. So there are subsets $R_j \subset \Gamma(\mathfrak{q})$ such that $R = \bigsqcup_j R_j \gamma_j$.

The $\delta \in R_j$ represent the orbit $\Gamma(\mathfrak{q})\kappa_j$. It is not hard to check that this implies that δ^{-1} runs through representatives of $\Gamma(\mathfrak{q})_{\kappa_j} \backslash \Gamma(\mathfrak{q})$ when δ runs through R_j.

Let us choose $g_{\kappa_j} = \gamma_j g_\kappa$. Then

$$g_\kappa n_\kappa(\gamma_j^{-1}\delta^{-1}g) a_\kappa(\gamma_j^{-1}\delta^{-1}g) k_\kappa(\gamma_j^{-1}\delta^{-1}g) = \gamma_j^{-1}\delta^{-1}g = \gamma_j^{-1} g_{\kappa_j} n_{\kappa_j}(\delta^{-1}g) a_{\kappa_j}(\delta^{-1}g) k_{\kappa_j}(\delta^{-1}g)$$

shows that $a_{\kappa_j}(\delta^{-1}g) = a_\kappa(\gamma_j^{-1}\delta^{-1}g)$, and hence

$$E_0(P^\kappa, \nu, i\mu; g) = \sum_j E_0^{\Gamma(\mathfrak{q})}(P^{\kappa_j}, \nu, i\mu; g)\,.$$

Any other choice of g_{κ_j} has the form $g_{\kappa_j} = \gamma_j g_\kappa p l$, with $p \in NA$, $l = \pm I$. This will indeed cause a factor $a[y]^{\rho+2\nu\rho+i\mu}$.

We turn to the estimation of the Fourier coefficients of $E_0^{\Gamma(\mathfrak{q})}(P^{\kappa_j},\nu,i\mu,g)$. We have seen that this suffices for our purpose.

Each cusp κ_j of $\Gamma(\mathfrak{q})$ has the form $-\delta/\gamma$, with $\delta, \gamma \in \mathcal{O}$, such that the ideal $\mathfrak{a} = (\delta, \gamma)$ is relatively prime to $\mathfrak{q}$, see for instance [Gu, Satz 1]. Let us fix one cusp κ_j among the cusps of $\Gamma(\mathfrak{q})$ above κ. There are $\alpha, \beta \in \mathfrak{a}^{-1}$ such that $g_{\kappa_j} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$ sends ∞ to $\kappa_j = -\frac{\delta}{\gamma}$. If γ_l runs through a set of representatives of $\Gamma(\mathfrak{q})_{\kappa_j} \backslash \Gamma(\mathfrak{q})$, then $\gamma_l^{-1}\kappa_j$ runs through the cuspidal orbit $\Gamma(\mathfrak{q})\kappa_j$. Gundlach, [Gu], gives in Satz 1 and Hilfsatz 1 the following description: The lower rows of the elements $g_j^{-1}\gamma_l$ run through the pairs $(c,d) \in \mathcal{O}^2$ that satisfy

$$\begin{cases} \mathcal{O}c + \mathcal{O}d = \mathfrak{a}, \quad c \equiv \gamma \mod \mathfrak{q}\mathfrak{a}, \quad d \equiv \delta \mod \mathfrak{q}\mathfrak{a}, \\ \text{from each class } \{\, (\varepsilon c, \varepsilon d) : \varepsilon \in \mathcal{O}^*,\, \varepsilon \equiv 1 \mod \mathfrak{q} \,\}, \\ \quad \text{exactly one pair occurs.} \end{cases} \tag{46}$$

This follows from [Gu, Hilfsatz 1, Satz 1] (see also [E1, Propositions 2.1 and 2.3]).

Let $z = x + iy \in \mathcal{H}^d$. For $\mathrm{Re}\,\nu > 1/2$,

$$E_0^{\Gamma(\mathfrak{q})}\big(P^{\kappa_j}, \nu, i\mu, n[x]a[y]\big) = \sum_{(c,d)} a\left[\frac{y}{|cz+d|^2}\right]^{\rho+2\nu\rho+i\mu} \tag{47}$$

where $\left(\frac{y}{|cz+d|^2}\right)_i = \frac{y_i}{|c^{\sigma_i} z_i + d^{\sigma_i}|^2}$ and where the pair (c,d) runs over a set satisfying the conditions in (46). By standard methods this gives the following description of the Fourier term of order $r \neq 0$, at the cusp ∞:

$$\frac{1}{\mathrm{vol}(\Gamma(\mathfrak{q})_N \backslash N)} \int_{\Gamma(\mathfrak{q})_N \backslash N} \chi_r(n)^{-1} E_0^{\Gamma(\mathfrak{q})}\left(P^{\kappa_j}, \nu, i\mu, na[y]\right) dn$$
$$= C_0 \sum_{(c,d)} |N(c)|^{-1-2\nu} |c|^{-2i\mu} e^{2\pi i S(rd/c)} |\pi r|^{\nu+i\mu} d_\infty^r(0, \nu+i\mu) W_{\infty,0}^{r,\nu+i\mu}(a[y])$$

with C_0 a constant, and (c,d) as in (48). (See [BMP, (9) and (16)] for d_∞^r and $W_{\infty,0}^{r,\nu+i\mu}$.)

Note that

$$|c|^{-2\rho-4\nu\rho-2i\mu} = \prod_j |c^{\sigma_j}|^{-1-2\nu-2i\mu_j} = |N(c)|^{-1-2\nu} |c|^{-2i\mu},$$

and that $c \mapsto |c|^{-2i\mu}$ is a character on F^*, vanishing on the group $\{\, \varepsilon \in \mathcal{O}^* : \varepsilon \equiv 1 \bmod \mathfrak{q} \,\}$, since $\mu \in \mathcal{L}_\kappa$.

The (c,d) in (46) correspond to a sum over $\Gamma(\mathfrak{q})_{\kappa_j} \backslash \Gamma(\mathfrak{q})$. Now we have to deal with a sum related to $\Gamma(\mathfrak{q})_{\kappa_j} \backslash \Gamma(\mathfrak{q}) / \Gamma(\mathfrak{q})_\infty$, with the class of 1 omitted if $\kappa_j = \infty$. The $(c,d) \in \mathfrak{a} \times \mathfrak{a}$ have to satisfy

$$\begin{cases} \mathcal{O}c + \mathcal{O}d = \mathfrak{a}, \quad c \equiv \gamma \bmod \mathfrak{q}, \quad c \neq 0, \quad d \equiv \delta \bmod \mathfrak{q} \\ \text{there is exactly one representative of each class } d \bmod \mathfrak{q}(c) \\ \text{there is exactly one representative of each class} \\ \quad c\,\{\, \varepsilon \in \mathcal{O}^* : \varepsilon \equiv 1 \bmod \mathfrak{q} \,\}. \end{cases} \tag{48}$$

Since the ideal $\mathfrak{a} = \mathcal{O}\gamma + \mathcal{O}\delta$ is relatively prime to $\mathfrak{q}$, the congruences $c \equiv \gamma$, $d \equiv \delta \mod \mathfrak{q}$ are equivalent to the same congruences $\mod \mathfrak{q}\mathfrak{a}$.

We shall denote by $\Psi(\gamma, \delta)$ a set of pairs (c,d) satisfying the conditions in (48) above.

In (10) and (18) of [BMP], we see that the quantity $D_\kappa^{\infty,r}(\nu, i\mu)$ in (44) is obtained by dividing the Fourier term given above by the factors d_∞^r and $W_{\infty,0}^{r,\nu+i\mu}$. We have thus arrived at a description of the following form

$$D_\kappa^{\infty,r}(\nu, i\mu) = \sum_{l=1}^{L} c_l a_l^{\nu+i\mu} \Phi_r(\nu+i\mu; \gamma_l, \delta_l) \tag{49}$$

with $c_l \in \mathbb{C}$, $a_l \in (0, +\infty)^d$ and where the Dirichlet series Φ_r is given, for $\mathrm{Re}\,\nu > 1/2$, by

$$\Phi_r(\nu+i\mu; \gamma, \delta) = \sum_{(c,d) \in \Psi(\gamma,\delta)} |N(c)|^{-1-2\nu} |c|^{-2i\mu} e^{2\pi i S(rd/c)}. \tag{50}$$

An estimate on the line $\mathrm{Re}\,\nu = 0$ for the meromorphic continuation of such a Dirichlet series will give an estimate of the same form for $D_\kappa^{\infty,r}(\nu, i\mu)$.

We will now consider one such series, depending on the pair γ and δ in $\mathcal{O}$ and where the ideal $\mathcal{O}\gamma + \mathcal{O}\delta = \mathfrak{a}$ is coprime to $\mathfrak{q}$. One difficulty is the condition $\mathcal{O}c + \mathcal{O}d = \mathfrak{a}$ in (48) that prevents d from running over a full set of representatives. The method to approach this problem can be found in [Kl], [F, Ch. III, §4], or in [E1, §2].

Let $I_\mathfrak{q}$ be the group of fractional ideals prime to $\mathfrak{q}$, and let $\mu_\mathfrak{q}$ be the Möbius function on $I_\mathfrak{q}$. Let $\Psi_1 := \Psi_1(\gamma, \delta)$ be the set of $(c, d) \in \mathfrak{a} \times \mathfrak{a}$ satisfying all the conditions in (48), except for $\mathcal{O}c+\mathcal{O}d = \mathfrak{a}$. For $(c, d) \in \Psi_1$, we only know that $\mathcal{O}c + \mathcal{O}d$ is an ideal contained in $\mathfrak{a}$. Now

$$\begin{aligned}\Phi_r(\nu + i\mu; \gamma, \delta) &= \sum_{(c,d)\in\Psi_1} |N(c)|^{-1-2\nu}|c|^{-2i\mu}e^{2\pi iS(rd/c)} \sum_{\mathfrak{b}|\mathfrak{a}^{-1}(\mathcal{O}c+\mathcal{O}d)} \mu_\mathfrak{q}(\mathfrak{b}) \\ &= \sum_{\mathfrak{b}\subset\mathcal{O}:\mathfrak{b}\in I_\mathfrak{q}} \mu_\mathfrak{q}(\mathfrak{b}) \sum_{(c,d)\in\Psi_1:c,d\in\mathfrak{ab}} |N(c)|^{-1-2\nu}|c|^{-2i\mu}e^{2\pi iS(rd/c)}.\end{aligned}$$

Let $F_\mathfrak{q} = \{\,(\alpha) \in I_\mathfrak{q} : \alpha \in F^*, \alpha \equiv 1 \bmod \mathfrak{q}, \alpha^{\sigma_j} > 0,\ \text{for } j = 1, \dots, d\,\}$. The quotient $I_\mathfrak{q}/F_\mathfrak{q}$ is the strict ray class group modulo $\mathfrak{q}$. We use this finite group to split up Φ_r into the sum over $\tau \in I_\mathfrak{q}/F_\mathfrak{q}$ of

$$\Phi_r^\tau(\nu + i\mu; \gamma, \delta) = \sum_{\mathfrak{b}\subset\mathcal{O}:\mathfrak{b}\in\tau} \mu_\mathfrak{q}(\mathfrak{b}) \sum_{(c,d)\in\Psi_1:c,d\in\mathfrak{ab}} |N(c)|^{-1-2\nu}|c|^{-2i\mu}e^{2\pi iS(rd/c)}.$$

Thus, to estimate Φ_r it suffices to estimate the finitely many terms Φ_r^τ.

For a given τ, we fix an integral ideal $\mathfrak{b}_0 \in \tau$. We write each $\mathfrak{b} \in \tau$ as $\mathfrak{b} = \mathfrak{b}_0(\vartheta)$, with $\vartheta = \vartheta_\mathfrak{b} \in F^*$ totally positive, $\vartheta \equiv 1 \bmod \mathfrak{q}$. For a given ϑ of this type, we replace c by ϑc and d by ϑd. The conditions for the new pair (c, d) are

$$\left\{\begin{array}{l} c \in \mathfrak{ab}_0, \quad c \neq 0, \quad c \equiv \gamma \bmod \mathfrak{q} \\ c \text{ modulo multiplication by } \varepsilon \in \mathcal{O}^*,\ \varepsilon \equiv 1 \bmod \mathfrak{q}, \\ d \in \mathfrak{ab}_0, \quad d \equiv \delta \bmod \mathfrak{q}, \quad d \bmod \mathfrak{q}(c). \end{array}\right. \tag{51}$$

Let us denote by $\Psi_2(\mathfrak{b}_0) := \Psi_2(\gamma, \delta; \mathfrak{b}_0)$ a set of pairs (c, d) as in (51).

The substitution $c \mapsto \vartheta_\mathfrak{b} c$, $d \mapsto \theta_\mathfrak{b} d$ leads to

$$\begin{aligned}\Phi_r^\tau(\nu + i\mu; \gamma, \delta) = \sum_{\mathfrak{b}\subset\mathcal{O}:\mathfrak{b}\in\tau} & \mu_\mathfrak{q}(\mathfrak{b})N(\mathfrak{b})^{-1-2\nu}N(\mathfrak{b}_0)^{1+2\nu}|\vartheta_\mathfrak{b}|^{-2i\mu} \\ & \cdot \sum_{(c,d)\in\Psi_2(\mathfrak{b}_0)} |N(c)|^{-1-2\nu}|c|^{-2i\mu}e^{2\pi iS(rd/c)}.\end{aligned}$$

We note that $\alpha \mapsto |\alpha|^{2i\mu}$ induces a unitary character of $F_\mathfrak{q}$. We fix a (unitary) extension λ_μ of this character to $I_\mathfrak{q}$. For $\mathfrak{b} = \mathfrak{b}_0\,(\vartheta)$ as above we have $|\vartheta|^{2i\mu} = \lambda_\mu(\mathfrak{b})\lambda_\mu(\mathfrak{b}_0)^{-1}$. So,

$$\Phi_r^\tau(\nu+i\mu;\gamma,\delta) = \lambda_\mu(\mathfrak{b}_0)N(\mathfrak{b}_0)^{1+2\nu}\,Q(\nu,\lambda_\mu,\tau)\,\psi_r^{\mathfrak{b}_0}(\nu+i\mu;\gamma,\delta)\,, \tag{52}$$

where

$$\begin{aligned} Q(\nu,\lambda_\mu;\tau) &= \sum_{\mathfrak{b}\in\tau,\,\mathfrak{b}\subset\mathcal{O}} \mu_{\mathfrak{q}}(\mathfrak{b})N(\mathfrak{b})^{-1-2\nu}\lambda_\mu(\mathfrak{b})^{-1}, \\ \psi_r^{\mathfrak{b}_0}(\nu+i\mu;\gamma,\delta) &= \sum_{(c,d)\in\Psi_2(\mathfrak{b}_0)} |N(c)|^{-1-2\nu}|c|^{-2i\mu}e^{2\pi i S(rd/c)}. \end{aligned} \tag{53}$$

We shall first estimate $\psi_r^{\mathfrak{b}_0}(\nu+i\mu;\gamma,\delta)$. If we fix $c\in\mathfrak{a}\mathfrak{b}_0$, $c\neq 0$, $c\equiv\gamma \bmod \mathfrak{q}$, then the d such that $(c,d)\in\Psi_2(\mathfrak{b}_0)$ run through representatives, $\bmod\,\mathfrak{q}(c)$, satisfying $d\equiv\delta' \bmod \mathfrak{q}\mathfrak{a}\mathfrak{b}_0$, for some δ' such that $\delta'\equiv\delta \bmod \mathfrak{q}$ and $\delta'\equiv 0 \bmod \mathfrak{a}\mathfrak{b}_0$. Such δ' exists, since $\mathfrak{q}$ and $\mathfrak{a}\mathfrak{b}_0$ are relatively prime. So $d=\delta'+\vartheta$ with ϑ running through all representatives of $\mathfrak{q}\mathfrak{a}\mathfrak{b}_0 \bmod \mathfrak{q}(c)$. Hence, if c is fixed and one sums over all ϑ, one gets a non-zero contribution if and only if the character $x\mapsto e^{2\pi i S(rx/c)}$ of $\mathfrak{q}\mathfrak{a}\mathfrak{b}_0 \bmod \mathfrak{q}(c)$ is trivial, that is, if and only if $r\mathfrak{d}\mathfrak{q}\mathfrak{a}\mathfrak{b}_0\subset(c)$, where $\mathfrak{d}$ is the different of $\mathcal{O}$. So $\psi_r^{\mathfrak{b}_0}(\nu+i\mu;\gamma,\delta)$ is given by a sum over finitely many c, determined by

$$\begin{cases} \mathfrak{a}\mathfrak{b}_0 \mid (c) \\ (c)\mid r\mathfrak{d}\mathfrak{q}\mathfrak{a}\mathfrak{b}_0 \\ c\equiv\gamma \bmod \mathfrak{q} \end{cases} \tag{54}$$

Hence

$$\psi_r^{\mathfrak{b}_0}(\nu+i\mu;\gamma,\delta) = \sum_{c \text{ as in } (54)} |N(c)|^{-2\nu}|c|^{-2i\mu}N(\mathfrak{a}\mathfrak{b}_0)^{-1} \tag{55}$$

is a finite sum, hence everywhere holomorphic. For $\mathrm{Re}\,\nu=0$ it is bounded by

$$N(\mathfrak{a}\mathfrak{b}_0)^{-1}\big|\,\{\,\mathfrak{b}\subset\mathcal{O}:\mathfrak{b}\mid r\mathfrak{d}\mathfrak{q}\,\}\,\big| = O_\varepsilon\big((|N(r)|N(\mathfrak{d})N(\mathfrak{q}))^\varepsilon\big)\,, \tag{56}$$

for each $\varepsilon>0$.

We now turn to $Q(\nu,\lambda_\mu;\tau)$, with $\tau\in I_{\mathfrak{q}}/F_{\mathfrak{q}}$. Set

$$L(s,\bar\lambda_\mu,\chi) = \sum_{\mathfrak{b}\in I_{\mathfrak{q}},\,\mathfrak{b}\subset\mathcal{O}} \frac{\overline{\lambda_\mu(\mathfrak{b})}\chi(\mathfrak{b})}{N(\mathfrak{b})^s}\,, \tag{57}$$

where χ is a character of the ray-class group $I_{\mathfrak{q}}/F_{\mathfrak{q}}$. Fourier analysis on $I_{\mathfrak{q}}/F_{\mathfrak{q}}$ allows to express $Q(\nu,\lambda_\mu;\tau)$ in terms of these L-functions,

$$\begin{aligned} |I_{\mathfrak{q}}/F_{\mathfrak{q}}|Q(\nu,\lambda_\mu;\tau) &= \sum_{\mathfrak{b}\in I_{\mathfrak{q}},\,\mathfrak{b}\subset\mathcal{O}} \frac{\overline{\lambda_\mu(\mathfrak{b})}\mu_{\mathfrak{q}}(\mathfrak{b})}{N(\mathfrak{b})^{1+2\nu}}\Big(\sum_{\chi\in(I_{\mathfrak{q}}/F_{\mathfrak{q}})^\wedge}\overline{\chi(\tau)}\chi(\mathfrak{b})\Big) \\ &= \sum_{\chi\in(I_{\mathfrak{q}}/F_{\mathfrak{q}})^\wedge} \frac{\overline{\chi(\tau)}}{L(1+2\nu,\bar\lambda_\mu,\chi)}\,. \end{aligned} \tag{58}$$

where $(I_\mathfrak{q}/F_\mathfrak{q})^\wedge$ denotes the character group of $I_\mathfrak{q}/F_\mathfrak{q}$. So the task of estimating $Q(\nu,\lambda_\mu;\tau)$ for $\mathrm{Re}\,\nu=0$, is reduced to estimating $L(s,\bar\lambda_\mu,\chi)^{-1}$ for all $\chi\in(I_\mathfrak{q}/F_\mathfrak{q})^\wedge$, for $\mathrm{Re}\,s=1$.

5.2 Estimates for certain *L*-functions. In this subsection we will obtain logarithmic estimates on the line $\mathrm{Re}\,s=1$ for the L-functions defined in (57). In the simplest case, $F=\mathbb{Q}$, $\mathfrak{q}=\mathbb{Z}$, we can use the estimate $\frac{1}{\zeta(1+it)}=O(\log^7 t)$ as $t\to\infty$, or the better estimate $\frac{1}{\zeta(1+it)}=O(\log t)$, see [T, (3.6.5) and (3.11.8)]. Such estimates are known for other L-functions, like Dedekind zeta functions. We include a proof for the general case we need, since we could not find it in the literature in the needed generality.

We shall follow classical methods of Landau, leading to a generalization of $\frac{1}{\zeta(1+it)}=O(\log^7 t)$. Y. Motohashi kindly pointed out to us a more involved method that should give a better estimate, comparable to $\frac{1}{\zeta(1+it)}=O(\log t)$. In the context of this paper we prefer to stick to an approach that is as simple as possible.

As seen in the previous subsection, these estimates will allow us to estimate the Fourier coefficients of Eisenstein series and therefore to get a bound for the contribution of the Eisenstein term of the sum formula.

We shall use the notation from §5.1, but to keep this section self-contained, we will recapitulate the main ingredients.

Let F be a totally real number field, $\mathfrak{q}$ an integral ideal in $\mathcal{O}_F$, $I_\mathfrak{q}$ the group of fractional ideals in F that are prime to $\mathfrak{q}$ and

$$F_\mathfrak{q}=\{\,(\alpha)\in I_\mathfrak{q}:\alpha\equiv 1 \bmod \mathfrak{q}\,,\ \alpha \text{ totally positive}\,\}\,.$$

So $I_\mathfrak{q}/F_\mathfrak{q}$ is the strict ray class group modulo $\mathfrak{q}$. We consider F^* embedded in $(\mathbb{R}^*)^d$ in the canonical way. Taking absolute values in each coordinate and denoting by V the composition of the absolute values with the embedding of F^*, we obtain a map $V:\mathcal{O}^*\to(\mathbb{R}_{>0})^d$ with a multiplicative lattice as its image in $(\mathbb{R}_{>0})^d$ contained in the subgroup defined by the equation $N(y)=1$. We have a unitary character of $F^*/\{\varepsilon\in\mathcal{O}^*:\varepsilon\equiv 1 \bmod \mathfrak{q}\}$ given by $\lambda_\mu(x)=V(x)^{2i\mu}$, where $y^{2i\mu}=\prod_{j=1}^d y_j^{2i\mu_j}$. We have assumed that $\sum_{j=1}^d\mu_j=0$ and we have extended the character λ_μ of $F_\mathfrak{q}$ to a unitary character of the group $I_\mathfrak{q}$. For simplification, we shall denote this character by λ in the sequel. If $\mu=0$, then we take the extension λ equal to 1.

Our aim is to obtain a lower bound of the absolute value of L-functions of the form

$$L(s,\bar\lambda,\chi)=\sum_{\mathfrak{b}\in I_\mathfrak{q},\,\mathfrak{b}\subset\mathcal{O}}\frac{\overline{\lambda(\mathfrak{b})}\chi(\mathfrak{b})}{N(\mathfrak{b})^s}\,, \tag{59}$$

where χ is a unitary character of the group $I_{\mathfrak{q}}/F_{\mathfrak{q}}$, on the line $\mathrm{Re}\, s = 1$. We follow the standard approach, which first establishes upper bounds on lines $\mathrm{Re}\, s = \sigma > 1$. The latter bounds are more easily obtained for ray class zeta functions:

$$Z(s, \lambda, \tau) := \sum_{\mathfrak{b}\in\tau,\, \mathfrak{b}\subset\mathcal{O}} \frac{\lambda(\mathfrak{b})}{N(\mathfrak{b})^s} = \sum_{n=1}^{\infty} \frac{f_\lambda(n)}{n^s}\,, \tag{60}$$

where τ is a class of $I_{\mathfrak{q}}/F_{\mathfrak{q}}$. The coefficients $f_\lambda(n)$ of this Dirichlet series have less satisfactory estimates than the partial sums

$$s_\lambda(n) := \sum_{k=1}^{n} f_\lambda(k) = \sum_{\mathfrak{b}\in\tau,\, \mathfrak{b}\subset\mathcal{O},\, N\mathfrak{b}\le n} \lambda(\mathfrak{b})\,. \tag{61}$$

In §2, Chap. IV of [J], it is shown that

$$s_1(n) = \alpha n + O_{F,\mathfrak{q}}(n^{1-1/d})\,, \tag{62}$$

with $\alpha \neq 0$ not depending on the ray class τ. We shall follow the same method to prove Lemma 5.1, which shows that $s_\lambda(n) \ll n^{1-1/d}$ if $\lambda \neq 1$ for $d > 1$. For $d = 1$, we have $s_1(n) = \alpha n + O(1)$.

It is more convenient to sum over numbers than over ideals. So we fix an integral ideal $\mathfrak{c}$ in the inverse class τ^{-1}, and write $\mathfrak{b} = (\alpha)\mathfrak{c}^{-1}$ with $\alpha \in F^*$, $\alpha \equiv 1 \bmod \mathfrak{q}$, $\alpha \gg 0$ (i.e., α totally positive). We set

$$s'_\lambda(n) := \sum_{\alpha} \lambda(\alpha) = \lambda(\mathfrak{c}) s_\lambda\big(n/N(\mathfrak{c})\big), \tag{63}$$

where $(\alpha)\mathfrak{c}^{-1}$ runs through the integral ideals in τ such that $N(\alpha) \le n$. The quantities $s_\lambda(n)$ and $s'_\lambda(nN(\mathfrak{c}))$ have the same growth behavior.

We proceed as in Lemma 2.5–2.7, Ch. IV, in [J]. The α in the sum satisfy

i) $\alpha \equiv \alpha_0 \bmod \mathfrak{qc}$, where $\alpha_0 \in \mathfrak{c}$ satisfies $\alpha_0 \equiv 1 \bmod \mathfrak{q}$,
ii) α is totally positive,
iii) we use only one element from each class $\{\, \alpha\varepsilon : \varepsilon \in \mathcal{O}^*,\ \varepsilon \gg 0,\ \varepsilon \equiv 1 \bmod \mathfrak{q} \,\}$,
iv) $N(\alpha) \le n$.

The α satisfying i) form a shifted lattice $\mathcal{M} = \alpha_0 + \mathfrak{qc} \subset F \subset \mathbb{R}^d$. Condition ii) requires that we restrict the sum to $\alpha \in \mathcal{M} \cap \mathbb{R}^d_{>0}$.

We write each $x \in \mathbb{R}^d_{>0}$ in the form $x = x_n x_u$, with x_u such that $N(x_u) = 1$, and $x_n = (N(x)^{1/d}, \cdots, N(x)^{1/d})$. Let $\mathcal{F}$ be a compact fundamental domain for the multiplicative group $U_{\mathfrak{q}} := \{\varepsilon \in \mathcal{O}^* : \varepsilon \equiv 1 \bmod \mathfrak{q},\ \varepsilon \gg 1\}$ in $N(x) = 1$. Later on we assume that the boundary of $\mathcal{F}$ is piecewise analytic. Let $\mathcal{X}_n$ be the set of all $x \in \mathbb{R}^d_{>0}$ satisfying $N(x) \le n$ and $x_u \in \mathcal{F}$. The α in the sum may be chosen as the elements of $\mathcal{M} \cap \mathcal{X}_n$.

For each totally positive $\alpha \in \mathcal{M}$, we have $\lambda(\alpha) = e^{\sum_j 2i\mu_j \log \alpha^{\sigma_j}}$. We extend this by defining $\lambda(x) = e^{\sum_j 2i\mu_j \log x_j}$ for all $x \in \mathbb{R}^d_{>0}$. We note that $\lambda = 1$ on the multiplicative group $U_{\mathfrak{q}}$, and $\lambda(x) = \lambda(x_u)$. Thus

$$s'_\lambda(n) = \sum_{x \in \mathcal{M} \cap \mathcal{X}_n} \lambda(x)\,.$$

Let Λ be a fundamental domain for the lattice $\mathfrak{qc}$, of which $\mathcal{M}$ is a shift. We replace the sum $s'_\lambda(n)$ by the integral

$$I_\lambda(n) := \frac{1}{\operatorname{vol}\Lambda} \int_{\mathcal{X}_n} \lambda(y)dy\,. \tag{64}$$

The difference between $I_\lambda(n)$ and $s'_\lambda(n)$ consists of two terms: The "interior error" occurs for each $x \in \mathcal{M}$ such that $x + \Lambda \subset \mathcal{X}_n$, whereas the boundary error is related to those $x \in \mathcal{M}$ for which $x + \Lambda$ meets the boundary $\partial\mathcal{X}_n$ of $\mathcal{X}_n$.

We start with the interior error. For $x \in \mathcal{M}$ such that $x + \Lambda \subset \mathcal{X}_n$, we consider

$$\begin{aligned}\delta(x) :=& \lambda(x) - \operatorname{vol}(\Lambda)^{-1} \int_{x+\Lambda} \lambda(y)dy \\ =& \frac{1}{\operatorname{vol}(\Lambda)} \int_\Lambda \big(\lambda(x) - \lambda(x+\eta)\big)d\eta\,.\end{aligned}$$

For $\eta \in \Lambda$, we have

$$\left(\frac{x_j + \eta_j}{x_j}\right)^{2i\mu_j} = 1 + O_\Lambda(\mu_j/x_j)\,.$$

The set $\mathcal{F}$ is compact, and has positive distance to all the coordinate hyperplanes. So we find that $x_j \geq C_1 N(x)^{1/d}$, for some $C_1 > 0$. We have

$$\begin{aligned}\left(\frac{x+\eta}{x}\right)^{i\mu} &= \prod_{j=1}^d \big(1 + O_{\Lambda,\mathcal{F}}(|\mu_j| N(x)^{-1/d})\big) \\ &= 1 + O_{\Lambda,\mathcal{F}}\big(\|\mu\| N(x)^{-1/d}\big)\end{aligned}$$

with $\|\mu\| = \max_{j=1}^d |\mu_j|$. Thus,

$$\delta(x) = \frac{\lambda(x)}{\operatorname{vol}(\Lambda)} \int_\Lambda O_{\Lambda,\mathcal{F}}\big(\|\mu\| N(x)^{-1/d}\big)d\eta \ll_{\Lambda,\mathcal{F}} \|\mu\| N(x)^{-1/d},$$

since $|\lambda(x)| = 1$. We find the following estimate for the interior error:

$$\sum_{x \in \mathcal{M}\cap\mathcal{X}_n} |\delta(x)| \ll_{\Lambda,\mathcal{F}} \|\mu\| \sum_{x \in \mathcal{M}\cap\mathcal{X}_n} N(x)^{-1/d}. \tag{65}$$

We use partial summation to handle the latter sum on the basis of the estimate $s'_1(n) \ll n$ for the case $\lambda = 1$ (see Statement 2.15, Chap. IV of [J]),

$$\begin{aligned}\sum_{x\in\mathcal{M}\cap\mathcal{X}_n} N(x)^{-1/d} &\le \sum_{m=1}^{n} m^{-1/d}\big(s_1'(m)-s_1'(m-1)\big)\\ &\le \sum_{m=1}^{n-1} s_1'(m)\big(m^{-1/d}-(m+1)^{-1/d}\big)+n^{-1/d}s_1'(n)\\ &\ll \sum_{m=1}^{n-1} m^{-1/d}+n^{-1/d+1} \ll n^{1-1/d}\,.\end{aligned}$$

Thus

$$\sum_{x\in\mathcal{M}\cap\mathcal{X}_n^a} \delta(x) = O_{\Lambda,\mathcal{F}}\big(\|\mu\| n^{1-1/d}\big)\,. \tag{66}$$

At the boundary $\partial\mathcal{X}_n$, for some $x\in\mathcal{M}\cap\mathcal{X}_n$, the set $x+\Lambda$ may jut out of $\mathcal{X}_n$. At other points we may have for $x\in\mathcal{M}\smallsetminus\mathcal{X}_n$ that $(x+\Lambda)\cap\mathcal{X}_n\neq\emptyset$. We note that $\mathcal{X}_n$ is obtained from $\mathcal{X}_1$ via multiplication by $t=n^{1/d}$. We apply the argument in the proof of Theorem 2, §2, Chap. VI, in [La1], to see that the number of $x\in\mathcal{M}$ such that $(x+\Lambda)\cap\partial\mathcal{X}_n\neq\emptyset$ is

$$O_{\Lambda,\mathcal{F}}(n^{1-1/d})\,. \tag{67}$$

The boundary error has the same estimate (not depending on μ).

We now turn to the integral $I_\lambda(n)$. It is convenient to change coordinates to $z_j=\log y_j$. Then the region of integration corresponds to the infinite region

$$\mathcal{Y}_n = \begin{cases} S(z)\le\log n & S(z)=\sum_{j=1}^d z_j,\\ z_u\in\mathcal{H} & z_u = z-\big(\frac{S(z)}{n},\dots,\frac{S(z)}{n}\big),\end{cases} \tag{68}$$

where $\mathcal{H}:=\log\mathcal{F}$. We note that $\mathcal{H}$ is a fundamental domain for a group of translations in $S=0$ for which $z\mapsto e^{2i\mu.z}$ is periodic. Therefore,

$$I_n = \frac{1}{\operatorname{vol}\Lambda}\int_{\mathcal{Y}_n} e^{2i\mu\cdot z}e^{S(z)}dz\,. \tag{69}$$

The factor $e^{S(z)}$ and the restriction $z_u\in\mathcal{H}$ are responsible for the absolute convergence. If we integrate first over a hyperplane $S(z)=C$, C a constant, we get zero; therefore $I_n=0$. In the light of (63), (66), (67) and (69), we get

Lemma 5.1. *If $\lambda=\lambda_\mu$ is non-trivial, then the sum $s_\lambda(n)$ defined in* (61) *satisfies*

$$s_\lambda(n) = O_{\Lambda,\mathcal{F},\mathfrak{c}}\big(\|\mu\| n^{1-1/d}\big)\,. \tag{70}$$

Now we estimate the zeta function $Z(s,\lambda,\tau)$, see (60). The argument will be essentially the same as in [L], and will only be sketched. We will follow the standard Mertens' scheme, using (70) to keep track of the influence of $\lambda = \lambda_\mu$. For $\operatorname{Re} s > 1$, one finds with (61),

$$Z(s,\lambda,\tau) = \sum_{n=1}^{m} \frac{f_\lambda(n)}{n^s} + \sum_{n=m+1}^{\infty} s_\lambda(n)\bigl(n^{-s} - (n+1)^{-s}\bigr) - \frac{s_\lambda(m)}{(m+1)^s}\,,$$

which stays valid for $\operatorname{Re} s > 1 - \frac{1}{d}$ if $\lambda \neq 1$. One takes $\lambda \neq 1$, $\sigma = \operatorname{Re} s \geq 1$ and uses (70) to obtain,

$$\Bigl|\sum_{n=1}^{m} \frac{f_\lambda(n)}{n^s}\Bigr| \leq \sum_{n=1}^{m-1} s_1(n)\left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{s_1(m)}{m} \ll_{F,\mathfrak{q}} \log m$$

$$\Bigl|\sum_{n=m+1}^{\infty} s_\lambda(n)\bigl(n^{-s} - (n+1)^{-s}\bigr)\Bigr| \ll_{\Lambda,\mathcal{F}} |s|\|\mu\| m^{-1/d}$$

$$\Bigl|\frac{s_\lambda(m)}{(m+1)^s}\Bigr| \ll_{\Lambda,\mathcal{F}} \|\mu\| m^{-1/d}\,.$$

A suitable choice is $m = [1 + (1+|s|)^d \|\mu\|^d]$. This gives

$$Z(s,\lambda,\tau) \ll_{F,\mathfrak{q}} \log\bigl(1+|s|\bigr) + \log\|\mu\|\,, \tag{71}$$

for $\lambda \neq 1$, $\operatorname{Re} s \geq 1$. The implicit constants depend also on the choice of $\mathfrak{c}$ and of $\mathcal{F}$.

Next, one estimates the derivative of the zeta function,

$$Z'(s,\lambda,\tau) = -\sum_{n=1}^{\infty} \frac{f_\lambda(n)\log n}{n^s}\,.$$

An argument entirely analogous to the one above gives for, $\operatorname{Re} s \geq 1$, $\lambda \neq 1$,

$$Z'(s,\lambda,\tau) \ll_{F,\mathfrak{q}} \log^2\bigl(1+|s|\bigr) + \log^2\|\mu\|\,. \tag{72}$$

The classical approach, c.f. Landau, [L, p. 91, p. 94], implies similar estimates for the case $\lambda = 1$, $1 \leq \operatorname{Re} s \leq 2$, with bounds $\frac{1}{|\operatorname{Im} s|^l} + \log^l |\operatorname{Im} s|$, $l = 1$ or 2, for Z and Z' respectively, instead of $\log^l(1+|s|) + \log^l \|\mu\|$.

We now turn to the L-series defined in (59). It is a linear combination of series $Z(s,\lambda,\tau)$, with τ running through the ray classes. The estimates for Z imply similar estimates for L. From (62) it follows that $Z(s,1,\tau) = \frac{\alpha}{s-1} + h_\tau(s)$, with h_τ holomorphic on $\operatorname{Re} s > 1 - \frac{1}{d}$. If the character χ of the ray class group $I_\mathfrak{q}/F_\mathfrak{q}$ is non-trivial, we have $L(s,1,\chi) = \sum_\tau \chi(\tau) h_\tau(s)$, holomorphic on $\operatorname{Re} s > 1 - \frac{1}{d}$. So the finitely many $L(s,1,\chi)$ with $\chi \neq 1$ have a common upper bound for $1 \leq \operatorname{Re} s \leq 2$, $|\operatorname{Im} s| \leq 1$, and so do their derivatives. Of course, $L(s,1,1)$ has a first order singularity at $s = 1$. We summarize:

$$L(s,\lambda,\chi) \ll_{F,\mathfrak{q}} \log\bigl(2+|t|\bigr) + \zeta_\lambda \log\|\mu\| + \delta_{\lambda,\chi}|t|^{-1} =: b(t,\mu,\chi)\,,$$

$$L'(s,\lambda,\chi) \ll_{F,\mathfrak{q}} \log^2\bigl(2+|t|\bigr) + \zeta_\lambda \log^2\|\mu\| + \delta_{\lambda,\chi}|t|^{-2} \ll b(t,\mu,\chi)^2,$$

for $s=\sigma+it$, $\sigma\ge 1$, $t\ne 0$, χ a character of the ray class group $I_{\mathfrak q}/F_{\mathfrak q}$. We use $\delta_{\lambda,\chi}=1$, if $\chi=1$, $\lambda=1$, and $\delta_{\chi,\lambda}=0$ otherwise, and $\zeta_\lambda=1$ if $\lambda\ne 1$, $\zeta_1=0$. In the case $\delta_{\lambda,\chi}=0$, these estimates extend to $s=1$ by continuity.

The product expansion of the L-function implies

$$\log\left|L(\sigma+it,\lambda,\chi)\right| = \sum_{n=1}^{\infty}\frac{1}{n}\sum_{\mathfrak p:(\mathfrak p,\mathfrak q)=1} N(\mathfrak p)^{-n\sigma}\,\mathrm{Re}\left(\lambda(\mathfrak p)^n\chi(\mathfrak p)^n N(\mathfrak p)^{-nit}\right)$$

From the inequality $3+4\cos\phi+\cos 2\phi\ge 0$ it follows in the standard way that for $\sigma>1$

$$\left|L(\sigma,1,1)\right|^3\left|L(\sigma+it,\lambda,\chi)\right|^4\left|L(\sigma+2it,\lambda^2,\chi^2)\right|\ge 1\,.$$

We suppose that $t\ne 0$ or $\delta_{\lambda,\chi}\ne 0$. Then we have, for $\sigma>1$,

$$\left|L(\sigma+it,\lambda,\chi)\right|\ge\frac{C_2'(\sigma-1)^{3/4}}{|L(\sigma+2it,\lambda^2,\chi^2)|^{1/4}}\,.$$

Leaving the exceptional case $\mu=0$, $\chi^2=1$, $|t|\le 1$ aside for the moment, we have $b(2t,2\mu,\chi^2)\ll b(t,\mu,\chi)$, and obtain

$$\left|L(\sigma+it,\lambda,\chi)\right|\ge C_2\frac{(\sigma-1)^{3/4}}{b(t,\mu,\chi)^{1/4}}. \tag{73}$$

Furthermore,

$$\begin{aligned}&\left|L(\sigma+it,\lambda,\chi)-L(1+it,\lambda,\chi)\right|\\ &\qquad\ll(\sigma-1)\left|L'(1+\theta+it,\lambda,\chi)\right|,\quad 0<\theta<\sigma-1\\ &\qquad\ll(\sigma-1)b(t,\mu,\chi)^2.\end{aligned}$$

For $t\ne 0$ if $\delta_{\lambda,\chi}=0$:

$$\begin{aligned}\left|L(1+it,\lambda,\chi)\right|&\ge\left|L(\sigma+it,\lambda,\chi)\right|-\left|L(\sigma+it,\lambda,\chi)-L(1+it,\lambda,\chi)\right|\\ &\ge\frac{C_2(\sigma-1)^{3/4}}{b(t,\mu,\chi)^{1/4}}-C_3(\sigma-1)b(t,\mu,\chi)^2.\end{aligned}$$

One may choose

$$\sigma=1+\left(\tfrac{C_2}{2C_3}\right)^4 b(t,\mu,\chi)^{-9},\qquad C_4=\left(\tfrac{C_2}{2C_3}\right)^3\tfrac{C_2}{2}\,,$$

to obtain

$$\left|L(1+it,\lambda,\chi)\right|\ge C_4 b(t,\mu,\chi)^{-7}. \tag{74}$$

In the exceptional cases when $\mu=0$ (for which we have chosen $\lambda=1$), $\chi^2=1$, $|t|\le 1$, we find

$$\left|L(\sigma+it,1,\chi)\right|\gg(\sigma-1)^{3/4}|\sigma+2it-1|^{1/4}.$$

So these $L(s,1,\chi)$ cannot have a zero at $\sigma\ge 1$, $0<|t|\le 1$. We know that there is a pole at $s=1$ if $\chi=1$ (see, e.g., [J, Ch. IV, Prop. 4.1]). If

$\chi \neq 1$, then $L(s,1,\chi)$ is holomorphic and non-zero at $s=1$, see [J, Ch. V, Prop. 10.2]. Thus, $L(s,1,\chi)^{-1}$ is bounded on the region $\sigma \geq 1$, $|t| \leq 1$. The ray class group $I_{\mathfrak{q}}/F_{\mathfrak{q}}$ is finite, so there are only finitely many exceptional χ.

We have thus proved

Proposition 5.2. *Let λ be a unitary character of $I_{\mathfrak{q}}$, which is given by $\lambda(\xi) = \prod_{j=1}^{d} |\xi^{\sigma_j}|^{2i\mu_j}$ on the ideals in $I_{\mathfrak{q}}$ of the form (ξ), $\xi \in F^*$, and which is equal to* 1 *if $\mu = 0$. Let χ be a character of the ray class group $I_{\mathfrak{q}}/F_{\mathfrak{q}}$. Then, for* $\operatorname{Re} s = 1$, *we have*

$$\frac{1}{|L(s,\lambda,\chi)|} \ll_{F,\mathfrak{q}} \begin{cases} \log^7\bigl(2+|\operatorname{Im} s|\bigr) + \log^7\|\mu\| & \text{if } \lambda \neq 1\,,\\ \log^7\bigl(2+|\operatorname{Im} s|\bigr) & \text{if } \lambda = 1\,. \end{cases}$$

In view of (58), the function $Q(\nu,\lambda,\tau)$ satisfies the same estimates,

$$Q(\nu,\lambda,\tau) \ll_{F,\mathfrak{q}} \log^7\bigl(2+|\operatorname{Im}\nu|\bigr) + \zeta_\lambda \log^7\|\mu\| \tag{75}$$

for $\operatorname{Re}\nu = 0$, with $\zeta_\lambda = 1$ if $\lambda \neq 1$, $\zeta_1 = 0$.

5.3 Estimation of the term corresponding to the continuous spectrum. We turn to the estimation of the term $\int_Y k(\nu)d\sigma_{r,r}^{\mathrm{cont}}(\nu)$ in (44), for the test function $k = \times_j k_j$ with arbitrary $k_j \in \mathcal{L}$ if $j \in E$, and k_j as indicated in (30), (31) for the other places. From (49), (52), (55), (56), and (75), we conclude for each cusp κ,

$$\begin{aligned} &D_\kappa^{\infty,r}(iy,i\mu) \ll |N(r)|^\varepsilon\bigl(\log^7(2+|y|) + \log^7\|\mu\|\bigr) \qquad \text{if } \mu \neq 0\,,\\ &D_\kappa^{\infty,r}(iy,0) \ll |N(r)|^\varepsilon \log^7\bigl(2+|y|\bigr)\,. \end{aligned} \tag{76}$$

The implicit constants in the estimates may depend on many quantities, for instance, the field F, the ideal $\mathfrak{q}$, the choice of the elements g_κ and g_{κ_j} describing cusps, and the choice of $\mathfrak{c} \in \tau^{-1}$ for elements of the ray class group.

In view of (44) and (31), the integral vanishes if $Q_- \neq 0$. So we need only look at the case $\{1,\dots,d\} = E \sqcup Q_+$.

$$\begin{aligned} \int_Y k\,d\sigma_{r,r}^{\mathrm{cont}} \ll |N(r)|^\varepsilon \sum_\kappa c_\kappa \sum_{\mu\in\mathcal{L}_\kappa} \int_{-\infty}^{\infty} \|k\|_{\alpha,a,E} \prod_{j\in E}\bigl(1+|y+\mu_j|\bigr)^{-a}\\ \cdot \prod_{j\in Q_+} e^{-s(y+\mu_j)^2-s/4}\cdot\bigl(\log^7(2+|y|)+\zeta_\mu\log^7\|\mu\|\bigr)dy\,, \end{aligned}$$

with $\zeta_0 = 0$, $\zeta_\mu = 1$ if $\mu \neq 0$. For each of the finitely many $\kappa \in \mathcal{P}$, we estimate the sum over $\mathcal{L}_\kappa$ by an integral over the hyperplane $\sum_j x_j = 0$. For $\alpha \in (1/2,\tau]$,

$$\int_Y k\,d\sigma^{\mathrm{cont}}_{r,r} \ll |N(r)|^{\varepsilon}\|k\|_{\alpha,a,E}\int_{\mathbb{R}^d}\prod_{j\in E}\big(1+|x_j|\big)^{-a}\prod_{j\in Q_+}e^{-sx_j^2}$$
$$\cdot\Big(\log^7\Big(2+\Big|\sum_j x_j\Big|\Big)+\log^7\Big(1+\max_j\Big|x_j-d^{-1}\sum_\ell x_\ell\Big|\Big)\Big)dx$$
$$\ll |N(r)|^{\varepsilon}\|k\|_{\alpha,a,E}\prod_{j\in E}\int_{-\infty}^{\infty}\big(1+|x|\big)^{-a+\varepsilon}dx$$
$$\cdot\prod_{j\in Q_+}\int_{-\infty}^{\infty}\big(1+|x|\big)^{\varepsilon}e^{-sx^2}dx$$
$$\ll |N(r)|^{\varepsilon}\|k\|_{\alpha,a,E}s^{(|E|-d)(\frac{1+\varepsilon}{2})},\quad \text{if } Q_-=\emptyset. \tag{77}$$

On the other hand, we have seen that $\int_Y k\,d\sigma^{\mathrm{cont}}_{r,r}=0$, if $Q_-\neq\emptyset$.

6 Proof of Theorem 3.1

The last term to be considered is given by the measure $d\sigma^{\mathrm{discr}}_{r,r} = d\sigma_{r,r}-d\sigma^{\mathrm{cont}}_{r,r}$:

$$\int_Y f(\nu)\,d\sigma^{\mathrm{discr}}_{r,r}(\nu)=\sum_{\varpi\neq\mathbf{1}}f(\nu_\varpi)\big|c^r(\varpi)\big|^2. \tag{78}$$

The sum formula (29), and the estimates in Proposition 4.1, Proposition 4.2, and (77), give for test functions k as chosen in §4, and $1/2<\alpha\le\tau$, $\varepsilon>0$ sufficiently small,

$$\int_Y k(\nu)\,d\sigma^{\mathrm{discr}}_{r,r}(\nu)=\frac{2^{1+|E|}}{(2\pi)^d}\sqrt{|D_F|}\,\mathsf{H}_E(k)s^{|E|-d}$$
$$+O_F\big(\|k\|_{\alpha,a,E}\ s^{|E|-d+c}\big)$$
$$+(\text{if } Q_-=\emptyset)\ O_{F,\mathfrak{q},\varepsilon}\big(\|k\|_{\alpha,a,E}|N(r)|^{\varepsilon}s^{(|E|-d)(\frac{1+\varepsilon}{2})}\big)$$
$$+O_{F,\alpha,\varepsilon}\big(\|k\|_{\alpha,a,E}N(r)^{2\alpha}s^{-(1+\varepsilon-\alpha)|Q_+|-(1-\alpha)|Q_-|}\big)\,. \tag{79}$$

Taking into account that

$$\int_y k(\nu)\,d\sigma^{\mathrm{discr}}_{r,r}(\nu)=\sum_{\substack{\varpi\neq\mathbf{1}\\ \lambda_{\varpi,j}\ge0,\,j\in Q_+\\ \lambda_{\varpi,j}<0,\,j\in Q_-}}\big|c^r(\varpi)\big|^2e^{-s\|\lambda_{\varpi,Q}\|_1}\prod_{j\in E}k_j(\nu_{\varpi,j})\,, \tag{80}$$

and that $d=|E|+|Q_+|+|Q_-|$, we obtain the following result:

Proposition 6.1. *Let $r\in\mathcal{O}'\smallsetminus\{0\}$. Choose a partition E, Q_+, Q_- of $\{1,\dots,d\}$ with $Q=Q_+\cup Q_-\neq\emptyset$ and let $\mathcal{R}$ be the corresponding set as* *in* (7)*. Let* $g = \times_{j\in E}\, g_j$*, with* $g_j \in \tilde{\mathcal{L}}$ *for each* $j \in E$*, let* $1/2 < \alpha \le \tau < 1$ *and* $\varepsilon > 0$ *sufficiently small. If* $Z_s(g)$ *is as in* (8)*, then we have*

$$Z_s(g) = \frac{2^{1+|E|}}{(2\pi)^d}\sqrt{|D_F|}\prod_{j\in E}\tilde{\mathsf{H}}(g_j)s^{-|Q|} + O_F\big(\|k\|_{\alpha,a,E}s^{-|Q|+c}\big)$$
$$+ (\textit{if } Q_- = \emptyset)O_{F,\mathfrak{q},\varepsilon}\big(\|k\|_{\alpha,a,E}|N(r)|^{\varepsilon}s^{-(1/2+\varepsilon)|Q|}\big)$$
$$+ O_{F,\alpha,\varepsilon}\big(\|k\|_{\alpha,a,E}|N(r)|^{2\alpha}s^{-((1+\varepsilon-\alpha)|Q_+| + (1-\alpha)|Q_-|)}\big)\,.$$

Here $c = 1/2$ *if* $Q_- \ne \emptyset$ *and* $c = 1$ *otherwise, and* $\tilde{\mathsf{H}}(g)$ *is as given in* (6)*. Also, if* $E = \emptyset$ *then the product factor on the right-hand side should be interpreted as* 1*.*

We see that all error terms are strictly smaller than $s^{-|Q|} = s^{-|Q_+|-|Q_-|}$. Taking into account the relation (5) between the test functions k and g, we obtain Theorem 3.1.

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