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Sums of Kloosterman sums for real quadratic number fields

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Abstract

We estimate sums of Kloosterman sums for a real quadratic number field F of the type

$$\mathbf{S} = \sum_{c} |N(c)|^{-1/2} S_F(r, r_1; c)$$

where c runs through the integers of F that satisfy $C \leqslant |N(c)| < 2C$, $A \leqslant |c/c'| < B$, with $A < B$ fixed and $C \to \infty$. By $x \mapsto x'$ we indicate the non-trivial automorphism of F. The Kloosterman sums are given by

$$S_F(r, r_1; c) = \sum_{d|c}{}^{*} e^{2\pi i \mathrm{Tr}_{F/\mathbf{Q}}(ra + r_1 d)/c},$$

with $ad \equiv 1|\ (c)$.

In the absence of exceptional eigenvalues for the corresponding Hilbert modular forms, our estimate implies that

$$\mathbf{S} = O_{F,\varepsilon,r,r_1,A,B}\left(C^{5/6+\varepsilon}\right)$$

for each $\varepsilon > 0$. An estimate not taking cancellation between Kloosterman sums into account would yield $O(C)$. The exponent $\frac{5}{6} + \varepsilon$ is less sharp than occurs in the bound $O_{F,r,r_1,\varepsilon}\left(C^{3/4+\varepsilon}\right)$, obtained in our paper in J. reine angew. Math. 535 (2001) 103–164 for sums of Kloosterman sums where c runs over integers satisfying $\sqrt{C} \leqslant |c| \leqslant 2\sqrt{C}$, $\sqrt{C} \leqslant |c'| \leqslant 2\sqrt{C}$. The proof is based on the Kloosterman-spectral sum formula for the corresponding Hilbert modular

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group. The Bessel transform in this formula has a product structure corresponding to the infinite places of F. This does not fit well to the bounds depending on $N(c)$ and c/c'. Nevertheless, we do obtain non-trivial bounds for **S**.

1. Introduction

Sums of Kloosterman sums have been estimated in two ways: By consideration of the Kloosterman–Selberg zeta functions, and with the use of sum formulas of Kuznetsov type. An example of the former method is given by Goldfeld and Sarnak [6]. For the latter, we refer to the example of Kuznetsov's treatment of the classical Kloosterman sums in [10]. The former method runs into serious trouble when one tries to apply it to number fields with infinitely many units. The sum formula, restricted to trivial K-types, has been extended to SL_2 over all number fields, see e.g., [3]. For estimates of sums of Kloosterman sums, the optimal version of the sum formula takes into account all K-types of the Lie group in question. We have done that in [4] for totally real number fields. In the real quadratic case, this leads to the estimate

$$\sum_{c\in\mathcal{O},\sqrt{C}\leqslant|c^{\sigma_j}|\leqslant 2\sqrt{C}} \frac{S_F(r,r_1;c)}{\sqrt{|N(c)|}} \ll_{F,\varepsilon,r,r_1} C^{3/4+\varepsilon} \quad (C\to\infty), \tag{1}$$

where we assume for the moment that there are no exceptional eigenvalues. F is a real quadratic number field, $\mathcal{O}$ its ring of integers, $N(\cdot)$ denotes the norm of F over $\mathbb{Q}$, and the numbers r and r_1 are non-zero elements of the different $\mathcal{O}'$. The Kloosterman sums over F are given by

$$S_F(r,r_1;c) = \sum_{d \bmod (c)}^{*} e^{2\pi i \mathrm{Tr}_{F/\mathbb{Q}}((rd+r_1a)/c)}, \tag{2}$$

where $ad\equiv 1 \bmod (c)$. The embeddings $F\to\mathbb{R}$ are denoted by $\sigma_1 : c\mapsto c$, and $\sigma_2 : c\mapsto c'$.

So c runs through four rectangular regions in $\mathcal{O}\subset\mathbb{R}^2$. This structure is imposed by the product structure of the sum formula in [4]. Peter Sarnak pointed out to us that it would be more natural to let c run over a region bounded by conditions on the multiplicative quantities $|N(c)|$ and $|c/c'|$. The purpose of this paper is to give estimates for sums of Kloosterman sums of this type. This we shall do in Theorems 5 and 6. Under the assumption that there are no exceptional eigenvalues, Theorem 5 implies the estimate

$$\sum_{c\in\mathcal{O},C\leqslant|N(c)|<2C,\ A\leqslant|c/c'|<B} \frac{S_F(r,r_1;c)}{\sqrt{|N(c)|}} \ll_{F,\varepsilon,r,r_1,A,B} C^{5/6+\varepsilon} \tag{3}$$

as $C \to \infty$. We pay for the transition from rectangular to multiplicative bounds on c by a lesser quality of the estimate. Nevertheless, Theorem 6 shows some cancellation among Kloosterman sums. It contains a parameter b bounding exceptional eigenvalues. For instance, if we use $b \leqslant \frac{1}{5}$ [12], Theorem 6 gives an estimate $O(C^{19/20+\varepsilon})$ for the sum in (3). This is better than $O(C^{1+\varepsilon})$ obtained in Proposition 7 on the basis of an estimate of the absolute values of the Kloosterman sums.

The sharp bounds in sum (3) can be smoothened by inserting a weight function. If we assume that there are no exceptional eigenvalues ($b = 0$), then (79) gives an estimate $O(C^{1/2} \log^2 C)$ for sum (3) with smooth bounds. Thus, we obtain an smooth analogue for the Linnik conjecture [11], which states $\sum_{c=1}^{N} S(n, m; c) = O_{n,m}(N^{1+\varepsilon})$ for the classical Kloosterman sums $S(n, m; c)$.

The test functions in the sum formula in [4] have a product structure corresponding to the product $\mathbb{R}^2 \supset F$. The first step in the proof of our results consists of an extension of the sum formula to a larger class of test functions. We carry this out in Section 2. Here it is no problem to work over a general totally real number field.

In the application in Section 3 of the sum formula to sums of Kloosterman sums with multiplicative bounds, we restrict ourselves to the real quadratic case.

2. Sum formula

2.1. Statement of the sum formula in [4]

Let F be a totally real number field of degree d over $\mathbb{Q}$, with embeddings $\sigma_j : F \to \mathbb{R}$, $1 \leqslant j \leqslant d$. By $\mathcal{O}$ we denote the ring of integers of F. By Γ we denote the corresponding Hilbert modular group $SL_2(\mathcal{O})$. The embeddings σ_j give an identification of Γ with a discrete subgroup of $G \coloneqq \mathrm{SL}_2(\mathbb{R})^d$.

We formulate the sum formula in Theorem 2.7.1 of [4] for the situation of the group Γ just fixed, and the cusp ∞ of Γ. With Proposition 2.5.6 of [4], it is possible to take the independent test function in the Kloosterman term. This leads to the following data:

(i) Two non-zero elements r, r_1 of the different $\mathcal{O}'$. These specify Fourier term orders of automorphic forms on $\Gamma \backslash G$.

This determines a vector $\mathbf{e} \in \{1, -1\}^d$ by $\mathbf{e}_j = \mathrm{sign}((rr_1)^{\sigma_j})$.

(ii) The test function $f \in C_c^\infty((0, \infty)^d)$ is of the form

$$f(t_1, \dots, t_d) = \prod_{j=1}^{d} f_j(t_j) \tag{4}$$

with $f_j \in C_c^\infty(0, \infty)$.

(iii) The Bessel transformation in (28) of [4] gives an even holomorphic function $h = \mathsf{B}_{\mathbf{e}}^{\leftarrow} f$ on $\mathbb{C}^d$, described by

$$h(v_1, \dots, v_d) = \prod_{j=1}^{d} h_j(v_j),$$

$$h_j(v) = \frac{1}{2} \int_0^\infty f_j(t) \frac{J_{-2v}^{\mathbf{e}_j}(4\pi\sqrt{t}) - J_{2v}^{\mathbf{e}_j}(4\pi\sqrt{t})}{\sin \pi v} \frac{dt}{t}, \tag{5}$$

where $J_u^1 = J_u,\ J_u^{-1} = I_u$, are Bessel functions, given by

$$J_u^{\pm 1}(y) = \sum_{n \geqslant 0} \frac{(\mp 1)^n}{n!\, \Gamma(u+n+1)} \left(\frac{y}{2}\right)^{u+2n}. \tag{6}$$

(iv) The *Kloosterman term* is a sum of Kloosterman sums as defined in (2):

$$K_{r,r_1}(f) = \sum_{c \in \mathcal{O}, c \neq 0} \frac{S_F(r, r_1; c)}{|N(c)|} f\left(\left|\left(\frac{rr_1}{c^2}\right)^{\sigma_1}\right|, \dots, \left|\left(\frac{rr_1}{c^2}\right)^{\sigma_d}\right|\right). \tag{7}$$

This sum converges absolutely for compactly supported test functions. In [4, Theorem 2.7.1], the Kloosterman term is $K_{-r,-r_1}^{\kappa,\kappa_1}(\cdot)$. We have specialized the cusps κ and κ_1 to ∞, and use $S_F(-r, -r_1; c) = S_F(r, r_1; c)$.

(v) The spectral term is given by a measure $d\sigma_{r,r_1}$ on the set $\mathcal{Y}$:

$$\mathcal{Y} := \left(i[0, \infty) \cup (0, 1/2) \cup \left\{ \frac{b-1}{2} \colon b \geqslant 2,\ b \in 2\mathbb{Z} \right\} \right)^d, \tag{8}$$

$$\int_{\mathcal{Y}} h(v)\, d\sigma_{r,r_1}(v) = \sum_{\varpi} h(v_\varpi) \overline{c^r(\varpi)} c^{r_1}(\varpi) + \sum_{\lambda \in \mathcal{P}} c_\lambda \sum_{\mu \in \mathcal{L}_\lambda} \int_{-\infty}^{\infty} h(iy + i\mu) \overline{D^r(iy, i\mu)} D^{r_1}(iy, i\mu)\, dy. \tag{9}$$

$\mathcal{P}$ is a system of representatives of the Γ-orbits of cusps. To each $\lambda \in \mathcal{P}$ is associated a lattice $\mathcal{L}_\lambda$ contained in the hyperplane $x_1 + x_2 + \cdots + x_d = 0$ in $\mathbb{R}^d$, and a positive constant $c_\lambda > 0$. By D_λ^r we denote a normalized Fourier coefficient at the cusp ∞ of the Eisenstein series associated to the cusp λ. We identify $iy \in i\mathbb{R}$ with $(iy, \dots, iy) \in \mathbb{C}^d$.

The ϖ run over an orthogonal family of closed irreducible subspaces of the cuspidal subspace of $L^2(\Gamma \backslash G)$. The $c^r(\varpi)$ are normalized Fourier coefficients of the automorphic forms in ϖ.

Each ϖ is the tensor product $\varpi_1 \otimes \cdots \otimes \varpi_d$ of even irreducible representations of $\mathrm{SL}_2(\mathbb{R})$. To each ϖ_j, we associate a *spectral parameter* $v_{\varpi,j}$ such that

$\frac{1}{4} - \nu_{\varpi,j}^2$ is the eigenvalue of the Casimir operator in ϖ_j:

$$\begin{array}{ll} \nu_{\varpi,j} \in i[0,\infty), & \text{unitary principal series;} \\ \nu_{\varpi,j} \in (0,1/2), & \text{complementary series;} \\ \nu_{\varpi,j} = \dfrac{b-1}{2} \in \dfrac{1}{2} + \mathbb{Z}_{\geqslant 0}, & \text{holomorphic or antiholomorphic} \\ & \text{discrete series.} \end{array}$$

We denote $\nu_{\varpi} = (\nu_{\varpi,1}, \dots, \nu_{\varpi,d}) \in \mathcal{Y}$.

We call $\nu_{\varpi,j} \in (0, \frac{1}{2})$ an *exceptional coordinate*. Luo, Rudnick and Sarnak have shown that all exceptional coordinates are contained in $(0, \frac{1}{5}]$, see [12]. In the papers [8,9], Kim and Shahidi bring this down to $b \leqslant \frac{5}{34}$.

Theorem 1 (See Bruggeman et al. [4, Theorem 2.7.1, Proposition 2.5.6]). *Let r, r_1, f and h be as above. Then the function h is integrable for the measure $d\sigma_{r,r_1}$ on $\mathcal{Y}$, and*

$$\int_{\mathcal{Y}} h \, d\sigma_{r,r_1} = K_{r,r_1}(f). \tag{10}$$

To prove Theorems 5 and 6 and estimate (3), we shall need to have the sum formula (10) for all $f \in C_c^\infty((0,\infty)^d)$, not only for f with product structure as in (4). That is the subject of this section. Theorem 4 states the result we need.

2.2. Continuity

Our approach is to consider a norm on $C_c^\infty((0,\infty)^d)$ for which all terms in (10) are continuous.

Proposition 5.1.2 in [4] states that the Kloosterman term $K_{r,r_1}(f)$ converges absolutely for any function f on $(0,\infty)^d$ satisfying

$$f(y) \ll b_{s,t}(y) := \prod_{j=1}^{d} \min(y_j^s, y_j^{-t}) \tag{11}$$

with $s,t \in \mathbb{R}$, $s+t>0$, $s>\frac{1}{2}$. This implies that the linear form $f \mapsto K_{r,r_1}(f)$ is continuous for each norm

$$N_{s,t}(f) := \sup_{y \in (0,\infty)^d} |f(y)| \prod_{j=1}^{d} \max(y_j^{-s}, y_j^t) = \sup_{y \in (0,1)^d} \frac{|f(y)|}{b_{s,t}(y)} \tag{12}$$

with $s+t>0$, $s>\frac{1}{2}$.

Let us turn to the Bessel transform $\mathsf{B}_{\mathbf{e}}^{\leftarrow}$ in (iii). We start by deriving a preliminary estimate:

Lemma 2. *Let* $s, t > \frac{1}{2}$. *For* $b_{s,t}$ *as in* (11), *we have*

$$\mathsf{B}_{\mathbf{e}}^{\leftarrow} b_{s,t}(v) \ll 1 + |\mathrm{Im}\, v|$$

uniformly for $v \in \mathscr{Y}$.

Proof. We use

$$\mathsf{B}_{\mathbf{e}}^{\leftarrow} b_{s,t}(v) = \prod_{j=1}^{d} \frac{1}{2}(b_{s,\mathbf{e}_j}^{0}(v_j) + b_{t,\mathbf{e}_j}^{\infty}(v_j)),$$

$$b_{s,\pm 1}^{0}(v) = \int_0^1 y^s \mathscr{B}_{\pm 1}(y, v) \frac{dy}{y},$$

$$b_{t,\pm 1}^{\infty}(v) = \int_1^\infty y^{-t} \mathscr{B}_{\pm 1}(y, v) \frac{dy}{y}, \tag{13}$$

$$\mathscr{B}_{\pm 1}(y, v) = \frac{J_{-2v}^{\pm 1}(4\pi\sqrt{y}) - J_{2v}^{\pm 1}(4\pi\sqrt{y})}{\sin \pi v}, \tag{14}$$

$$\mathscr{B}_1\left(y, \frac{b-1}{2}\right) = 2(-1)^{b/2} J_{b-1}^{1}(4\pi\sqrt{y}) \quad (b \in 2\mathbb{Z},\ b \geqslant 2). \tag{15}$$

We show that $b_{s,\pm 1}^{0}(v)$ and $b_{t,\pm 1}^{\infty}(v)$ are $O(1 + |\mathrm{Im}\, v|)$ for the relevant values of v, by considering the following cases:

$\pm$	ν	$b_{s,\pm 1}^{0}(\nu)$	$b_{t,\pm 1}^{\infty}(\nu)$
$+$	$\left\{ \frac{b-1}{2} : b \in 2\mathbb{Z}, b \geq 2 \right\}$	$b)$	$b)$
$+$	$i[0, 1/2) \cup (0, 1/2)$	$f)$	$g)$
$+$	$i[1/2, \infty)$	$a)$	$e)$
$-$	$i[0, 1/2) \cup (0, 1/2)$	$f)$	$c)$
$-$	$i[1/2, \infty)$	$a)$	$d)$

(a) Lemma 11.1 of [3] contains several estimates of Bessel functions that we can use here. In (38), [3], we find

$$J_u^{\pm 1}(y) \ll y^{\mathrm{Re}\, u} / |\Gamma(u+1)|, \tag{16}$$

uniformly for $\mathrm{Re}\, u \geqslant 0$, and $0 < y \leqslant y_0$. We use this to obtain for $\mathrm{Re}\, v = 0$, $\mathrm{Im}\, v \geqslant 1/2$:

$$\mathscr{B}_{\pm 1}(y, v) \ll (1 + |v|)^{-1/2} \quad (y \leqslant 1). \tag{17}$$

From $s > \frac{1}{2} > 0$, we obtain $b^0_{s,\pm 1}(v) = O(|v|^{-1/2}) = O(1)$ for $v \in i[1/2, \infty)$.

(b) In [3], Lemma 11.1, (40), we find

$$J^1_u(y) \ll \frac{1}{u} \quad (u > 0, y > 0). \tag{18}$$

This covers the case $v = \frac{b-1}{2}$, $b \in 2\mathbb{Z}$, $b \geqslant 2$.

(c) We have

$$\mathscr{B}_{-1}(y, v) = \frac{4}{\pi} \cos \pi v\, K_{2v}(4\pi\sqrt{y}). \tag{19}$$

Estimate (42) in [3] gives for $\mathrm{Re}\, v > -\frac{1}{2}$:

$$\mathscr{B}_{-1}(y, v) \ll |\cos \pi v\, \Gamma(2v + \tfrac{3}{2})| y^{-\mathrm{Re}\, v - 1/2}. \tag{20}$$

Consider $|v| \leqslant \frac{1}{2}$. Then the factors $\cos \pi v$ and $\Gamma(2v + 3/2)$ are $O(1)$. We obtain

$$b^\infty_{t,-1}(v) \ll \int_1^\infty y^{-1/2-t} \frac{dy}{y} \ll 1. \tag{21}$$

(d) For $\mathrm{Re}\, v = 0$, $\mathrm{Im}\, v \geqslant \frac{1}{2}$, estimate (20) gives $\mathscr{B}_{-1}(y, v) \ll (1 + |v|) y^{-1/2}$, and $b^\infty_{t,-1}(v) = O(1 + |\mathrm{Im}\, v|)$.

(e) Let us use (4.1.4) in [1] for $v = iu \in i\mathbb{R}$, $|u| \geqslant \frac{1}{2}$, $y \geqslant 1$:

$$\begin{aligned} J_{2v}(y) &= \frac{1}{4\pi i} \int_{\mathrm{Re}\, s = -1/2} \left(\frac{y}{2}\right)^{-s} \frac{\Gamma(v + s/2)}{\Gamma(1 + v - s/2)}\, ds + \frac{(y/2)^{2v}}{\Gamma(2v + 1)} \\ &\ll y^{1/2} \int_{-\infty}^{\infty} \frac{e^{(\pi/2)(|u - \tau/2| - |u + \tau/2|)}\, d\tau}{(1 + |2u + \tau|)^{3/4} (1 + |2u - \tau|)^{3/4}} + e^{\pi|u|} |u|^{-1/2} \\ &\ll y^{1/2} e^{\pi|u|}. \end{aligned} \tag{22}$$

Hence $\mathscr{B}_1(y, v) = O(y^{1/4})$ and $b^\infty_{t,1}(iu) \ll 1$.

(f) A refinement of (16) is

$$J^{\pm 1}_u(y) = \frac{(y/2)^u}{\Gamma(u + 1)} + O\left(\frac{y^{\mathrm{Re}\, u + 2}}{\Gamma(u + 2)}\right) \tag{23}$$

for $|u| = 1$, and $y \leqslant y_0$. This implies for $|v| = \frac{1}{2}$, $y \leqslant 1$:

$$\mathscr{B}_{\pm 1}(y, v) \ll y^{-1/2}. \tag{24}$$

By holomorphy, this estimate extends to $|v| \leqslant \frac{1}{2}$, and yields $b^0_{s,\pm 1}(v) \ll 1$ for $|v| \leqslant \frac{1}{2}$.

(g) From 3.61, (1), (2), in [13]:

$$\mathscr{B}_1(y, v) = ie^{\pi iv} H_{2v}^{(1)}(4\pi\sqrt{y}) - ie^{-\pi iv} H_{2v}^{(2)}(4\pi\sqrt{y}). \tag{25}$$

We use this for the remaining case $|v| \leqslant \frac{1}{2}$. The factors $e^{\pm\pi iv}$ are $O(1)$. In 6.21, (2), (3) of [13], we find:

$$\begin{aligned} H_{2v}^{(1 \text{ or } 2)}(y) &= \frac{\pm 1}{\pi i} \int_{-\infty}^{0} e^{y \sinh w - 2vw}\, dw \\ &\quad + \frac{1}{\pi} \int_0^{\pi} e^{\pm iy \sin w \mp 2ivw}\, dw \\ &\quad \pm \frac{e^{\mp 2\pi iv}}{\pi i} \int_0^{\infty} e^{-y \sinh w - 2vw}\, dw \\ &\ll 2 \int_0^{\infty} e^{-y \sinh x + x/2}\, dx + 1 \ll 1 \end{aligned} \tag{26}$$

for $y \geqslant y_0$. This gives first $\mathscr{B}_1(y, v) \ll 1$ for $|v| = \frac{1}{2},\ y \geqslant 1$, and hence for all v such that $|v| \leqslant \frac{1}{2}$. Hence $b_{s,1}^{\infty}(v) \ll 1$ for $|v| \leqslant \frac{1}{2}$. $\square$

We conclude that for $f \in C_c^{\infty}((0, \infty)^d)$ and $s, t > \frac{1}{2}$,

$$\mathsf{B}_{\mathbf{e}}^{\leftarrow} f(v) = \int_{(0,\infty)^d} f(y) \prod_{j=1}^{d} \left(\frac{1}{2} \mathscr{B}_{\mathbf{e}_j}(y_j, v_j) \right) \frac{dy_1}{y_1} \cdots \frac{dy_d}{y_d} \tag{27}$$

defines a continuous function on $\mathscr{Y}$, and

$$\sup_{v \in \mathscr{Y}} \left| \mathsf{B}_{\mathbf{e}}^{\leftarrow} f(v) \prod_{j=1}^{d} (1 + |\mathrm{Im}\, v_j|)^{-1} \right| \ll_{s,t} N_{s,t}(f). \tag{28}$$

We need to do better than (28) to obtain continuity in L^1-sense for the measure $d\sigma_{r,r_1}$.

Lemma 3. *Let* ℓ_j *be the differential operator* $4y_j^2 \partial_{y_j}^2 + 4y_j \partial_{y_j} + 16\pi^2 \mathbf{e}_j y_j$ *on* $(0, \infty)^d$. *For all* $f \in C_c^{\infty}((0, \infty)^d)$, *we have*:

$$\mathsf{B}_{\mathbf{e}}^{\leftarrow} \ell_j f = 4v_j^2 \mathsf{B}_{\mathbf{e}}^{\leftarrow} f. \tag{29}$$

Proof. By partial integration, this is a direct consequence of the Bessel differential equation. $\square$

This leads to the following norm on $C_c^\infty((0,\infty)^d)$:

$$M_{s,t}(f) := \max_{E\subset\{1,\dots,d\}} N_{s,t}\left(\left(\prod_{j\in E}\ell_j^2\right)f\right). \tag{30}$$

So if $s,t>\frac{1}{2}$, then

$$\sup_{v\in\mathscr{Y}} |\mathsf{B}_{\mathbf{e}}^{\leftarrow} f(v)| \ll_{s,t} M_{s,t}(f) \prod_{j=1}^{d}\left(\frac{1+|\mathrm{Im}\, v_j|}{1+|v_j|^4}\right) \tag{31}$$

for all $C_c^\infty((0,\infty)^d)$.

Let $\tau\in(1/2,1)$, and put

$$\eta(v_1,\dots,v_d) := \prod_{j=1}^{d} \eta_{\mathbf{e}_j}(v_j),$$

$$\eta_1(v) := \begin{cases} (1-v^2)^{-3/2} & \text{for } |\mathrm{Re}\, v|\leqslant\tau,\\ b^{-4} & \text{for } v=\frac{b-1}{2},\ b\in 2\mathbb{Z},\ b\geqslant 2,\end{cases}$$

$$\eta_{-1}v := (1-v^2)^{-5/2}(1/4-v^2).$$

This defines a test function η as in Definition 2.5.1 of [4], for which Theorem 2.7.1 in [4] holds. In particular, it is an integrable function for $d\sigma_{r,r}$ and for $d\sigma_{r_1,r_1}$. Moreover, this function is positive on $\mathscr{Y}$, and

$$\prod_{j=1}^{d}\left(\frac{1+|\mathrm{Im}\, v_j|}{1+|v_j|^4}\right) \ll \eta(v_1,\dots,v_d)$$

on $\mathscr{Y}$. Hence $\mathsf{B}_{\mathbf{e}}^{\leftarrow} f$ is integrable for these measures for each $f\in C_c^\infty((0,\infty)^d)$. Lemma 3.1.1 in [4] and (31) imply that $\mathsf{B}_{\mathbf{e}}^{\leftarrow} f$ is integrable for $d\sigma_{r,r_1}$, and that

$$\left|\int_{\mathscr{Y}} \mathsf{B}_{\mathbf{e}}^{\leftarrow} f\, d\sigma_{r,r_1}\right| \ll_{s,t,r,r_1} M_{s,t}(f). \tag{32}$$

2.3. Extension

The continuity results allow us to extend the sum formula to the completion with respect to $M_{s,t}$ $(s,t>\frac{1}{2})$ of the space spanned by the functions f in (ii) of Section 2.1. Here we are content to extend the sum formula to $C_c^\infty((0,\infty)^d)$.

The fact that $|K_{r,r_1}(f)| \ll N_{s,t}(f)$ for $f \in C_c^\infty((0,\infty)^d)$ implies that $f \mapsto K_{r,r_1}(f)$ is a distribution of order zero on $(0,\infty)^d$. From (32) and (30), we conclude that $f \mapsto \int_{\mathscr{Y}} \mathsf{B}_{\mathbf{e}}^{\leftarrow} f \, d\sigma_{r,r_1}$ is a distribution of order at most $4d$. These distributions coincide on the test functions of product type, hence they are equal (see, e.g., [7, Theorem 5.2.1, p. 128]). Thus, we have proved:

Theorem 4. *Let $r, r_1 \in \mathcal{O} \setminus \{0\}$. For each $f \in C_c^\infty((0,\infty)^d)$ the Bessel transform*

$$\mathsf{B}_{\mathbf{e}}^{\leftarrow} f(v) \coloneqq 2^{-d} \int_{(0,\infty)^d} f(t) \prod_{j=1}^{d} \mathscr{B}_{\mathbf{e}_j}(v_j, t_j) \frac{dt_1}{t_1} \cdots \frac{dt_d}{t_d} \tag{33}$$

converges absolutely, it defines an integrable function on $\mathscr{Y}$ for the measure $d\sigma_{r,r_1}$, and

$$K_{r,r_1}(f) = \int_{\mathscr{Y}} \mathsf{B}_{\mathbf{e}}^{\leftarrow} f \, d\sigma_{r,r_1}. \tag{34}$$

3. Sums of Kloosterman sums for real quadratic number fields

Now we assume that F is a real quadratic number field, so $d = 2$, and we revert to the notations of the Introduction. We shall apply Theorem 4, and will arrive at the estimates in Theorems 5 and 6, from which (3) will follow.

3.1. Sums of Kloosterman sums

Instead of the sum of Kloosterman sums in (3), we consider

$$\Lambda_{r,r_1}(A,B;C) \coloneqq \sum_{c \in \mathcal{O}, C \leqslant |N(c)| < 2C, A \leqslant |c/c'| < B} \frac{S_F(r,r_1;c)}{|N(c)|}. \tag{35}$$

Any bound $\Xi(A,B;C)$ for this sum gives a bound $C^{1/2}\Xi(A,B;C)$ for the sum in (3).

We view $C \geqslant 1$ as the main parameter. We keep track of the influence of the other parameters $0 < A < B$, $r, r_1 \in \mathcal{O}' \setminus \{0\}$. In doing so, it is convenient to put:

$$R \coloneqq |N(rr_1)|, \tag{36}$$

$$U \coloneqq \max\left(\left| \frac{rr_1}{(rr_1)'} \right|^{1/4} A^{-1/2}, \left| \frac{(rr_1)'}{rr_1} \right|^{1/4} B^{1/2} \right), \tag{37}$$

$$L \coloneqq \log(B/A). \tag{38}$$

We note that $L > 0$, $U \geqslant 1$, and that R is bounded from below by a positive constant depending only on F.

The sum formula gives good results when both $|c|$ and $|c'|$ are not too small. This is arranged by the following condition:

$$C \geqslant \max(R^{1/2}U^2, R^{-1/2}U). \tag{39}$$

To formulate our result, we define bounds for the singly and doubly exceptional coordinates. With the notations in v), Section 2.1, we define:

$$\beta(1) := \sup\{v_{\varpi,j}\colon\ v_{\varpi,j} \in (0,1/2), v_{\varpi,3-j} \notin (0,1/2)\}, \tag{40}$$

$$\beta(2) := \sup\{v_{\varpi,j}\colon\ v_{\varpi,j} \in (0,1/2), v_{\varpi,3-j} \in (0,1/2)\}. \tag{41}$$

Here, we take the supremum equal to 0 in case the set is empty. From [12], it follows that $\beta(1), \beta(2) \leqslant \frac{1}{5}$.

Theorem 5. *Take $r,\ r_1,\ A,\ B$ as above and let $\varepsilon > 0$. With the notations in* (36)–(38), *for each C satisfying condition* (39) *and $C^{1/6} \geqslant \max(2, L^{-1})$, we have*:

$$\begin{aligned}\Lambda_{r,r_1}(A,B;C) \ll_\varepsilon & R^{1/4+\varepsilon}(L+1)C^{1/3+\varepsilon} + R^{1/4}\max(A^{-1/2}, B^{1/2})C^\varepsilon \\ & + R^{1/4-\beta(1)/2+\varepsilon}U^{2\beta(1)}(L+1)C^{1/4+\beta(1)} \\ & + R^{1/4-\beta(2)+\varepsilon}U^{4\beta(2)}(L+1)C^{1/6+2\beta(2)}.\end{aligned}$$

This result is optimal in the C-aspect only if $\beta(1) \leqslant \frac{1}{12}$ and $\beta(2) \leqslant \frac{1}{12}$. If there are no exceptional coordinates, $\beta(1) = \beta(2) = 0$, then the last two terms can be omitted.

The statement in the theorem becomes more transparent, if we leave implicit the influence or $r,\ r_1$, and the sector $|c/c'| \in [A, B)$:

$$\Lambda_{r,r_1}(A,B;C) \ll_{r,r_1,A,B,\varepsilon} C^{1/3+\varepsilon} + C^{1/4+\beta(1)} + C^{1/6+2\beta(2)}. \tag{42}$$

If we knew that $\beta(j) \leqslant \frac{1}{12}$, then we would find the bound $O(C^{1/3+\varepsilon})$, which leads to (3).

The following result is optimal under the present knowledge concerning $\beta(1)$ and $\beta(2)$ (take $b = \frac{1}{5}$).

Theorem 6. *Take $r,\ r_1,\ A,\ B$ as above. Assume that $\beta(1) \leqslant b,\ \beta(2) \leqslant b$ for some $b \in (\frac{1}{12}, \frac{1}{4})$. With the notations in* (36)–(38), *we have for each C satisfying condition* (39) *and*

$$C^{1/4-b} \geqslant R^{-b/2}U^{2b}\max(2, L^{-1}, \log^2 C),$$

and for each $\varepsilon > 0$:

$$\begin{aligned} \Lambda_{r,r_1}(A,B;C) \ll_\varepsilon & R^{1/4-b/2+\varepsilon} U^{2b}(L+1)C^{1/4+b+\varepsilon} \\ & + R^{1/4+b/4+\varepsilon} U^{-b}(L+1)C^{3/8-b/2} \\ & + R^{1/4+b+\varepsilon} U^{-4b}(L+1)C^{1/4-2b} \\ & + R^{1/4}\max(A^{-1/2}, B^{1/2})C^{\varepsilon}. \end{aligned}$$

If we leave out the explicit dependence on the parameters, we find for $\frac{1}{12} < b < \frac{1}{2}$:

$$\Lambda_{r,r_1}(A,B;C) \ll_{A,B,R,\varepsilon} C^{1/4+b+\varepsilon}. \tag{43}$$

Let us compare this estimate with what can be said on the basis of the Weil–Salié estimate

$$|S_F(r,r_1;c)| \ll_\varepsilon \min(\sqrt{|N(r)|}, \sqrt{|N(r_1)|})\ |N(c)|^{1/2+\varepsilon} \quad (\varepsilon > 0); \tag{44}$$

see, e.g., Theorem 10 in [2]. We shall show, in Section 3.6, that this estimate implies:

Proposition 7. *Let* $r, r_1 \in \mathcal{O}' \setminus \{0\},\ 0 < A < B$. *For all* $C \geqslant 1$:

$$\begin{aligned} \Lambda_{r,r_1}(A,B;C) \ll_\varepsilon & \min(\sqrt{|N(r)|}, \sqrt{|N(r_1)|}) \\ & \times (LC^{1/2+\varepsilon} + \max(A^{-1/2}, B^{1/2})C^{\varepsilon}) \quad (\varepsilon > 0). \end{aligned}$$

This gives $\Lambda_{r,r_1}(A,B;C) \ll_{A,B,R,\varepsilon} C^{1/2+\varepsilon}$. The sum formula gives an improvement of $C^{1/4-b}$ in the C-aspect. Thus, Theorem 6 implies cancellation among Kloosterman sums for any $b < \frac{1}{4}$.

The remainder of this section gives the proofs of Theorems 5 and 6 and Proposition 7.

3.2. Kloosterman term

Let τ_{sh}, respectively σ_{sh}, be the characteristic function of the interval $(2,4]$, respectively $(\alpha, \beta]$, where

$$\alpha := -\log B + \frac{1}{2}\log\left|\frac{rr_1}{(rr_1)'}\right|, \quad \beta := -\log A + \frac{1}{2}\log\left|\frac{rr_1}{(rr_1)'}\right|. \tag{45}$$

We have

$$\Lambda_{r,r_1}(A,B;C) = K_{r,r_1}(f_{\mathrm{sh}}), \tag{46}$$

$$f_{\mathrm{sh}}\left(\frac{y_1^2}{(4\pi)^2}, \frac{y_2^2}{(4\pi)^2}\right) := \tau_{\mathrm{sh}}\left(\frac{y_1y_2}{X}\right)\sigma_{\mathrm{sh}}\left(\log\frac{y_1}{y_2}\right), \tag{47}$$

$$X := \frac{4\pi^2 R^{1/2}}{C}. \tag{48}$$

Note that condition (39) implies that $X \leqslant 4\pi^2 U^{-2} \leqslant 4\pi^2$.

This test function f_{sh} with "sharp bounds" allows us to write the sum of Kloosterman sums as a Kloosterman term in the sum formula. We apply the sum formula to a smooth approximation f of f_{sh}. The proofs of Theorems 5 and 6 consist of finding good bounds for $K_{r,r_1}(f - f_{\mathrm{sh}})$ and for $K_{r,r_1}(f) = \int_{\mathscr{Y}} \mathsf{B}_{\mathbf{e}}^{\leftarrow} f \, d\sigma_{r,r_1}$.

We take $\tau \in C_c^\infty(0,\infty)$ such that $0 \leqslant \tau \leqslant 1$, $\mathrm{Supp}(\tau) \subset [2, 4+\frac{2}{Y}]$, $\tau = 1$ on $[2+\frac{1}{Y}, 4]$, and $\sigma \in C_c^\infty(\mathbb{R})$ such that $0 \leqslant \sigma \leqslant 1$, $\mathrm{Supp}(\sigma) \subset [\alpha, \beta + \frac{1}{Y}]$, $\sigma(q) = 1$ if $\alpha + \frac{1}{Y} \leqslant q \leqslant \beta$. The parameter Y governs the steepness of τ and σ. We take $Y \geqslant \max(2, \frac{1}{\beta-\alpha})$. We arrange the choice of τ and σ in such a way that

$$\begin{cases} \int_0^\infty |\tau'(y)|\,dy & = \int_{-\infty}^\infty |\sigma'(y)|\,dy = 2, \\ \int_0^\infty |\tau^{(\ell)}(y)|\,dy & \ll_\ell Y^{\ell-1} \quad (\ell \geqslant 2), \\ \int_{-\infty}^\infty |\sigma^{(\ell)}(y)|\,dy & \ll_\ell Y^{\ell-1} \quad (\ell \geqslant 2). \end{cases} \tag{49}$$

Theorem 4 can be applied to the following test function:

$$f\left(\frac{y_1^2}{(4\pi)^2}, \frac{y_2^2}{(4\pi)^2}\right) := \tau\left(\frac{y_1y_2}{X}\right)\sigma\left(\log\frac{y_1}{y_2}\right). \tag{50}$$

3.3. Bessel transform

The hardest part of the proof is a good estimation of the Bessel transform (see (33) and (14)):

$$\begin{aligned} h(v_1,v_2) &:= \mathsf{B}_{\mathbf{e}}^{\leftarrow} f(v_1,v_2) \\ &= \frac{1}{4}\int_0^\infty \int_0^\infty f(t_1,t_2)\mathscr{B}_{\mathbf{e}_1}(t_1,v_1)\mathscr{B}_{\mathbf{e}_2}(t_2,v_2)\,\frac{dt_1}{t_1}\frac{dt_2}{t_2} \\ &= \int_0^\infty \int_0^\infty \tau\left(\frac{y_1y_2}{X}\right)\sigma\left(\log\frac{y_1}{y_2}\right)\tilde{\mathscr{B}}_{\mathbf{e}_1}(y_1,v_1)\tilde{\mathscr{B}}_{\mathbf{e}_2}(y_2,v_2)\,\frac{dy_1}{y_1}\frac{dy_2}{y_2}, \end{aligned} \tag{51}$$

$$\tilde{\mathscr{B}}_{\pm 1}(y,v) := \frac{J_{-2v}^{\pm 1}(y) - J_{2v}^{\pm 1}(y)}{\sin \pi v}. \tag{52}$$

We follow the approach of Deshouillers and Iwaniec, [5], expanded a bit in [1]. There the spectral set $\mathscr{Y}$ has dimension one. It is split into various pieces. For each piece an integral representation of Bessel functions is chosen that gives a good estimate of the Bessel transform. Here the set $\mathscr{Y}$ has dimension two, and the number of cases gets too large to be considered piece by piece.

We systematize our approach by noting the following form of the integral representations in [1,5]

$$\tilde{\mathscr{B}}_{\pm 1}(y, v) = \int_W e^{-yD(w)} y^{E(w)}\, d\mu(w). \tag{53}$$

Here $d\mu$ is some measure on some set W. The functions D and E on W satisfy $\operatorname{Re} D \geqslant 0$, $\operatorname{Re} E \geqslant 0$, and either $E = 0,\ D(w) \neq 0$ for all $w \in W$, or $D = 0,\ E(w) \neq 0$ for all $w \in W$. The dependence of $d\mu$ on v is not visible in the notation. (See the proof of Lemma 9 for the actual choices we shall use.) All integral representations that we use converge absolutely, and allow the following reformulation for suitable values of n_j:

$$\begin{aligned} h(v_1, v_2) = \int_0^\infty \int_0^\infty \int_{W_1} \int_{W_2} & e^{-y_1 D_1(w_1)} y_1^{E_1(w_1)} e^{-y_2 D_2(w_2)} y_2^{E_2(w_2)} \\ & \times \tau(y_1 y_2 / X) \sigma(\log(y_1/y_2))\, d\mu_2(w_2)\, d\mu_1(w_1) \frac{dy_1}{y_1} \frac{dy_2}{y_2} \end{aligned} \tag{54}$$

$$\begin{aligned} = & \int_{W_1} \int_{W_2} \int_0^\infty \int_0^\infty \eta_1(y_1, w_1, n_1) \eta_2(y_2, w_2, n_2) \\ & \times \partial_{y_1}^{n_1} \partial_{y_2}^{n_2} (\tau(y_1 y_2/X) \sigma(\log(y_1/y_2)) y_1^{-a_1} y_2^{-a_2}) \\ & \times dy_1\, dy_2\, d\mu_2^{n_2}(w_2)\, d\mu_1^{n_1}(w_1), \end{aligned} \tag{55}$$

$$\text{if } D = 0\text{:} \quad \begin{cases} \eta(y, w, n) & = \quad y^{E(w)+n-1}, \\ a & = \quad 0, \\ d\mu^n(w) & = \quad \dfrac{(-1)^n}{(E(w))_n}\, d\mu(w) \\ & = \quad \frac{(-1)^n}{E(w)(E(w)+1)\cdots(E(w)+n-1)}\, d\mu(w), \end{cases} \tag{56}$$

$$\text{if } E = 0\text{:} \quad \begin{cases} \eta(y, w, n) \quad = \quad e^{-yD(w)}, \\ a = 1, \\ d\mu^n(w) \quad = \quad D(w)^{-n}\, d\mu(w). \end{cases} \tag{57}$$

The integral in (55) has product structure, except for the factor involving partial derivatives. We try to find a bound for this factor that has as much product structure as possible.

3.3.1. General estimation scheme

By induction, we see that

$$\begin{aligned}\partial_{y_1}^{n_1}\partial_{y_2}^{n_2}&\left(\frac{\tau(y_1y_2/X)\sigma(\log(y_1/y_2))}{y_1^{a_1}y_2^{a_2}}\right)\\ &=\sum_{b,c\geqslant 0,b+c\leqslant n_1+n_2} C_{n_1,n_2}^{b,c}Q_{n_1,n_2}(b,c),\end{aligned} \tag{58}$$

$$Q_{n_1,n_2}(b,c) := X^{-b}\,\frac{\tau^{(b)}(y_1y_2/X)\sigma^{(c)}(\log(y_1/y_2))}{y_1^{a_1+n_1-b}y_2^{a_2+n_2-b}}, \tag{59}$$

where the coefficients $C_{n_1,n_2}^{b,c}$ are not important, as we will restrict ourselves to $n_1+n_2\leqslant 4$.

$$\begin{aligned}\int_0^\infty\int_0^\infty &\eta_1(y_1,w_1,n_1)\eta_2(y_2,w_2,n_2)Q_{n_1,n_2}(b,c)\,dy_1\,dy_2\\ &= X^{-b}\int_0^\infty\int_0^\infty e^{-y_1D_1(w_1)-y_2D_2(w_2)}y_1^{E_1(w_1)-a_1-n_1+b}\\ &\quad\times\begin{Bmatrix} y_1^{n_1-1} & \text{if } D_1=0\\ 1 & \text{if } E_1=0\end{Bmatrix}\begin{Bmatrix} y_2^{n_2-1} & \text{if } D_2=0\\ 1 & \text{if } E_2=0\end{Bmatrix}\\ &\quad\times y_2^{E_2(w_2)-a_2-n_2+b}\tau^{(b)}(y_1y_2/X)\sigma^{(c)}(\log(y_1/y_2))\,dy_1\,dy_2.\end{aligned}$$

We substitute $y_1=\sqrt{pq}$ and $y_2=\sqrt{p/q}$. The factors $e^{-y_jD_j(w_j)}$ are bounded by 1. The variable p runs effectively over an interval contained in $[2X,8X]$, and the variable q over $[e^\alpha, e^{\beta+1/Y}]$. With (49), we find:

$$\begin{aligned}&\ll X^{-b}\int_0^\infty\int_0^\infty p^{\mathrm{Re}(E_1(w_1)+E_2(w_2)-n_1a_1-n_2a_2+2b-2)/2}\\ &\quad\times q^{\mathrm{Re}(E_1(w_1)-E_2(w_2)-n_1a_1+n_2a_2)/2}|\tau^{(b)}(p/X)|\,|\sigma^{(c)}(\log q)|\,\frac{dq}{q}\,dp\\ &\ll X^{\mathrm{Re}(E_1(w_1)+E_2(w_2)-n_1a_1-n_2a_2)/2}\end{aligned} \tag{60}$$

$$\begin{aligned}\times\max(&e^{\mathrm{Re}(E_1(w_1)-E_2(w_2)-n_1a_1+n_2a_2)\alpha/2},\\ &e^{\mathrm{Re}(E_1(w_1)-E_2(w_2)-n_1a_1+n_2a_2)(\beta+1/Y)/2}),\end{aligned} \tag{61}$$

$$\times\begin{Bmatrix} 1 & \text{if } b=0\\ Y^{b-1} & \text{if } b\geqslant 1\end{Bmatrix}\begin{Bmatrix} \beta+1/Y-\alpha & \text{if } c=0,\\ Y^{c-1} & \text{if } c\geqslant 1.\end{Bmatrix} \tag{62}$$

Let us put

$$\tilde{U} := Ue^{1/Y} = \max(e^{-\alpha/2}e^{1/Y}, e^{\beta/2}e^{1/Y}) \geqslant 1, \tag{63}$$

$$V := L + 1/Y = \log(B/A) + 1/Y. \tag{64}$$

The maximum over $b, c \geqslant 0,\ b + c \leqslant n_1 + n_2$, of factor (62) is

$$\ll \begin{cases} V & \text{if } n_1 + n_2 = 0, \\ (V+1)\, Y^{n_1+n_2-1} & \text{if } n_1 + n_2 \geqslant 1. \end{cases}$$

It seems that we might weaken the estimate, if we would try to bring this factor into product form.

For factor (61), we use the following bound:

$$\tilde{U}^{|\mathrm{Re}\ E_1(w_1) - n_1 a_1| + |\mathrm{Re}\ E_2(w_2) - n_2 a_2|}.$$

Thus, we have obtained:

Lemma 8. *We put*:

$$F(w, n) := \begin{cases} \dfrac{X^{\mathrm{Re}\ E(w)/2}\, \tilde{U}^{\mathrm{Re}\ E(w)}}{|E(w)_n|} & \textit{if } D = 0, \\ X^{-n/2}\, \tilde{U}^n |D(w)|^{-n} & \textit{if } E = 0, \end{cases} \tag{65}$$

$$G(N) := \begin{cases} V & \textit{if } N = 0, \\ (V+1)\, Y^{N-1} & \textit{if } N \geqslant 1. \end{cases} \tag{66}$$

Then for $n_1, n_2 \in \mathbb{N}_{\geqslant 0},\ n_1 + n_2 \leqslant 4$:

$$h(v_1, v_2) \ll G(n_1 + n_2) \int_{W_1} F_1(w_1, n_1)\, |d\mu_1(w_1)| \times \int_{W_2} F_2(w_2, n_2)\, |d\mu_2(w_2)|. \tag{67}$$

In some cases, we shall need to use different orders of integration in different parts of W. Then we divide up $W = \bigcup_{n \in N} W(n)$, and use the slightly more general estimate:

$$h(v_1, v_2) \ll \sum_{n_1 \in N_1} \sum_{n_2 \in N_2} G(n_1 + n_2) \times \int_{W_1(n_1)} F_1(w_1, n_1)\, |d\mu_1(w_1)| \int_{W_2(n_2)} F_2(w_2, n_2)\, |d\mu_2(w_2)|. \tag{68}$$

3.3.2. Local estimates

The spectral set $\mathscr{Y}$ in (8) is the Cartesian product $\mathscr{Y}_{\mathbf{e}_1} \times \mathscr{Y}_{\mathbf{e}_2}$. We split up $\mathscr{Y}_{\pm 1}$ into four regions, depending on a parameter $T \geqslant 2$, and $\beta \in (0, 1/4)$ such that all relevant exceptional coordinates are in $(0, \beta]$:

$$\begin{array}{r|cc} & \pm 1 = 1 & \pm 1 = -1 \\ \hline \mathbf{A} := & i[0,2) \cup \left\{\frac{1}{2}, \frac{3}{2}\right\} & i[0,2] \\ \mathbf{B} := & i[2,T) \cup \left\{ \nu \in \frac{1}{2} + \mathbb{Z} \, : \, \frac{5}{2} \leq \nu < T \right\} & i[2,T) \\ \mathbf{C} := & i[T,\infty) \cup \left\{ \nu \in \frac{1}{2} + \mathbb{Z} \, : \, \nu \geq T \right\} & i[T,\infty) \\ \mathbf{E} := & (0,\beta] & (0,\beta] \end{array} \tag{69}$$

We shall use $\beta = \beta(2)$ when considering $\mathbf{E} \times \mathbf{E} \subset \mathscr{Y}$, and $\beta = \beta(1)$ for $\mathbf{X} \times \mathbf{E}$ and $\mathbf{E} \times \mathbf{X}$ with $\mathbf{X} \in \{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$. We have already remarked that $\beta \leqslant \frac{1}{5}$ is known.

Lemma 9. (a) *For each* $\nu \in \mathbf{A}$, *there is a choice* $W = W(0) \cup W(1)$ *such that*

$$\int_{W(n)} F(w,n)\, d\mu(w) \ll \begin{cases} \log(UX^{-1/2}) & \text{if } n = 0, \\ 1 & \text{if } n = 1. \end{cases}$$

(b) *For each* $\nu \in \mathbf{B}$, *there is a choice of* W *such that*

$$\int_{W} F(w,1)\, d\mu(w) \ll |\nu|^{-3/2}.$$

(c) *For each* $\nu \in \mathbf{C}$, *there is a choice of* W *such that*

$$\int_{W} F(w,2)\, d\mu(w) \ll |\nu|^{-5/2}.$$

(d) *For each* $\nu \in \mathbf{E}$, *there is a choice* $W = W(0) \cup W(1)$ *such that*

$$\int_{W(n)} F(w,n)\, d\mu(w) \ll \begin{cases} X^{-\beta} U^{2\beta} \log(UX^{-1/2}) & \text{if } n = 0, \\ X^{-\beta} U^{2\beta} & \text{if } n = 1. \end{cases}$$

Proof. We consider six different cases. Our aim is to establish estimates, not explicit formulas. So if the integrand in (53) is the sum of two complex conjugate quantities, we can work with an estimate of one of the resulting terms. By (cst.) we denote an explicit constant.

We note that $U \geqslant 1$, $X = 4\pi^2 R^{1/2} C^{-1} \leqslant 4\pi^2$, $UX^{-1/2} \gg 1$, and $XU^2 \leqslant 4\pi^2 \ll 1$; see (39). We shall also use $\tilde{U} = Ue^{1/Y} \ll U$.

(i) *Equal sign*, $v \in i[0,2] \subset \mathbf{A}$, *or* $v \in \mathbf{E}$. As in (25), (26), we have

$$\begin{aligned}\tilde{B}_1(y,v) = \sum_{\pm} (\text{cst.}) e^{\pm \pi i v} \int_0^\infty e^{-y \sinh \tau + 2v\tau}\, d\tau \\ + \sum_{\pm} (\text{cst.}) \int_{-\pi/2}^{\pi/2} e^{\pm iy \cos\theta \mp 2iv\theta}\, d\theta \\ + \sum_{\pm} (\text{cst.}) e^{\mp \pi i v} \int_0^\infty e^{-y \sinh \tau - 2v\tau}\, d\tau. \qquad (70)\end{aligned}$$

The factors $e^{\pm \pi i v}$ are $O(1)$ for the values of v under consideration. We take $W = W(0) \cup W(1)$, $W(0) = (0, Z] \cup i(-\pi/2, \pi/2)$, $W(1) = [Z, \infty)$ for some $Z > 0$ still to be determined. On $(0, \infty)$ we take

$$d\mu(w) = \cosh(2vw)\, dw, \quad D(w) = \sinh w;$$

and on $i(-\pi/2, \pi/2)$:

$$d\mu(w) = e^{-2vw} \frac{dw}{i}, \quad D(w) = -i \cos(w/i).$$

We have

$$F(w,0) \ll 1, \quad F(w,1) \ll X^{-1/2} \tilde{U} \begin{cases} 1/\cos(w/i) & \text{if } w \in i[-\pi/2, \pi/2], \\ 1/\sinh w & \text{if } w \in (0, \infty). \end{cases}$$

It seems sensible to choose Z such that both bounds are equal at $w = Z$:

$$Z = \log(\tilde{U} X^{-1/2} + \sqrt{\tilde{U}^2 X^{-1} + 1}) \ll \log(\tilde{U} X^{-1/2}).$$

We obtain:

$$\begin{aligned}\int_{W(0)} F(w,0)\, d\mu(w) &\ll \int_{-\pi/2}^{\pi/2} e^{\pi |\mathrm{Im}\, v|}\, d\theta + \int_0^Z e^{2|\mathrm{Re}\, v|\tau}\, d\tau \\ &\ll 1 + Z e^{2|\mathrm{Re}\, v| Z} \ll 1 + \log(\tilde{U} X^{-1/2}) (\tilde{U} X^{-1/2})^{2|\mathrm{Re}\, v|} \\ &\ll \begin{cases} \log(U X^{-1/2}) & \text{if } v \in \mathbf{A}, \\ \log(U X^{-1/2}) X^{-\beta} U^{2\beta} & \text{if } v \in \mathbf{E}, \end{cases}\end{aligned}$$

$$\begin{aligned}\int_{W(1)} F(w,1)\, d\mu(w) &\ll \int_Z^\infty X^{-1/2}\tilde{U}\frac{e^{2|\mathrm{Re}\,v|\tau}}{\sinh\tau}\, d\tau\\ &\ll X^{-1/2}\tilde{U}\int_Z^\infty e^{(2|\mathrm{Re}\,v|-1)\tau}\, d\tau\\ &\ll \begin{cases} 1 & \text{if } v\in\mathbf{A},\\ X^{-\beta}U^{2\beta} & \text{if } v\in\mathbf{E}.\end{cases}\end{aligned}$$

In the case $v\in\mathbf{E}$, the implicit constant of the estimate contains $\frac{1}{1-2\beta}$. As $\beta\leqslant\frac{1}{5}$, this causes no harm.

(ii) *Equal sign case*, $v=\frac{1}{2}$ *or* $\frac{3}{2}\in\mathbf{A}$. In [13], 6.2, (7), we find for $\mathrm{Re}\,u>0$:

$$J_u(y) = \frac{1}{\pi}\int_0^{\pi/2}\cos(u\theta - y\sin\theta)\, d\theta + \frac{1}{\pi}\int_0^\infty e^{-u\tau}\sin(y\cosh\tau - \pi u/2)\, d\tau. \tag{71}$$

With (15), we see that a good choice is $W = i[-\pi/2,0)\cup i(0,\pi/2]\cup[0,\infty)$, with, for $w=i\theta\in i[-\pi/2,\pi/2]$, $\theta\neq 0$:

$$d\mu(w) = e^{2vi\theta}\, d\theta,\quad D(i\theta) = i\sin\theta;$$

and for $w=\tau\in[0,\infty)$:

$$d\mu(w) = e^{-2v\tau}\, d\tau,\quad D(\tau) = -i\cosh\tau.$$

We have $F(w,0)\ll 1$, $F(\tau,1)\ll X^{-1/2}\tilde{U}/\cosh\tau$ for $\tau\geqslant 0$, and $F(i\theta,1)\ll X^{-1/2}\tilde{U}/\sin\theta$ for $\theta\in[-\pi/2,\pi/2]$. We take

$$Z = \log(\tilde{U}X^{-1/2} + \sqrt{\tilde{U}^2X^{-1}-1}) \ll \log(\tilde{U}X^{-1/2}),$$

and $W(0) = i[-\pi/2,\pi/2]\cup[0,Z]$, $W(1) = [Z,\infty)$. We obtain

$$\int_{W(0)} F(w,0)\, d\mu(w) \ll 1 + \int_0^Z e^{-2v\tau}\, d\tau \ll 1,$$

$$\int_{W(1)} F(w,1)\, d\mu(w) \ll X^{-1/2}\tilde{U}\int_Z^\infty \frac{e^{-2v\tau}}{\cosh\tau}\, d\tau \ll U^{-1}X^{1/2}.$$

This is better than required in the lemma.

(iii) *Unequal sign*, $v \in i[0,2] \subset \mathbf{A}$ *or* $v \in \mathbf{E}$. From (19) and [13], 6.22, (7):

$$\tilde{\mathscr{B}}_{-1}(y,v) = (\text{cst.}) \cos \pi v \int_{-\infty}^{\infty} e^{-y \cosh w - 2vw}\, dw. \tag{72}$$

We consider this for v in a bounded set, so $\cos \pi v = O(1)$.

We take $W = [0, \infty)$, $d\mu(w) = \cosh(2vw)\, dw$, $D(w) = \cosh w$, and have $F(w,0) \ll 1$, $F(w,1) \ll X^{-1/2}\tilde{U}/\cosh w$. We take

$$Z = \log(\tilde{U}X^{-1/2} + \sqrt{\tilde{U}^2 X^{-1} - 1}) \ll \log(\tilde{U}X^{-1/2}),$$

and split up W into $W(0) = [0, Z)$, $W(1) = [Z, \infty)$.

$$\int_{W(0)} F(w,0)\, d\mu(w) \ll \int_0^Z e^{2|\mathrm{Re}\, v|w}\, dw \ll \log(UX^{-1/2})U^{2\, \mathrm{Re}\, v} X^{-\mathrm{Re}\, v},$$

$$\int_{W(1)} F(w,1)\, d\mu(w) \ll \int_Z^{\infty} X^{-1/2}\tilde{U}\frac{e^{2|\mathrm{Re}\, v|w}}{\cosh w}\, dw \ll U^{2|\mathrm{Re}\, v|} X^{-\mathrm{Re}\, v}.$$

This satisfies the requirements in parts (a) and (e) of the lemma.

(iv) *Equal sign*, $v \geqslant \frac{5}{2}$, *cases* **B** *and* **C**. As in (4.1.3) of [1], we use for $u > 1$:

$$J_u(y) = \frac{1}{4\pi i}\int_{\mathrm{Re}\, s = -\sigma} \left(\frac{y}{2}\right)^{-s} \frac{\Gamma((u+s)/2)}{\Gamma(1+(u-s)/2)}\, ds. \tag{73}$$

Take $\sigma = n \in \{1, 2\}$, and $u = b - 1 = 2v$, $b \in 2\mathbb{Z}$, $b \geqslant 4$. Take $W = \mathbb{R}$, $d\mu(w) = 2^{-\sigma + iw}\Gamma(v + \frac{iw-n}{2})\Gamma(1 + v + \frac{n-iw}{2})^{-1}\, dw$, and $E(w) = n - iw$. We have

$$F(w,n) \ll \frac{X^{n/2}\tilde{U}^n}{(1+|w|)^n}$$

and

$$\begin{aligned}
&\int_W F(w,n)\, dw \\
&\quad \ll \int_{-\infty}^{\infty} \frac{X^{n/2}\tilde{U}^n}{(1+|w|)^n\, |(v - \frac{n}{2} - \frac{iw}{2})_{n+1}|}\left|\frac{\Gamma(v - \frac{n}{2} + \frac{iw}{2})}{\Gamma(v - \frac{n}{2} - \frac{iw}{2})}\right| dw \\
&\quad \ll X^{n/2}\tilde{U}^n \int_0^{\infty} \frac{dx}{(1+x)^n (v - 1 + x)^{n+1}}
\end{aligned}$$

$$\ll \begin{cases} X^{1/2}\tilde{U}v^{-2}\log v & \text{if } n=1, \\ X\tilde{U}^2v^{-3} & \text{if } n=2; \end{cases}$$

$$\ll \begin{cases} v^{-3/2} & \text{if } n=1, \\ v^{-5/2} & \text{if } n=2. \end{cases}$$

(v) *Equal sign*, $v\in i[2,\infty)$, *cases* **B** *and* **C**. From (73) we obtain, with $\operatorname{Re} u = 0$:

$$J_u(y) = \frac{1}{4\pi i}\int_{\operatorname{Re} s=-1}\left(\frac{y}{2}\right)^{-s}\frac{\Gamma\left(\frac{u+s}{2}\right)}{\Gamma\left(1+\frac{u-s}{2}\right)}\,ds + \frac{(y/2)^u}{\Gamma(u+1)}. \tag{74}$$

For $\operatorname{Re} v = 0$, $\operatorname{Im} v \geqslant 2$, $v = it$:

$$\tilde{B}_1(y,v) \ll e^{-\pi t}\left|\int_{-\infty}^{\infty}\left(\frac{y}{2}\right)^{1-iw}\frac{\Gamma\left(it+\frac{iw-1}{2}\right)}{\Gamma\left(it+\frac{3-iw}{2}\right)}\,dw + \frac{(y/2)^{2it}}{\Gamma(1+2it)}\right|.$$

We use $W = \mathbb{R}\cup\{i\}$, $D=0$, $E(w) = 1-iw$ for $w\in\mathbb{R}$, $E(i) = 2it$, and the measure

$$d\mu(w) = e^{-\pi t}2^{iw}\frac{\Gamma\left(it+\frac{iw-1}{2}\right)}{\Gamma\left(it+\frac{3-iw}{2}\right)}\,dw \quad \text{on } \mathbb{R},$$

plus $e^{-\pi t}\Gamma(1+2it)^{-1}$ times a delta measure at i. With $n\in\{1,2\}$:

$$\begin{aligned}
&\int_W F(w,n)\,d\mu(w) \\
&\quad \ll \int_{-\infty}^{\infty}\frac{X^{1/2}\tilde{U}}{|(1-iw)_n|}\,e^{-\pi t}\left|\frac{\Gamma\left(\frac{2it+iw+1}{2}\right)}{\Gamma\left(\frac{2it-iw+1}{2}\right)}\right|\frac{dw}{|it+\frac{iw-1}{2}|\,|it+\frac{1-iw}{2}|} \\
&\qquad + \frac{1}{|(2it)_n|}\,t^{-1/2} \\
&\quad \ll X^{1/2}U\int_{-\infty}^{\infty}\frac{e^{-\pi t-\pi|t+w/2|/2+\pi|t-w/2|/2}\,dw}{(|1+|w|)^n(1+|t+w/2|)(1+|t-w/2|)} + t^{-n-1/2}.
\end{aligned}$$

We have

$$\int_{-\infty}^{-2t} \frac{dw}{(1-w)^n(1-t-w/2)(1+t-w/2)} \ll t^{-n-1}\log t,$$
$$\int_{-2t}^{0} \frac{e^{-\pi t-\pi w/2}\,dw}{(1-w)^n(1+t+w/2)(1+t-w/2)}$$
$$\ll t^{-n-1}e^{-\pi t}\int_{-2t}^{-t} e^{-\pi w}\,dw + e^{-\pi t/2}\int_{-t}^{0}\frac{dw}{(1-w)^n t} \ll t^{-n-1},$$
$$\int_{0}^{\infty} \frac{e^{-\pi t-\pi w/2}\,dw}{(1+w)^n(1+t+w/2)(1+|t-w/2|)} \ll e^{-\pi t}O(1).$$

All contributions have the bound $X^{1/2}U|v|^{-n-1}\log|v| + |v|^{-n-1/2} \ll |v|^{-n-1/2}$, in agreement with parts (b) and (c) in the lemma.

(vi) *Unequal sign*, $v\in i[2,\infty)$, *cases* **B** *and* **C**. Let $v=it$, $t\geqslant 2$, and $n=1$, respectively 2. As in pp. 308–309 of [1]:

$$\tilde{\mathscr{B}}_{-1}(y,v) = (\text{cst.})e^{\pi t}\int_{\operatorname{Re} s=-2n+1}\left(\frac{y}{2}\right)^{-s}\Gamma(it+s/2)\Gamma(-it+s/2)\,ds$$
$$+ (\text{cst.})e^{\pi t}\sum_{\pm}\sum_{l=0}^{n-1}\left(\frac{y}{2}\right)^{2l\mp 2it}\Gamma(-l\pm 2it). \tag{75}$$

Take $W=\mathbb{R}\cup\{l\pm i:\ 0\leqslant l\leqslant n-1\}$, $D=0$, $E(w)=2n-1-iw$ for $w\in\mathbb{R}$, $E(l\pm i)=2l\mp 2it$. On $\mathbb{R}$ the measure is

$$e^{\pi t}2^{1-2n+iw}\Gamma(it-n+\tfrac{1+iw}{2})\Gamma(-it-n+\tfrac{1+iw}{2})\,dw;$$

at $l\pm i$ we have delta measures with the factors $e^{\pi t}2^{-2l\pm 2it}\Gamma(-l\pm 2it)$.

The points $l\pm i$ give the following contribution to $\int_W F(w,n)\,dw$:

$$\frac{X^l\tilde{U}^{2l}}{(2l\mp 2it)_n}e^{\pi t}2^{-2l\pm 2it}\Gamma(-l\pm 2it) \ll X^l\tilde{U}^{2l}t^{-l-1/2-n}$$
$$\ll \begin{cases} |v|^{-3/2} & \text{if } n=1,\\ |v|^{-5/2} & \text{if } n=2.\end{cases}$$

Finally, we estimate the contribution of $\mathbb{R}\subset W$:

$$\int_{-\infty}^{\infty}\frac{X^{n-1/2}\tilde{U}^{2n-1}}{|(2n-1-iw)_n|}\frac{e^{\pi t-\pi|t+w/2|/2-\pi|t-w/2|/2}\,dw}{(1+|2t+w|)^n(1+|2t-w|)^n}$$

$$
\begin{aligned}
&\ll X^{n-1/2}\tilde{U}^{2n-1}\int_0^\infty \frac{e^{\pi t-\pi(t+w/2)/2-\pi|t-w/2|/2}\,dw}{(1+w)^n(1+2t+w)^n(1+|2t-w|)^n}\\
&\ll X^{n-1/2}\tilde{U}^{2n-1}\bigg(\int_{2t}^\infty \frac{e^{\pi t-\pi w/2}\,dw}{(1+w)^n(1+2t+w)^n(1-2t+w)^n}\\
&\quad+\int_0^{2t}\frac{dw}{(1+w)^n(1+2t+w)^n(1+2t-w)^n}\bigg)\\
&\ll X^{n-1/2}U^{2n-1}\left(t^{-2n}\int_{2t}^\infty e^{\pi t-\pi w/2}\,dw+\begin{cases} t^{-2}\log t & \text{if } n=1,\\ t^{-4} & \text{if } n=2;\end{cases}\right)\\
&\ll t^{-n-1/2}. \qquad \square
\end{aligned}
$$

Lemma 8 shows that going from **B** to **C** causes an additional factor Y in the estimate of h. In Lemma 9, the step from **B** to **C** amounts to an additional factor $|v|$. It seems appropriate to take the boundary T between **B** and **C** equal to Y.

3.4. Integration

The sum formula in Theorem 4 shows that $K_{r,r_1}(f)=\int_{\mathscr{Y}} h\,d\sigma_{r,r_1}$. Corollary 3.3.2 in [4] implies for $x_1,x_2\geqslant\frac{1}{4}$, and $\varepsilon>0$:

$$
S(x_1,x_2):=\int_{v\in\mathscr{Y},|v_j|<x_j}|d\sigma_{r,r_1}|\ll_\varepsilon R^{1/4+\varepsilon}(x_1x_2)^2. \tag{76}
$$

Let $p_1,p_2\geqslant 0$, and consider intervals $[a_1,b_1]$, $[a_2,b_2]\subset(0,\infty)$. Partial integration leads to the following estimate:

$$
\int_{v\in\mathscr{Y},a_j\leqslant|v_j|<b_j}|v_1|^{-p_1}|v_2|^{-p_2}\,|d\sigma_{r,r_1}(v_1,v_2)|\ll R^{1/4+\varepsilon}\prod_{j=1}^{2}\begin{cases} b_j^{2-p_j} & \text{if } p_j<2,\\ a_j^{2-p_j} & \text{if } p_j>2.\end{cases}
$$

Indeed, we have the following:

$$
\begin{aligned}
&\int_{v\in\mathscr{Y},a_j\leqslant|v_j|<b_j}|v_1|^{-p_1}|v_2|^{-p_2}\,|d\sigma_{r,r_1}(v_1,v_2)|\\
&\quad=\int_{x=a_1}^{b_1}x^{-p_1}\,d_x\left(\int_{y=a_2}^{b_2}y^{-p_2}\,d_yS(x,y)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{x=a_1}^{b_1} x^{-p_1}\, d_x \left(y^{-p_2} S(x,y)\big|_{y=a_2}^{b_2} + p_2 \int_{y=a_2}^{b_2} y^{-p_2-1} S(x,y)\, dy \right) \\
&= x^{-p_1} y^{-p_2} S(x,y)\big|_{y=a_2}^{b_2}\big|_{x=a_1}^{b_1} + p_2 x^{-p_1} \int_{y=a_2}^{b_2} y^{-p_2-1} S(x,y)\, dy \Bigg|_{x=a_1}^{b_1} \\
&\quad + p_1 \int_{x=a_1}^{b_1} x^{-p_1-1} y^{-p_2} S(x,y)\, dx \Bigg|_{y=a_2}^{b_2} \\
&\quad + p_1 p_2 \int_{x=a_1}^{b_1} \int_{y=a_2}^{b_2} x^{-p_1-1} y^{-p_2-1} S(x,y)\, dy\, dx \\
&\leqslant b_1^{-p_1} b_2^{-p_2} S(b_1,b_2) + a_1^{-p_1} a_2^{-p_2} S(a_1,a_2) \\
&\quad + p_2 b_1^{-p_1} \int_{y=a_2}^{b_2} y^{-p_2-1} S(b_1,y)\, dy + p_1 b_2^{-p_2} \int_{x=a_1}^{b_1} x^{-p_1-1} S(x,b_2)\, dx \\
&\quad + p_1 p_2 \int_{x=a_1}^{b_1} \int_{y=a_2}^{b_2} x^{-p_1-1} y^{-p_2-1} S(x,y)\, dy\, dx \\
&\ll_{p_1,p_2} R^{1/4+\varepsilon} (b_1^{2-p_1} b_2^{2-p_2} + a_1^{2-p_1} a_2^{2-p_2} + b_1^{2-p_1} \max(b_2^{2-p_2}, a_2^{2-p_2}) \\
&\quad + b_2^{2-p_2} \max(b_1^{2-b_1}, a_1^{2-p_1}) \\
&\quad + \max(b_2^{2-p_2}, a_2^{2-p_2}) \max(b_1^{2-p_1}, a_1^{2-p_1})) \\
&\ll R^{1/4+\varepsilon} \prod_{j=1}^{2} \begin{cases} b_j^{2-p_j} & \text{if } p_j < 2, \\ a_j^{2-p_j} & \text{if } p_j > 2. \end{cases}
\end{aligned}
$$

If $p_j > 2$, then the same estimate holds for integration over $a_j \leqslant |v_j| < \infty$.

With Lemma 8, its extension (68), and Lemma 9, we apply this to estimate the contribution

$$I(\mathbf{X}_1, \mathbf{X}_2) := \int_{\mathbf{X}_1 \times \mathbf{X}_2} |h|\, |d\sigma_{r,r_1}|$$

for $\mathbf{X}_1, \mathbf{X}_2 \in \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{E}\}$. If exactly one of $\mathbf{X}_1$ and $\mathbf{X}_2$ is equal to $\mathbf{E}$, we take β in Lemma 9 equal to $\beta(1)$, and if $\mathbf{X}_1 = \mathbf{X}_2 = \mathbf{E}$, then we take $\beta = \beta(2)$. The estimate for $I(\mathbf{X}_1, \mathbf{X}_2)$ is symmetric in $\mathbf{X}_1$ and $\mathbf{X}_2$.

$$I(\mathbf{C}, \mathbf{C}) \ll R^{1/4+\varepsilon} (V+1)\, Y^2,$$

$$I(\mathbf{B}, \mathbf{C}) \ll R^{1/4+\varepsilon} (V+1)\, Y^2,$$

$$I(\mathbf{B}, \mathbf{B}) \ll R^{1/4+\varepsilon} (V+1)\, Y^2,$$

$$I(\mathbf{A},\mathbf{C}) \ll R^{1/4+\varepsilon}(V+1)(\log(UX^{-1/2})\,Y^{1/2}+Y^{3/2}),$$

$$I(\mathbf{A},\mathbf{B}) \ll R^{1/4+\varepsilon}(V+1)(\log(UX^{-1/2})\,Y^{1/2}+Y^{3/2}),$$

$$\begin{aligned} I(\mathbf{A},\mathbf{A}) \ll{} & R^{1/4+\varepsilon} \\ & \times (V\log^2(UX^{-1/2})+(V+1)\log(UX^{-1/2})+(V+1)\,Y), \end{aligned}$$

$$\begin{aligned} I(\mathbf{A},\mathbf{E}) \ll{} & R^{1/4+\varepsilon}X^{-\beta(1)}U^{2\beta(1)} \\ & \times (V\log^2(UX^{-1/2})+(V+1)\log(UX^{-1/2})+(V+1)\,Y), \end{aligned}$$

$$I(\mathbf{B},\mathbf{E}) \ll R^{1/4+\varepsilon}X^{-\beta(1)}U^{2\beta(1)}(V+1)(Y^{1/2}\log(UX^{-1/2})+Y^{3/2}),$$

$$I(\mathbf{C},\mathbf{E}) \ll R^{1/4+\varepsilon}X^{-\beta(1)}U^{2\beta(1)}(V+1)(Y^{1/2}\log(UX^{-1/2})+Y^{3/2}),$$

$$\begin{aligned} I(\mathbf{E},\mathbf{E}) \ll{} & R^{1/4+\varepsilon}X^{-2\beta(2)}U^{4\beta(2)} \\ & \times (V\log^2(UX^{-1/2})+(V+1)\log(UX^{-1/2})+(V+1)\,Y). \end{aligned}$$

We take together all contributions, and use (39) to find $\log(UX^{-1/2}) = -\log(2\pi) + \frac{1}{6}\log(R^{1/2}U^2) + \frac{2}{3}\log(R^{-1/2}U) + \frac{1}{2}\log C \ll \log C$. We obtain under assumption (39):

$$\begin{aligned} K_{r,r_1}(f) = \int_{\mathscr{Y}} h\,d\sigma_{r,r_1} \ll{} & R^{1/4+\varepsilon}(V\log^2(UX^{-1/2}) \\ & + (V+1)(Y^2+Y^{1/2}\log(UX^{-1/2}))) \\ & + R^{1/4+\varepsilon}X^{-\beta(1)}U^{2\beta(1)} \\ & \times (V\log^2(UX^{-1/2})+(V+1)(Y^{3/2}+Y^{1/2}\log(UX^{-1/2}))) \\ & + R^{1/4+\varepsilon}X^{-2\beta(2)}U^{4\beta(2)} \end{aligned}$$

$$
\begin{aligned}
&\times (V \log^2(UX^{-1/2}) + (V+1)(Y + \log(UX^{-1/2})))\\
\ll\ & R^{1/4+\varepsilon}(V \log^2 C + (V+1)(Y^2 + Y^{1/2} \log C))\\
&+ R^{1/4-\beta(1)/2+\varepsilon} U^{2\beta(1)} C^{\beta(1)}\\
&\times (V \log^2 C + (V+1)(Y^{3/2} + Y^{1/2} \log C))\\
&+ R^{1/4-\beta(2)+\varepsilon} U^{4\beta(2)} C^{2\beta(2)}\\
&\times (V \log^2 C + (V+1)(Y + \log C)).
\end{aligned} \tag{77}
$$

Condition (39) implies that $X^{-1}U^2 \geqslant \frac{1}{4\pi^2}$. So the conclusion stays valid if we replace all $\beta(j)$ by $b \in [\beta(j), \frac{1}{4})$.

3.5. Smooth bounds

The sum of Kloosterman sums $K_{r,r_1}(f)$ is an approximation of the sum $\Lambda_{r,r_1}(A, B; C)$: the bounds on (c, c') have been replaced by smooth conditions. We fix Y and find:

$$
\begin{aligned}
K_{r,r_1}(f) \ll_Y\ & R^{1/4+\varepsilon}(V \log^2 C + (V+1)\log C)\\
&+ R^{1/4-\beta(1)/2+\varepsilon} U^{2\beta(1)} (V+1) C^{\beta(1)} \log C\\
&+ R^{1/4-\beta(2)+\varepsilon} U^{4\beta(2)} C^{2\beta(2)} (V \log^2 C + (V+1)\log C).
\end{aligned} \tag{78}
$$

If there are no exceptional coordinates at all, the sum with smooth bounds is, in the C-aspect,

$$K_{r,r_1}(f) \ll_{A,B,r,r_1} \log^2 C. \tag{79}$$

In particular, this says that the analog of the Linnik conjecture holds, with smooth bounds, in the context of this paper.

3.6. Direct estimation of sums of Kloosterman sums

For $\Omega \subset \mathbb{R}^2$, we denote by $M(\Omega)$ the number of $c \in \mathcal{O}$ such that $(c, c') \in \Omega$. Let us assume that Ω is connected and has a piecewise smooth boundary.

To estimate $M(\Omega)$, we choose a compact fundamental domain Π for the lattice $\mathcal{O} \subset \mathbb{R}^2$ such that $0 \in \Pi$. By $M_1(\Omega)$, we denote the number of translates $c + \Pi$, $c \in \mathcal{O}$, that intersect Ω. Hence $M(\Omega) \leqslant M_1(\Omega) = M(\Omega_1)$, where

$$\Omega_1 = \bigcup_{c \in \mathcal{O}, c+\Pi \cap \Omega \neq \emptyset} (c + \Pi).$$

Let $\delta := \sup\{2||x||\colon x \in \Pi\}$, and let $\Omega[\delta]$ be the δ-neighborhood of Ω. Then $\Omega_1 \subset \Omega[\delta]$, and $M(\Omega) \ll_\Pi \mathrm{vol}(\Omega[\delta])$.

For $0<a<b$, $0<t<u$, we estimate $\mathrm{vol}(\Omega(a,b,t,u)[\delta])$ with

$$\Omega(a,b,t,u) := \{(x_1,x_2) \in \mathbb{R}^2\colon\ t \leqslant |x_1x_2| \leqslant u, a \leqslant |x_1/x_2| \leqslant b\}. \tag{80}$$

We use that

$$\mathrm{vol}(\Omega(a,b,t,u)[\delta]) \ll \mathrm{vol}(\Omega(a,b,t,u)) + \delta\ \mathrm{length}(\partial\Omega(a,b,t,u)) + \delta^2. \tag{81}$$

In fact, for convex Ω with smooth boundary, we have the equality

$$\mathrm{vol}(\Omega[\delta]) = \mathrm{vol}(\Omega) + \delta \mathrm{length}(\partial\Omega) + \pi\delta^2.$$

This follows from the fact that there is an obvious bijection between $\partial\Omega[\delta]$ and $\partial\Omega$, to be used in an evaluation of the area of $\Omega[\delta]\backslash\Omega$ with Stokes' theorem. In a similar way one obtains (81) for $\Omega = \Omega(a,b,t,u)$. The integrals along the concave parts of the boundary give less than δ times the length of that part of the boundary. Each of the 16 corner points of $\partial\Omega$ contributes less than $\pi\delta^2$.

Let $\Xi = \Omega(a,b,t,u) \cap (0,\infty)^2$. We have $\mathrm{vol}(\Omega(a,b,t,u)[\delta]) \leqslant 4\,\mathrm{vol}(\Xi[\delta])$.

$$\mathrm{vol}(\Xi) = \int_{p=t}^{u}\int_{q=a}^{b} \frac{dq}{2q}\,dp = \log\left(\frac{b}{a}\right)\frac{u-t}{2}.$$

The two straight parts of the boundary $\partial\Xi$ have the lengths:

$$\begin{aligned}
&||(\sqrt{ua}, \sqrt{u/a}) - (\sqrt{ta}, \sqrt{t/a})|| + ||(\sqrt{ub}, \sqrt{u/b}) - (\sqrt{tb}, \sqrt{t/b})||\\
&\quad = ((a+a^{-1})^{1/2} + (b+b^{-1})^{1/2})(\sqrt{u}-\sqrt{t})\\
&\quad \ll \max(a^{-1/2}, b^{1/2})(\sqrt{u}-\sqrt{t}).
\end{aligned}$$

The piece of the hyperbola $x_1x_2 = p$ with $x_1, x_2 > 0$ and $\frac{x_1}{x_2} \in [a,b]$ has length

$$\begin{aligned}
\int_a^b \sqrt{\frac{1}{4}\frac{p}{q} + \frac{1}{4}\frac{p}{q^3}}\,dq = \frac{\sqrt{p}}{2}\int_a^b \sqrt{q+q^{-1}}\,\frac{dq}{q} \ll \sqrt{p}\int_{\log a}^{\log b} e^{|x|/2}\,dx\\
\ll \sqrt{p}\max(a^{-1/2}, b^{1/2}).
\end{aligned}$$

So $\mathrm{length}(\partial\Xi) \ll \sqrt{u}\max(a^{-1/2}, b^{1/2})$. We have obtained:

Lemma 10. *For* $0<a<b$, $0<t<u$:

$$M(\Omega(a,b,t,u)) \ll_F (u-t)\log\frac{b}{a} + u^{1/2}\max(a^{-1/2}, b^{1/2}) + 1.$$

Proof of Proposition 7. All terms in the sum $\Lambda_{r,r_1}(A,B;C)$ are

$$O_\varepsilon(\min(\sqrt{|N(r)|},\sqrt{|N(r_1)|})C^{-1/2+\varepsilon}),$$

see (44). According to Lemma 10, the number of terms is

$$M(\Omega(A,B,C,2C)) \ll C\log\frac{B}{A} + C^{1/2}\max(A^{-1/2},B^{1/2}) + 1. \qquad \square$$

3.7. Difference between sums with sharp and with smooth bounds

The difference $K_{r,r_1}(f) - K_{r,r_1}(f_{\mathrm{sh}})$ can be estimated by a sum in which all terms are bounded by $\min(\sqrt{|N(r)|},\sqrt{|N(r_1)|})C^{-1/2+\varepsilon}$, where we use the Weil–Salié bound (44). As $K_{r,r_1}(f)$ is estimated in terms of R, it seems harmless to replace $\min(\sqrt{|N(r)|},\sqrt{|N(r_1)|})$ by $\sqrt{\sqrt{|N(r)|}\sqrt{|N(r_1)|}} = R^{1/4}$.

The definitions of $\tau_{\mathrm{sh}},\ \sigma_{\mathrm{sh}},\ \tau$ and σ in Section 3.2 imply that the number of terms in the sum is equal to the number of $c \in \mathcal{O}$ for which

$$\begin{aligned}
(|cc'|,|c/c'|) \in &\left[\frac{C}{1+\frac{1}{2Y}},2C\right] \times [Ae^{-1/Y},B] \\
&\times \left(C,\frac{2C}{1+\frac{1}{2Y}}\right) \times \left(A,Be^{-1/Y}\right) \\
&= \Omega\left(Ae^{-1/Y},B,\frac{C}{1+\frac{1}{2Y}},C\right) \cup \Omega\left(Ae^{-1/Y},B,\frac{2C}{1+\frac{1}{2Y}},2C\right) \\
&\quad \cup \Omega\left(Ae^{-1/Y},A,C,\frac{2C}{1+\frac{1}{2Y}}\right) \cup \Omega\left(Be^{-1/Y},B,C,\frac{2C}{1+\frac{1}{2Y}}\right).
\end{aligned}$$

Lemma 10 gives an estimate for the number of terms:

$$\begin{aligned}
&\ll CY^{-1}\log\left(\frac{B}{A}e^{1/Y}\right) + C^{1/2}\max(A^{-1/2}e^{1/2Y},B^{1/2}) + 1 \\
&\quad + C\frac{1}{Y} + C^{1/2}\max(A^{-1/2}e^{1/2Y},A^{1/2}) \\
&\quad + C\frac{1}{Y} + C^{1/2}\max(B^{-1/2}e^{1/2Y},B^{1/2}) \\
&\ll (L+1)Y^{-1}C + \max(A^{-1/2},B^{1/2})C^{1/2}.
\end{aligned}$$

We have obtained

$$K_{r,r_1}(f) - K_{r,r_1}(f_{\rm sh}) \\ \ll R^{1/4}((L+1)Y^{-1}C^{1/2+\varepsilon} + \max(A^{-1/2}, B^{1/2})C^{\varepsilon}). \tag{82}$$

3.8. Final estimates

We obtain a bound for $\Lambda_{r,r_1}(A,B;C)$ by adding the bounds in (77) and (82). We still have to choose the parameter $Y \geqslant \max(2, L^{-1})$ governing the steepness of the test functions.

Ignoring the exponents ε, the logarithms, and the influence of L and V, we have four main terms:

$$\begin{array}{ll} R^{1/4}Y^2, & R^{1/4-\beta(1)/2}U^{2\beta(1)}Y^{3/2}C^{\beta(1)}, \\ R^{1/4-\beta(2)}U^{4\beta(2)}YC^{2\beta(2)}, & R^{1/4}Y^{-1}C^{1/2}. \end{array} \tag{83}$$

In the C-aspect, the optimal choice is $Y = Y_0 C^{\gamma}$, where Y_0 does not depend on C, and with

$$\gamma = \begin{cases} \frac{1}{6} & \text{if } \beta(1) \leqslant \frac{1}{12}, \beta(2) \leqslant \frac{1}{12}, & \text{(i)} \\ \frac{1}{5} - \frac{2}{5}\beta(1) & \text{if } \beta(1) \geqslant \frac{1}{12}, \beta(2) \leqslant \frac{1}{20} + \frac{2}{5}\beta(1), & \text{(ii)} \\ \frac{1}{4} - \beta(2) & \text{if } \beta(1) \leqslant -\frac{1}{8} + \frac{5}{2}\beta(2), \beta(2) \geqslant \frac{1}{12}. & \text{(iii)} \end{cases} \tag{84}$$

In case (i), the middle terms in (83) are smaller in the C-aspect than the largest term. But it seems not feasible to use this to get rid of the powers of U. We take $Y_0 = 1$, and obtain Theorem 5, if we convert the condition $Y \geqslant \max(2, L^{-1})$ into a condition on C. Indeed, noting that $V = L + Y^{-1} \ll L + 1$, we have

$$\begin{aligned} \Lambda_{r,r_1}(A,B;C) \ll\, & R^{1/4+\varepsilon}(L+1)(\log^2 C + C^{1/3} + C^{1/12}\log C) \\ & + R^{1/4-\beta(1)/2+\varepsilon}U^{2\beta(1)}C^{\beta(1)}(L+1) \\ & \times (\log^2 C + C^{1/4} + C^{1/12}\log C) \\ & + R^{1/4-\beta(2)+\varepsilon}U^{4\beta(2)}C^{2\beta(2)}(L+1)(\log^2 C + C^{1/6} + \log C) \\ & + R^{1/4}((L+1)C^{1/3+\varepsilon} + \max(A^{-1/2}, B^{1/2})C^{\varepsilon}). \end{aligned} \tag{85}$$

We do not completely work out the other cases, but consider the case that $\beta(1),\ \beta(2) \leqslant b \in (\frac{1}{12}, \frac{1}{4})$ as in Theorem 6. We note that the bound in (77) stays valid if we replace the $\beta(j)$ by b. So we are in case (iii) of (84). We choose $Y = R^{b/2}U^{-2b}C^{1/4-b}$. This has to be at least $\max(2, L^{-1})$. That is ensured by the condition in Theorem 6. We have also assumed in the theorem that $Y \geqslant \log^2 C$, which simplifies many terms in (77). This leads to the estimate in Theorem 6.

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