QUANTUM MAASS FORMS

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Abstract. We propose a definition of the concept of quantum Maass forms. On the basis of this definition, we discuss the relation with Maass forms and with cohomology groups.

In [9], D.Zagier defined the function $\varphi : \mathbb{Q} \to \mathbb{C}$ by

$$\varphi(\xi) \;=\; e^{\pi i \xi/12} \sum_{n=0}^{\infty} \prod_{j=1}^{n} \left(1 - e^{2\pi i j \xi}\right) . \tag{1}$$

For $\gamma = \pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$, $c > 0$, this function satisfies

$$\varphi(\xi) - v(\gamma)(c\xi + d)^{-3/2}\varphi(\xi) \;=\; g_\gamma(\xi) , \tag{2}$$

with a multiplier system v of weight $-\frac{3}{2}$. The function g_γ is the restriction to $\mathbb{Q}$ of an element of $C^\infty(\mathbb{R})$.

The name *quantum modular forms* is used for functions on $\mathbb{Q}$ that satisfy a modular transformation property modulo simpler functions on $\mathbb{R}$, in Zagier's example the smooth functions g_γ.

The word "quantum" has no physical meaning here. Quantum modular forms are not modular forms, but have some relation to holomorphic modular forms (to the Dedekind eta function in Zagier's example).

The subject of this paper is the definition of *quantum Maass forms*, related to $\mathrm{PSL}_2(\mathbb{Z})$-invariant real analytic eigenfunction of the Laplace operator on the upper half plane that are not holomorphic. A definition is proposed in §1.1.5, and extended in §1.2.2. Section 2 gives results that suggest that these definitions are sensible.

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1. Quantum Maass forms associated to Maass cusp forms and Eisenstein series

We work with the discrete subgroup $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ of $G = \mathrm{PSL}_2(\mathbb{R})$. By $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we denote $\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

The space $\mathscr{E}_s^\Gamma$ consists of the complex valued functions u on the upper half plane $\mathfrak{H}$ that satisfy

(i) Γ-*invariance:* $u(\gamma z) = u(z)$ for all $\gamma \in \Gamma$.

(ii) *Eigenfunction of Laplace operator:* $-y^2\left(\partial_x^2 + \partial_y^2\right) u = s(1-s)u$ for some $s \in \mathbb{C}$, the *spectral parameter*.

Our main interested is in the case $0 < \mathrm{Re}\, s < 1$. For convenience, we assume $s \neq \frac{1}{2}$.

The space $\mathscr{E}_s^\Gamma$ has infinite dimension. The finite dimensional subspace space Mf_s of *Maass forms* is defined by the condition

(iii) *Polynomial growth:* $u(z) = O\left(y^A\right)$ as $y \to \infty$ for some $A > 0$.

The space $\mathsf{Mf}_s^0 \subset \mathsf{Mf}_s$ of *Maass cusp forms* consists of u satisfying

(iii') *Quick decay:* $u(z) = O\left(y^{-A}\right)$ as $y \to \infty$ for all $A > 0$.

1.1. **Quantum Maass forms associated to Maass cusp forms.** The space Mf_s^0 is non-zero only for a discrete set of $s \in \frac{1}{2} + i\mathbb{R}$. Each $u \in \mathsf{Mf}_s^0$ has a Fourier expansion

$$u(z) \;=\; \sum_{\neq 0} a_n \sqrt{y} K_{s-1/2}(2\pi|n|y) e^{2\pi i n x}\,. \tag{3}$$

A Maass cusp form is *even* if $a_{-n} = a_n$ for all n, and *odd* if $a_{-n} = -a_n$. This gives a decomposition $\mathsf{Mf}_s^0 = \mathsf{Mf}_s^{0,+} \oplus \mathsf{Mf}_s^{0,-}$, where $\mathsf{Mf}_s^{0,+}$, respectively $\mathsf{Mf}_s^{0,-}$, is the subspace of even, respectively odd, Maass cusp forms.

The main theorem in [3] associates to each $u \in \mathsf{Mf}_s^{0,\pm}$ objects of three other types:

(a) (See Maass, Chap. IV and V of [5].) With a_n as in (3), the *L-function* is $L(\rho) = 2\sum_{n=1}^\infty a_n n^{-\rho}$. Let $\gamma_s(\rho) = \frac{1}{4}\pi^{-\rho}\Gamma\left(\frac{\rho-s+\frac{1}{2}}{2}\right)\Gamma\left(\frac{\rho+s-\frac{1}{2}}{2}\right)$. The completed L-function, $L^*(\rho) = \gamma_s\left(\rho + \frac{1\mp 1}{2}\right) L(\rho)$, has a holomorphic continuation to $\rho \in \mathbb{C}$, and satisfies $L^*(1-\rho) = \pm L^*(\rho)$.

(b) The *periodic function* on $\mathbb{C} \smallsetminus \mathbb{R}$

$$f(\tau) \;=\; \frac{1}{2}\pi^s \Gamma(1-s) \cdot \begin{cases} \sum_{n=1}^\infty n^{s-\frac{1}{2}} a_n e^{2\pi i n \tau} & \text{if } \tau \in \mathfrak{H}\,, \\ \mp \sum_{n=1}^\infty n^{s-\frac{1}{2}} a_n e^{-2\pi i n \tau} & \text{if } \tau \in \mathfrak{H}^-\,, \end{cases} \tag{4}$$

is holomorphic and 1-periodic. It is not Γ-invariant.

(c) The *period function*

$$\psi(\tau) = f(\tau) - \tau^{-2s} f(-1/\tau)\,,$$

which extends holomorphically to $\mathbb{C}' = \mathbb{C} \smallsetminus (-\infty, 0]$. It satisfies

$$\psi(\tau) \;=\; \psi(\tau+1) + (\tau+1)^{-2s}\psi\left(\frac{\tau}{\tau+1}\right), \quad \tau^{-2s}\psi\left(\frac{1}{\tau}\right) \;=\; \pm\psi(\tau)\,. \tag{5}$$

The following theorem is not explicitly stated in [3]. Don Zagier pointed it out to me.

Theorem 1. *The periodic function f associated to a Maass cusp form can be extended to $\mathbb{Q}$ such that for all $\xi \in \mathbb{Q}$:*

$$f(\tau) \;=\; f(\xi) + o(1) \qquad (\tau \xrightarrow{\text{ga}} \xi)\,. \tag{6}$$

It satisfies

$$\begin{aligned} f(\xi+1) \;&=\; f(\xi) && \text{for all } \xi \in \mathbb{Q}\,,\\ \xi^{-2s} f\left(\frac{-1}{\xi}\right) \;&=\; f(\xi) - \psi(\xi) && \text{for all } \xi \in \mathbb{Q} \cap (0,\infty)\,. \end{aligned}$$

Geodesic approach. Here $\tau \xrightarrow{\text{ga}} \xi$ means that τ approaches ξ along a geodesic in $\mathfrak{H}$ or $\mathfrak{H}^-$. So τ tends to ξ along a piece of a vertical line through ξ, or along a euclidean circle with its center on the real axis. The implicit constant in $o(1)$ may depend on the geodesic.

Proof. In §I.4 of [3], the integral representation

$$f(\tau) \;=\; \frac{1}{2\pi i} \int_{\operatorname{Re}\rho = C} \tilde{f}_\pm(\rho) e^{\pm i\pi\rho/2} \tau^{-\rho}\, d\rho$$

is given for $\pm \operatorname{Im} \tau > 0$, with $C > 0$. Moving the line of integration to $\operatorname{Re}\rho = -\varepsilon$ for some small $\varepsilon > 0$, we pick up a residue at $\rho = 0$. The residue gives a constant $f(0)$, which can be expressed in terms of the L-function; see (1.16b) in [3]. An estimate of the L-function shows that the integral over $\operatorname{Re}\rho = -\varepsilon$ contributes $o(1)$ as $\tau \xrightarrow{\text{ga}} 0$. This gives (6) for $\xi = 0$. Repeatedly applying the periodicity and the relation $\psi(\tau) = f(\tau) - \tau^{-2s} f(-1/\tau)$, gives (6) at all $\xi \in \mathbb{Q}$. We use that $\tau \mapsto \tau + 1$ and $\tau \mapsto \frac{-1}{\tau}$ map geodesics to geodesics. □

1.1.1. *Explicit formula.* A direct extension of the reasoning in [3] gives an expression in terms of the twisted L-series

$$L_o(\rho, \xi) \;=\; \sum_{n \neq 0} n a_n e^{2\pi i n \xi} |n|^{1-\rho}\,, \tag{7}$$

which has an analytic continuation and a functional equation. For $\xi = \frac{a}{c}$, $a, c \in \mathbb{Z}$, $c > 0$, $(a, c) = 1$, $a\tilde{a} \equiv 1 \bmod c$:

$$f\left(\frac{a}{c}\right) \;=\; -\frac{1}{4}\pi^{-s-\frac{1}{2}} \Gamma\left(s + \frac{1}{2}\right) c^{2s} L_0\left(s + \frac{1}{2}, -\frac{\tilde{a}}{c}\right), \tag{8}$$

One needs the meromorphic continuation of both L_o and $L_e(\rho, \xi) = \sum_{n \neq 0} a_n e^{2\pi i n \xi} |n|^{-\rho}$. This continuation is obtained from integrals of $u(\xi + iy)$ and $\partial_x u(\xi + iy)$ over $y \in (0, \infty)$. The functional equations relate these L-series for $\xi = \frac{a}{c}$ and $\xi = \frac{\tilde{a}}{c}$, and imply that $L_e\left(\pm(s - \frac{1}{2}), \xi\right) = 0$ for all $\xi \in \mathbb{Q}$.

1.1.2. *Automorphic distribution.* As we shall discuss in §2.1, the periodic function f represents a hyperfunction on $\mathbb{R}$, which for f coming from a Maass cusp form is actually a distribution. The limit behavior in Theorem 1 indicates that this distribution is rather regular. Schmid and Miller, [7], [6], have studied this distribution as the distribution derivative of a continuous function.

1.1.3. *Smooth extension of the period function.* The proposition in §III.3, [3] implies that there is $\tilde{\psi} \in C^\infty(\mathbb{R})$ such that $\psi(x) = \tilde{\psi}(x)$ for $x > 0$, and $|x|^{-2s}\tilde{\psi}(-1/x) = \tilde{\psi}(x)$ for $x \in \mathbb{R} \smallsetminus \{0\}$. So $\tilde{\psi}$ is real analytic on $\mathbb{R} \smallsetminus \{0\}$. At $x < 0$, the value of $\tilde{\psi}(x)$ has no direct relation to the values of $\psi(\tau)$ with $\tau \in \mathbb{C}'$ near x.

We have $|\xi|^{-2s} f(-1/\xi) = f(\xi) - \tilde{\psi}(\xi)$ for almost all $\xi \in \mathbb{Q}$.

1.1.4. *Principal series.* This leads to a reformulation in terms of the smooth vectors in the principal series representation with spectral parameter s.

Let $\mathcal{V}_s^\infty$ be the space of $f \in C^\infty(\mathbb{R})$ for which $x \mapsto |x|^{-2s} f(-1/x)$ extends as an element of $C^\infty(\mathbb{R})$. This space is invariant under the following right action of G:

$$(f|_{2s}g)(x) \;=\; |cx+d|^{-2s} f(gx) \qquad \text{for } g = \begin{bmatrix} a\, b \\ c\, d \end{bmatrix} \in G\,. \tag{9}$$

The extended period function $\tilde{\psi}$ is an element of $\mathcal{V}_s^\infty$. There is a 1-cocycle $\gamma \mapsto c_\gamma$ on Γ with values in $\mathcal{V}_s^\infty$ such that for all $\gamma \in \Gamma$

$$f(\xi) - (f|_{2s}\gamma)(\xi) \;=\; c_\gamma(\xi) \qquad \text{for almost all } \xi \in \mathbb{Q}\,. \tag{10}$$

So indeed, the function on $\mathbb{Q}$ induced by the periodic function satisfies a modular transformation property, modulo an element of $\mathcal{V}_s^\infty$.

Actually, $\tilde{\psi}$ is real analytic outside 0 and ∞. So the cocycle c takes values in the G-invariant subspace $\mathcal{V}_s^{\omega,\infty}$ of semi-analytic vectors in $\mathcal{V}_s^\infty$: vectors that are real analytic except at finitely many points. A vector f is real analytic at ∞ if $x \mapsto |x|^{-2s} f(-1/x)$ is given by a power series in $\frac{-1}{x}$ that converges on a neighborhood of $\frac{-1}{x} = 0$.

1.1.5. *Definition of quantum Maass forms.* Let $\mathcal{R}_s$ be the space of (equivalence classes of) functions $\mathbb{Q} \smallsetminus E \to \mathbb{C}$ where $E \subset \mathbb{Q}$ is a finite set. We consider two such functions to be equal in $\mathcal{R}_s$ if their difference is zero outside a finite subset of $\mathbb{Q}$. The formula (9) defines a right action of Γ in R_s. By restriction, $\mathcal{V}_s^{\omega,\infty}$ determines a Γ-submodule of $\mathcal{R}_s$. Define the Γ-module $\mathcal{Q}_s$ as the quotient:

$$0 \longrightarrow \mathcal{V}_s^{\omega,\infty} \longrightarrow \mathcal{R}_s \longrightarrow \mathcal{Q}_s \longrightarrow 0\,. \tag{11}$$

The periodic function f coming from the Maass cusp form u determines an element of $\mathcal{R}_s$. Its class in $\mathcal{Q}_s$ is Γ-invariant. The function f is not in $\mathcal{R}_s^\Gamma$. Actually, $\mathcal{R}_s^\Gamma \cong \mathbb{C}$ is not very interesting. Let us define the space of quantum Maass forms with spectral parameter s as

$$\mathsf{qMf} := \mathcal{Q}_s^\Gamma / \mathcal{R}_s^\Gamma\,. \tag{12}$$

Theorem 1 shows that there is a map $\mathbf{q}_s : \mathsf{Mf}_s^0 \to \mathsf{qMf}_s$.

1.1.6. *Map to the semi-analytic cohomology.* To (11) is associated a long exact sequence, of which we use the following part:

$$0 \longrightarrow 0 \longrightarrow \mathcal{R}_s^\Gamma \longrightarrow \mathcal{Q}_s^\Gamma \longrightarrow H^1\left(\Gamma, \mathcal{V}_s^\infty\right)\,. \tag{13}$$

(The space $\left(\mathcal{V}_s^\infty\right)^\Gamma$ is zero.) This implies that there is an injective map $\mathbf{c}_s : \mathsf{qMf}_s \to H^1\left(\Gamma, \mathcal{V}_s^{\omega,\infty}\right)$.

In [4] it is shown that there is an injective map $\mathbf{r}_s : \mathsf{Mf}_s^0 \to H^1(\Gamma, \mathcal{V}_s^{\omega,\infty})$. The image is the parabolic cohomology subgroup $H^1_{\mathrm{par}}(\Gamma, \mathcal{V}_s^{\omega,\infty})$. (For the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, the parabolic cohomology group $H^1_{\mathrm{par}}(\Gamma, A)$ can be defined as the subgroup of classes in $H^1(\Gamma, A)$ that can be represented by a cocycle c that satisfies $c_T = 0$ for $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \in \Gamma$.)

It turns out that $\mathbf{r}_s$ is equal to the composition $\mathbf{c}_s \circ \mathbf{q}_s$. So the quantum Maass forms constitute a space through which we can factorize the map from Maass cusp forms to cohomology. The injectivity of $\mathbf{r}_s$ implies that $\mathbf{q}_s$ is injective.

1.2. **Eisenstein series, extended quantum Maass forms.** The Eisenstein series

$$\begin{aligned} E_s(z) &= \Lambda(2s)y^s + \Lambda(2s-1)y^{1-s} \\ &\quad + 2\sum_{n\neq 0} |n|^{\frac{1}{2}-s}\sigma_{2s-1}(|n|)\,\sqrt{y}K_{s-1/2}(2\pi|n|y)e^{2\pi i n x}\,, \\ \Lambda(u) &= \pi^{-\frac{u}{2}}\Gamma\left(\frac{u}{2}\right)\zeta(u)\,, \qquad \sigma_u(n) = \sum_{d|n} d^u\,. \end{aligned} \tag{14}$$

gives a meromorphic family of elements of Mf_s, complementary to the cusp forms. It is holomorphic in the region $0 < \mathrm{Re}\, s < 1$, $s \neq \frac{1}{2}$.

A modification of the theory of Lewis and Zagier works for the Eisenstein series; see §IV.1, [3]. The holomorphic periodic function associated to E_s can be chosen as

$$f_s(\tau) = \pm\frac{\sqrt{\pi}\Gamma(1-s)\Lambda(2s)}{2\Gamma\left(\frac{1}{2}-s\right)} \pm \pi^s\Gamma(1-s)\sum_{n=1}^{\infty}\sigma_{2s-1}(n)e^{\pm 2\pi i n\tau}\,, \tag{15}$$

with the convention $\pm 1 = \mathrm{sign}\,\mathrm{Im}\,\tau$. The period function $\psi_s(\tau) = f_s(\tau) - \tau^{-2s} f_s(-1/\tau)$ is holomorphic on $\mathbb{C}'$. However, it has no smooth extension through 0. The function $\tilde{\psi}_s$ defined by $\tilde{\psi}_s(x) = \psi_s(x)$ for $x > 0$, and $\tilde{\psi}_s(x) = |x|^{-2s}\psi_s(-1/x)$ is real analytic on $\mathbb{R} \smallsetminus \{0\}$, but it is not an element of $\mathcal{V}_s^{\omega,\infty}$.

The expression (15) is explicit, and implies that at the rational point $\frac{a}{c}$, $a, c \in \mathbb{Z}$, $c > 0$, $(a, c) = 1$, we do not have a limit value, but an expansion:

$$\begin{aligned} f_s(\tau) &= \frac{i}{2}c^{2s-2}\Lambda(2s-1)\,\frac{1}{\tau - \frac{a}{c}} \\ &\quad \pm \frac{1}{2}\pi^{-1/2}\Gamma(1-s)\Gamma\left(s+\frac{1}{2}\right)c^{-2s}\Lambda(2s)\left(\mp i\left(\tau - \frac{a}{c}\right)\right)^{-2s} \\ &\quad - i\pi^{\frac{1}{2}-s}\Gamma\left(s+\frac{1}{2}\right)\sum_{x=1}^{c-1}\left(\!\!\left(\frac{dx}{c}\right)\!\!\right)\zeta(2s, x/c) + o(1) \qquad \tau \xrightarrow{\text{ga}} \frac{a}{c}\,, \end{aligned} \tag{16}$$

where $\pm 1 = \sin\mathrm{Im}\,\tau$, and where $\zeta(\cdot,\cdot)$ denotes the Hurwitz zeta function. The computations are similar to those for cusp forms. We use the fact that the twisted L-series associated to (15) have expressions in terms of the Hurwitz zeta function.

The example of the Eisenstein series shows that there are two difficulties: (1) There are no limit values, but expansions. (2) The cocycle has values in a larger space than $\mathcal{V}_s^{\omega,\infty}$.

If we do not want to end the study of quantum Maass forms here, generalizations are needed.

1.2.1. *Type of quantum Maass forms.* $\mathscr{V}_s^{\omega,\infty}$ consists of the semi-analytic vectors in $\mathscr{V}^\infty$. It contains the G-invariant subspace $\mathscr{V}_s^\omega$ of vectors that are analytic everywhere. Analytic means analytic on $\mathbb{R}$, and $x \mapsto |x|^{-2s} f(-1/x)$ given by a power series in $-1/x$ converging if $|x|$ is sufficiently large. Another space, containing $\mathscr{V}_s^{\omega,\infty}$, is $\mathscr{V}_s^{\omega,\mathrm{fs}}$ consisting of vectors that are analytic except at finitely many points. The function $\tilde\psi_s$ for Eisenstein series is in $\mathscr{V}_s^{\omega,\mathrm{fs}}$, and $\tilde\psi$ for cusp forms is in $\mathscr{V}_s^{\omega,\infty} = \mathscr{V}_s^\infty \cap \mathscr{V}_s^{\omega,\mathrm{fs}}$.

Let t be one of ω, (ω,∞), (ω,fs), or let t denote another Γ-invariant space of functions inside the principal series with spectral parameter s. Then we define $\mathscr{Q}_s^t$ as the quotient in

$$0 \longrightarrow \mathscr{V}_s^t \longrightarrow \mathscr{R}_s \longrightarrow \mathscr{Q}_s^t \longrightarrow 0\,. \tag{17}$$

We define the space of quantum Maass forms of type t as $\mathsf{qMf}_s^t = \left(\mathscr{Q}_s^t\right)^\Gamma / \mathscr{R}_s^\Gamma$. In §1.1, we have omitted $t = (\omega,\infty)$ from the notations.

The inclusions between space of vectors in the principal series induces the following natural maps

$$\begin{array}{ccccc} & & & \nearrow & \mathsf{qMf}_s^{\omega,\mathrm{fs}} \\ \mathsf{qMf}_s^\omega & \longrightarrow & \mathsf{qMf}_s^{\omega,\infty} & & \\ & & & \searrow & \mathsf{qMf}_s^\infty \end{array}$$

1.2.2. *Extended quantum Maass forms.* Let $\tilde{\mathscr{R}}_s$ be the space of systems p that associate to almost all $\xi \in \mathbb{Q}$ an expansion $p(\xi,\tau)$ holomorphic in τ near ξ. Two such systems of expansions are equal if $p_1(\xi,\tau) = p_2(\xi,\tau) + o(1)$ ($\tau \xrightarrow{\mathrm{ga}} \xi$) for almost all ξ. A constant is a holomorphic function, so $\mathscr{R}_s \subset \tilde{\mathscr{R}}_s$. The group Γ acts in $\tilde{\mathscr{R}}_s$ by $(p|_{2s}\gamma)(\xi,\tau) = j_\gamma(\tau)^{-s} p(\gamma\xi,\gamma\tau)$, where

$$j_\gamma(\tau)^{-s} = \left((c\tau+d)^2\right)^{-s} \qquad \text{for } \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{18}$$

is a holomorphic extension of $x \mapsto |cx+d|^{-2s}$. For any type t we define $\tilde{\mathscr{Q}}_s^t = \tilde{\mathscr{R}}_s / \mathscr{V}_s^t$. This is a huge space. We refrain from giving a name to the elements of $\left(\tilde{\mathscr{Q}}_s^t\right)^\Gamma / \tilde{\mathscr{R}}_s^\Gamma$.

To the Eisenstein series E_s, we can associate an element of $\left(\tilde{\mathscr{Q}}_s^{\omega,\mathrm{fs}}\right)^\Gamma / \tilde{\mathscr{R}}_s$, represented by the system of expansions in (16). It turns out that the middle term in (16), with $\pm\left((\mp i(\tau - a/c)\right)^{-2s}$, determines a system of expansions in $\tilde{\mathscr{R}}_s^\Gamma$. So this term may be considered as trivial and should be divided out.

It seems sensible to define $\dot{\mathscr{R}}_s \subset \tilde{\mathscr{R}}_s$ as the systems of expansions of the form

$$p(\xi,\tau) \;=\; \frac{d_\xi}{\tau - \xi} + c_\xi + o(1) \quad (\tau \xrightarrow{\mathrm{ga}} \xi)\,. \tag{19}$$

It is a Γ-invariant subspace. We define $\dot{\mathscr{Q}}_s^t = \dot{\mathscr{R}}_s / \mathscr{V}_s^t$, and call elements of $\mathsf{eqMf}_s^t = \left(\dot{\mathscr{Q}}_s^t\right)^\Gamma / \dot{\mathscr{R}}_s^\Gamma$ *extended quantum Maass forms* of *type* t. If necessary, we shall call elements of qMf^t_s *genuine* quantum Maass forms. By definition, there is an injective map $\mathbf{c}^t_s : \mathsf{eqMf}^t_s \to H^1(\Gamma, \mathcal{V}^t_s)$.

We have seen that there is a map $\mathbf{q}^{\omega,\mathrm{fs}}_s : \mathsf{Mf}_s \to \mathsf{eqMf}^{\omega,\mathrm{fs}}_s$.

2. Quantum Maass forms associated to invariant eigenfunctions

The aim is to attach extended quantum Maass forms to all invariant eigenfunctions in $\mathcal{E}^\Gamma_s$.

2.1. Invariant hyperfunctions. Up till now, we have used the line model of the principal series. When dealing with hyperfunctions it is more comfortable to use the projective model. In that model, $\mathcal{V}^\omega_s$ and $\mathcal{V}^\infty_s$ correspond to analytic, respectively smooth, functions on $\mathbb{P}^1_\mathbb{R}$. The transformation behavior is

$$f(x)|^{\mathrm{pr}}_{2s} g(x) \;=\; j^{\mathrm{pr}}_g(x)^{-s} f(gx)\,, \qquad j^{\mathrm{pr}}_{\left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right]}(x) \;=\; \frac{(ax+b)^2+(cx+d)^2}{1+x^2}\,. \tag{20}$$

If f is in the line model, then $f^{\mathrm{pr}}(x) = \left(x^2+1\right)^s f(x)$ is in the projective model.

The results of §1 can be formulated in the projective model: Principal series functions have to be multiplied by $\left(1+x^2\right)^s$. To an (extended) quantum Maass form represented by p we associate the corresponding object in the projective model, given by $p^{\mathrm{pr}}(\xi,\tau) = \left(1+\tau^2\right)^s p(\xi,\tau)$. When dealing with p^{pr}, there is no problem in also considering $p(\infty,\tau)$. We assume that from now on, this reformulation in the projective model has been done. A drawback of the projective model is that the simple relation $f(\xi+1) = f(\xi)$ in Theorem 1 becomes $\left(\frac{1+\xi^2}{1+(\xi+1)^2}\right)^s f^{\mathrm{pr}}(\xi+1) = f^{\mathrm{pr}}(\xi)$.

Let $\mathcal{E}_s$ be the space of all solutions of $-y^2\left(\partial_x^2+\partial_y^2\right)u = s(1-s)u$. Helgason, [2], has shown that there is an isomorphism of G-spaces $\Pi_s : \mathcal{V}^{-\omega}_s \to \mathcal{E}_s$, the Poisson integral. For further explanations, see [8], [1], or [4]. Under this isomorphism, $(\mathcal{V}^{-\omega}_s)^\Gamma$ corresponds to $\mathcal{E}^\Gamma_s$.

Each $u \in \mathcal{E}^\Gamma_s$ has a Fourier expansion

$$\begin{aligned} u(z) \;=\; & b_0 y^{1-s} + a_0 y^s \\ & + \sum_{n\in\mathbb{Z}\smallsetminus\{0\}} \sqrt{y}\left(a_n K_{s-1/2}(2\pi|n|y) + b_n I_{1/2-s}(2\pi|n|y)\right) e^{2\pi i n x}\,. \end{aligned} \tag{21}$$

The convergence of this series implies that $b_n = O\left(e^{-A|n|}\right)$ for each $A > 0$, and $a_n = O\left(e^{\varepsilon|n|}\right)$ for each $\varepsilon > 0$.

The hyperfunction α_u such that $\Pi_s\alpha_u = u$ is represented on $\mathbb{R}$ by the function $f^{\mathrm{pr}}_u(\tau) = \left(1+\tau^2\right)^s f_u(\tau)$, where f_u is given by

$$f_u(\tau) \;=\; \pm\frac{\sqrt{\pi}\,\Gamma(1-s)}{2\Gamma\left(\frac12 - s\right)} a_0 \pm \frac{\pi^s}{2}\Gamma(1-s)\sum_{n=1}^\infty n^{s-\frac12} a_{\pm n} e^{\pm 2\pi i n\tau}\,, \tag{22}$$

with $\pm\mathrm{Im}\,\tau > 0$. See Proposition 2.1 in [1].

A representative g of α_u is a holomorphic function on $U \smallsetminus \mathbb{P}^1_{\mathbb{R}}$ for some neighborhood U of $\mathbb{P}^1_{\mathbb{R}}$ in $\mathbb{P}^1_{\mathbb{C}}$. Two representatives differ by a holomorphic function on a full neighborhood of $\mathbb{P}^1_{\mathbb{R}}$. That difference corresponds to an element of $\mathcal{V}_s^\omega$.

If $\gamma \in \Gamma$, then $f|_{2s}^{\mathrm{pr}}\gamma$ is an other representative of α_u. So $\gamma \mapsto r_\gamma = g - g|_{2s}^{\mathrm{pr}}\gamma$ is a cocycle with values in $\mathcal{V}_s^\omega$. This induces a map $\mathbf{r}_s^\omega : \mathcal{E}_s^\Gamma \to H^1(\Gamma, \mathcal{V}_s^\omega)$, which is shown in [4] to be injective.

2.2. **Quantum Maass forms of analytic type.**

Theorem 2. *There is an injective linear map* $\mathbf{q}_s^\omega : \mathcal{E}_s^\Gamma \to \mathsf{eqMf}_s^\omega$ *such that* $\mathbf{c}_s^\omega \circ \mathbf{q}_s^\omega = \mathbf{r}_s^\omega$.

Proof. Let $u \in \mathcal{E}_s^\Gamma$, $\alpha \in (\mathcal{V}_s^{-\omega})^\Gamma$, $u = \Pi_s\alpha$. A representative g of α determines $p \in \tilde{\mathcal{R}}_s$ by $p(\xi,\tau) = g(\tau) + o(1)$ $(\tau \overset{\mathrm{ga}}{\to} \xi)$. The Γ-invariance of α implies that $p|_{2s}^{\mathrm{pr}}\gamma = p + r_\gamma$, with $r_\gamma \in \mathcal{V}_s^\omega$. We have obtained an element of $\left(\tilde{\mathcal{Q}}_s^\omega\right)^\Gamma / \tilde{\mathcal{R}}_s^\Gamma$, which does not depend on the choice of the representative g. We want to subtract an element of $\tilde{\mathcal{R}}_s^\Gamma$ from p such that we are left with an element of $\dot{\mathcal{R}}_s$. Then we shall have an element of eqMf_s^ω that is mapped under $\mathbf{c}_s^\omega$ to $\mathbf{r}_s^\omega u$.

To do this, we investigate the behavior of the representative g near ∞. We use §4.3 in [1]. We can decompose α as the sum of three Γ_∞-invariant hyperfunctions, corresponding to various parts of the Fourier expansion (21). Γ_∞ denotes the subgroup of Γ generated by $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$.

Lemma 4.2 gives a hyperfunction α^c corresponding to the terms with $K_{s-1/2}$ in (21). A representative of α^c is given by an integral involving the terms with $n \neq 0$ in f_u. Its behavior is $c_\infty + o(1)$ as $\tau \overset{\mathrm{ga}}{\to} \infty$.

To $a_0 y^s$ corresponds a hyperfunction α^0 that is near ∞ represented by a multiple of $m_0(\infty,\tau) = \pm\frac{e^{\mp\pi i s}}{2\cos\pi s}\tau^{2s}\left(1+\tau^{-2}\right)^s$, with $\pm\mathrm{Im}\,\tau > 0$. It turns out that this representative is invariant under T. So we can define an element of $\tilde{\mathcal{R}}_s^\Gamma$ by defining $m_0(\xi,\gamma\tau) = j_\gamma^{\mathrm{pr}}(\tau)^s m_0(\infty,\tau)$, independent of the choice of $\gamma \in \Gamma$ such that $\xi = \gamma\infty$.

The remaining part α^∞ is a hyperfunction supported in $\{\infty\}$. The term $b_0 y^{1-s}$ corresponds to a hyperfunction that is near ∞ represented by a multiple of $\tau \mapsto \tau$. This has the behavior of the expansion at ∞ of an element of $\dot{\mathcal{R}}_s$.

The terms with $I_{1/2-s}$ correspond to a hyperfunction that is near ∞ represented by

$$B(\tau) = -\frac{i\pi^{\frac12 - s}}{2\Gamma\left(\frac32 - s\right)} \sum_{n\neq 0} |n|^{\frac12 - s} b_n \tau \left(1+\tau^{-2}\right)^s {}_1F_1(1; 2-2s; 2\pi i n\tau)\,. \tag{23}$$

This last contribution is not explicitly given in §4.3 in [1]; so we give more details.

First we have to check that the hyperfunction β induced by B is sent to the $I_{1/2-s}$-part of (21). The estimate $b_n = O\left(e^{-A|n|}\right)$ for each $A > 0$ implies the absolute convergence in (23). The support of β is contained in $\{\infty\}$. The Poisson map is given by an integral around ∞:

$$\Pi_s\beta(z) = \frac{1}{\pi}\int_{|\tau|=R} B(\tau)\left(\frac{y(1+\tau^2)}{(\tau-z)(\tau-\bar{z})}\right)^{1-s} \frac{d\tau}{1+\tau^2}\,,$$

with $R > 1$, $R > |z|$. After a change of variables:

$$\Pi_s\beta(z) = \frac{-i\pi^{-1/2-s}}{2\Gamma\left(\frac{3}{2}-s\right)} \sum_{n\neq 0} |n|^{s-1/2} b_n y^{1-s} \int_{|\tau|=R} \left(1+\frac{y^2}{\tau^2}\right)^{s-1} \left(1+\frac{x}{\tau}\right)^{1-2s} \cdot {}_1F_1(1; 2-2s; 2\pi i n(\tau+x)) \frac{d\tau}{\tau}$$

for sufficiently large R. Insert the power series expansion for ${}_1F_1$ and the binomial series for $\left(1+\frac{y^2}{\tau^2}\right)^{s-1}$ and $\left(1+\frac{x}{\tau}\right)^{1-2s}$. Some computations lead to the power series expansion of $I_{1/2-s}$, and give the desired terms in (21).

It turns out that $B|^{\mathrm{pr}}_{2s}T(\tau) = B(\tau) + o(1)$ as $\tau \stackrel{\mathrm{ga}}{\to} \infty$. To check this, we use again the power series expansion of ${}_1F_1$ and the binomial formula for powers of $1+\frac{1}{\tau}$. So the contribution of $B(\tau)$ to $p(\infty,\tau)$ can be transported to all $\xi \in \mathbb{Q}$ to give an element of $\tilde{\mathscr{R}}^\Gamma_s$.

Now we have a system of expansions $p_1 \in \tilde{\mathscr{R}}_s$ given by $p_1(\xi,\tau) = g(\tau) - m(\xi,\tau)$ with $m \in \tilde{\mathscr{R}}^\Gamma_s$, and $p(\infty,\tau) = d_\infty\tau + c_\infty + o(1)$ ($\tau \stackrel{\mathrm{ga}}{\to} \infty$). Let $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$ such that $\xi = \gamma\infty \in \mathbb{Q}$. As $\tau \stackrel{\mathrm{ga}}{\to} \xi$:

$$\begin{aligned} p_1(\xi,\tau) &= j_\gamma(\gamma^{-1}\tau)^s \left(p_1|^{\mathrm{pr}}_{2s}\right)(\infty,\gamma^{-1}\tau) \\ &= j_\gamma(\gamma^{-1}\tau)^s \left(g|^{\mathrm{pr}}_{2s}\gamma(\gamma^{-1}\tau) - \left(m|^{\mathrm{pr}}_{2s}\gamma\right)(\infty,\gamma^{-1}\tau)\right) \\ &= j_\gamma(\gamma^{-1}\tau)\left(g(\gamma^{-1}\tau) + r_\gamma(\gamma^{-1}\tau) - m(\infty,\gamma^{-1}\tau)\right) \\ &= j_\gamma(\gamma^{-1}\tau)^s \left(r_\gamma(\gamma^{-1}\tau) + p_1(\infty,\gamma^{-1}\tau)\right) \\ &= (\mathrm{Anal}) + (\mathrm{Anal})\left(d_\infty\left(-\frac{d}{c} - \frac{1}{c^2(\tau-\xi)}\right) + c_\infty + o(1)\right), \end{aligned}$$

where (Anal) means a function that is holomorphic in τ on a neighborhood of $\mathbb{P}^1_{\mathbb{R}}$. This shows that $p_1 \in \dot{\mathscr{R}}_s$. □

The construction also gives a criterion for obtaining a genuine quantum Maass form:

Proposition 3. *Let $u \in \mathscr{E}^\Gamma_s$. Then $\mathbf{q}^\omega_s u$ is in qMf^ω_s if and only if $b_0 = 0$ in* (21).

Parabolic representative. Let us consider the image $\mathbf{q}^{\omega,\mathrm{fs}}_s u$ of $\mathbf{q}^\omega_s u$ under the natural map $\mathsf{eqMf}^\omega_s \to \mathsf{eqMf}^{\omega,\mathrm{fs}}_s$. This image can be represented by $p + \eta$ where η is any element of $\mathscr{V}^{\omega,\mathrm{fs}}_s$.

If the function g represents α_u on $\mathbb{P}^1_{\mathbb{R}}$, it differs by the representative $f^{\mathrm{pr}}_u : \tau \mapsto \left(1+\tau^2\right)^s f_u(\tau)$ by an element $\eta \in \mathscr{V}^{\omega,\mathrm{fs}}_s$. Indeed, $\eta = g - f^{\mathrm{pr}}_u$ is holomorphic on a neighborhood of $\mathbb{R}$. So $p_1 = p + \eta$ represents $\mathbf{q}^{\omega,\mathrm{fs}}_s u$. It has the special property that $p_1|^{\mathrm{pr}}_{2s}T = p_1$; this we call a *parabolic representative*.

2.3. **Quantum Maass forms and analytic cohomology.** In [4] it is shown that the injective map $\mathbf{r}_s : \mathscr{E}^\Gamma_s \to H^1(\Gamma, \mathscr{V}^\omega_s)$ is not surjective. So the question arises whether $\mathbf{q}_s^\omega \mathscr{E}_s^\Gamma$ is equal to the whole of eqMf_s^ω. The answer is that eqMf_s^ω is as large as possible:

Theorem 4. *Let* $0 < \operatorname{Re} s < 1$, $s \neq \frac{1}{2}$. *The map* $\mathbf{c}_s^\omega : \mathsf{eqMf}_s^\omega \to H^1(\Gamma, \mathscr{V}_s^\omega)$ *is bijective.*

Up till now, the assumption $s \neq \frac{1}{2}$ was convenient. Here it is essential.

Proof. Let f be holomorphic on a neighborhood of ∞. For $|\operatorname{Im} \tau|$ sufficiently large, we consider the one-sided average

$$f|_{2s}^{\mathrm{pr}} \mathsf{Av}_T^+ \;=\; \sum_{n=0}^{\infty} f|_{2s}^{\mathrm{pr}} T^n : \tau \mapsto \sum_{n=0}^{\infty} \left(\frac{1+\tau^2}{1+(\tau+n)^2} \right)^s f(\tau+n)\,. \tag{24}$$

The sum converges absolutely for $\operatorname{Re} s > \frac{1}{2}$, and defines a holomorphic function that satisfies

$$f|_{2s}^{\mathrm{pr}} \mathsf{Av}_T^+ |_{2s}^{\mathrm{pr}} (1-T) \;=\; f|_{2s}^{\mathrm{pr}} (1-T) |_{2s}^{\mathrm{pr}} \mathsf{Av}_T^+ \;=\; f\,. \tag{25}$$

If $f(\infty) = 0$, the convergence and relation (25) hold for $\operatorname{Re} s > 0$. For $f_0(\tau) = \left(1+\tau^{-2}\right)^s$ we find

$$f_0|_{2s}^{\mathrm{pr}} \mathsf{Av}_T^+(\tau) \;=\; \tau^{2s} \left(1+\tau^{-2}\right)^s \zeta(2s, \tau)\,, \tag{26}$$

where $\zeta(\cdot,\cdot)$ is the Hurwitz zeta function. This gives the meromorphic continuation of $f_0|_{2s}^{\mathrm{pr}} \mathsf{Av}_T^+$ to $s \in \mathbb{C}$, with a first order pole at $s = \frac{1}{2}$ as the sole singularity in $0 < \operatorname{Re} s$. Writing a general f as $f = f(\infty) f_0 + f_1$, we have $f_1(\infty) = 0$, and find $f|_{2s}^{\mathrm{pr}} \mathsf{Av}_T^+$ for all $s \neq \frac{1}{2}$, $0 < \operatorname{Re} s < 1$. The relation (25) extends as well. The asymptotic behavior of the Hurwitz zeta function implies that there are A_0 and A_1 such that

$$\left(f|_{2s}^{\mathrm{pr}} \mathsf{Av}_T^+ \right)(\tau) \;=\; A_1 \tau + A_0 + o(1) \qquad (\tau \xrightarrow{\mathrm{ga}} \infty)\,. \tag{27}$$

For $\xi = \gamma\infty$, the subgroup $\Gamma_\xi \subset \Gamma$ fixing ξ is generated by $\pi_\xi = \gamma T \gamma^{-1}$. The one-sided average $\mathsf{Av}_{\pi_\xi}^+$ is obtained from Av_T^+ by conjugation.

Let $c \in Z^1(\Gamma, \mathscr{V}_s^\infty)$. We define $p \in \dot{\mathscr{R}}_s$ by

$$p(\xi, \tau) \;=\; -\left(c_\pi |_{2s}^{\mathrm{pr}} \mathsf{Av}_{\pi_\xi}^+ \right)(\tau) + o(1) \qquad (\tau \xrightarrow{\mathrm{ga}} \xi)\,. \tag{28}$$

For $\delta \in \Gamma$, we find

$$\begin{aligned}
(p|_{2s}^{\mathrm{pr}} \delta)(\xi, \tau) \;&=\; j_\delta(\tau)^{-s} p(\delta\xi, \delta\tau) \\
&=\; -j_\delta(\tau)^{-s} c_{\delta\pi_\xi\delta^{-1}} |_{2s}^{\mathrm{pr}} \mathsf{Av}_{\delta\pi_\xi\delta^{-1}}^+ (\delta\tau) + o(1) \\
&=\; -c_{\delta\pi_\xi\delta^{-1}} |_{2s}^{\mathrm{pr}} \delta | \mathsf{Av}_{\pi_\xi}^+ |_{2s}^{\mathrm{pr}} \delta^{-1} |_{2s}^{\mathrm{pr}} \delta(\tau) + o(1) \\
&=\; -c_\delta |_{2s}^{\mathrm{pr}} (\pi_\xi - 1) |_{2s}^{\mathrm{pr}} \mathsf{Av}_{\pi_\xi}^+(\tau) - c_{\pi_\xi} |_{2s}^{\mathrm{pr}} \mathsf{Av}_{\pi_\xi}^+(\tau) + o(1) \\
&=\; c_\delta(\tau) + p(\xi, \tau)
\end{aligned}$$

So p represents $[p] \in \mathsf{eqMf}_s^\omega$ and $\mathbf{c}_s^\omega [p] = [c]$. □

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