Math. Z. 192, 297-328 (1986)

Mathematische Zeitschrift

Modular Forms of Varying Weight. II

Roelof W. Bruggeman

Mathematisch Instituut, Rijksuniversiteit Utrecht, Postbus 80.010, 3508 TA Utrecht, the Netherlands

1. Introduction

In [2] we started, for the full modular group, a study of families of automorphic forms parametrized by the weight. The following aims were given:

i) Give analytic continuation and functional equations for Eisenstein and Poincaré series in two variables parametrizing eigenvalue and multiplier system.

ii) Give, as far as possible, all square integrable modular forms as values of combinations of Eisenstein and Poincaré series in two variables.

iii) Get as much information as possible on the dependence of the modular spectrum on the multiplier system.

In this note i) is carried out. The main tools are results from [2]; the idea comes from [3].

A result concerning ii) is given. Not only square integrable modular forms are considered, but forms with growth bounded by some exponential function. It is not clear whether it is the best possible result.

To obtain these results we consider in §4, in a more general context, families of subspaces of a fixed linear space which give rise to modules over sheaves of holomorphic functions.

2. Definitions and Results

(2.1) *Notations.* We denote by $\mathfrak{h}$ the upper half plane, $G=\mathrm{SL}_2(\mathbb{R})$ acts on $\mathfrak{h}$ and $\Gamma\subset G$ is the full modular group.

F is the standard fundamental domain of Γ. For $r\in\mathbb{C} \bmod 12\,\mathbb{Z}$ we denote by v_r the $(2r)$-th power of the multiplier system of the Dedekind eta-function.

(2.2) *Notation.* For $r\in\mathbb{C}$

is an elliptic differential operator on $\mathfrak{h}$ coming from the Casimir operator of G acting in weight r. Any eigendistribution of L_r is given by a real analytic function.

(2.3) *Notation.* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\iota} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ is an automorphism of G; it leaves Γ invariant and $v_r(\gamma^{\iota}) = v_{-r}(\gamma)$, unless $\gamma = \begin{pmatrix} -1 & * \\ 0 & -1 \end{pmatrix}$. Put $Jf(z) = f(-\bar{z})$ for functions f on $\mathfrak{h}$. Then

$$L_r J = J L_{-r}.$$

(2.4) **Definition.** A *modular form* of *weight* r and *eigenvalue* $\frac{1}{4} - s^2$ is a function $f: \mathfrak{h} \to \mathbb{C}$ satisfying

$(a)_r$ $\quad f(\gamma \cdot z) = v_r(\gamma)\, e^{ir \arg(cz+d)} f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $-\pi < \arg \leqq \pi$

and

$(e)_{r,s}$ $\quad L_r f = (\frac{1}{4} - s^2) f$.

By $A[r,s]$ we denote the space of all such modular forms.

(2.5) *Remarks.* In this note we let the r in v_r coincide with the weight, see the discussion in (2.4) of [2].

It is convenient to parametrize the eigenvalue of L_r by $s \mapsto \frac{1}{4} - s^2$; clearly $A[r, -s] = A[r,s]$.

J maps $A[r,s]$ into $A[-r,s]$; it is its own inverse. Conjugation maps $A[r,s]$ into $A[-\bar{r}, \bar{s}]$.

(2.6) **Definition.** For $\nu \in \mathbb{Z}$ and $f \in A[r,s]$ put

$$F_\nu(z) = \int_0^1 e^{-2\pi i \left(\nu + \frac{1}{12} r\right) x'} f(z + x')\, dx'.$$

$F_\nu f$ satisfies $(e)_{r,s}$ (see 2.4) and

$(w^\nu)_r$ $\quad h(z + x') = e^{2\pi i \left(\nu + \frac{1}{12} r\right) x'} h(z)$.

We call the space of functions satisfying $(e)_{r,s}$ and $(w^\nu)_r$:

$$W^\nu[r,s].$$

By separation of variables one sees that for $h \in W^\nu[r,s]$ condition $(e)_{r,s}$ amounts to a regular differential equation on $(0, \infty)$ of order 2; so $\dim W^\nu[r,s] = 2$.

As in (2.5) we have $W^\nu[r, -s] = W^\nu[r,s]$,

$$J: W^\nu[r,s] \to W^{-\nu}[-r,s],$$

(2.7) **Definition.** For $\nu\in\mathbb{Z}$, $r,s\in\mathbb{C}$, $\operatorname{Re}(\nu+\frac{1}{12}r)\neq 0$, put

$$\omega^\nu(r,s;z)=e^{2\pi i(\nu+\frac{1}{12}r)x}\,W_{\frac{1}{2}r\varepsilon,s}(4\pi(\nu+\tfrac{1}{12}r)\,\varepsilon\,y),$$

with $\varepsilon=\operatorname{sign}(\operatorname{Re}(\nu+\frac{1}{12}r))$ and $W_{k,s}$ the exponentially decreasing Whittaker function, see [13].

(2.8) *Facts*, to be found in [13], or in some general book on special functions.

i) $\omega^\nu(r,s)\in W^\nu[r,s]$

ii) $\omega^\nu(r,-s)=\omega^\nu(r,s)$

$J\,\omega^\nu(r,s)=\omega^{-\nu}(-r,s)$

$\overline{\omega^\nu(r,s;z)}=\omega^{-\nu}(-\bar r,\bar s;z)$

iii) $\omega^\nu(r,s;i\,y)\sim(4\pi(\nu+\frac{1}{12}r)\,\varepsilon\,y)^{\frac{1}{2}r\varepsilon}\,e^{-2\pi\varepsilon(\nu+\frac{1}{12}r)y}$ for $y\to\infty$.

(2.9) *Notation.* ${}^\circ W^\nu[r,s]=\mathbb{C}\cdot\omega^\nu(r,s)$.

(2.10) **Definition.** Let $r,s\in\mathbb{C}$, $B\subset\mathbb{Z}$, B finite, $|\operatorname{Re} r|<12$, if $\operatorname{Re} r=0$ then $0\in B$. Define:

$$A^B[r,s]=\{f\in A[r,s]\colon\ F_\nu f\in{}^\circ W^\nu[r,s]\ \text{for all}\ \nu\in\mathbb{Z}\setminus B\},$$
$$S^B[r,s]=\{f\in A^B[r,s]\colon\ F_\nu f=0\ \text{for all}\ \nu\in B\}.$$

(2.11) *Remarks.* The elements of $S^B[r,s]$ are *cusp forms*. In Proposition (2.8) of [2] one sees that $\dim S^B[r,s]<\infty$ for (r,s) in some neighbourhood of $(-12,12)\times\mathbb{C}$.

If $\operatorname{Re} r\neq 0$, one may take $B=\emptyset$. One has $A^\emptyset[r,s]=S^\emptyset[r,s]$. If $\operatorname{Re} r=0$, in general $S^{\{0\}}[0,s]\subsetneqq A^{\{0\}}[0,s]$.

(2.12) *Example.* Let $|\operatorname{Re} r|<12$ and put $\eta_r(z)=y^{\frac{1}{2}r}\cdot\eta(z)^{2r}$, with η the Dedekind eta-function, see [9], Ch. IX.

Then

$$\eta_r\in\begin{cases}S^\emptyset[r,\pm(r-1)/2] & \text{if}\quad 0<\operatorname{Re} r<12\\ A^{\{0\}}[r,\pm(r-1)/2] & \text{if}\ -12<\operatorname{Re} r\leq 0\end{cases}$$

η_0 is a constant.

(2.13) *Facts*, see e.g. [13]. Let $\nu\in\mathbb{Z}$, $r,s\in\mathbb{C}$, $s\notin-\frac{1}{2}\mathbb{N}$. Put

$$\mu^\nu(r,s;z)=e^{2\pi i(\nu+\frac{1}{12}r)x}\,y^{\frac{1}{2}+s}\,e^{-2\pi(\nu+\frac{1}{12}r)y}\cdot$$
$${}_1F_1\left[\begin{matrix}\frac{1}{2}+s-\frac{1}{2}r\\ 1+2s\end{matrix}\middle|\,4\pi(\nu+\tfrac{1}{12}r)\,y\right],$$

with ${}_1F_1$ the confluent hypergeometric function

$${}_1F_1\left[\begin{matrix}a\\ b\end{matrix}\middle|\,u\right]=\sum_{n=0}^{\infty}\frac{\Gamma(b)\,\Gamma(a+n)\,u^n}{\Gamma(b+n)\,\Gamma(a)\,n!}.$$

iii) For $\mathrm{Re}(\nu+\frac{1}{12}r)\neq 0$ put $\varepsilon=\operatorname{sign}\mathrm{Re}(\nu+\frac{1}{12}r)$ and

$$v^\nu(r,s)=(4\pi\varepsilon(\nu+\tfrac{1}{12}r))^{\frac{1}{2}+s}\,\Gamma(-2s)\,\Gamma(\tfrac{1}{2}-s-\tfrac{1}{2}\varepsilon r)^{-1}.$$

Then $\omega^\nu(r,s)=v^\nu(r,s)\,\mu^\nu(r,s)+v^\nu(r,-s)\,\mu^\nu(r,-s)$ for (r,s) in a dense subset.

iv) $|\mu^\nu(r,s;iy)|\ll y^{\frac{1}{2}|r|}\,e^{2\pi|\mathrm{Re}(\nu+\frac{1}{12}r)|y}$ for $y\to\infty$.

v) Let $f\colon \mathfrak{h}\to\mathbb{C}$ be integrable and satisfy

$f(z)=0$ unless $|x|\leqq a$ and $y\geqq b$ for some $a,b>0$

$|f(z)|\ll e^{-2\pi cy}$ on $\mathfrak{h}$ for some $c>0$.

Then

$$\langle\mu^\nu,f\rangle(r,s)=\int_{\mathfrak{h}}\mu^\nu(r,s;z)\,\overline{f(z)}\,y^{-2}\,dx\,dy$$

defines a meromorphic function $\langle\mu^\nu,f\rangle$ on

$$\{(r,s)\in\mathbb{C}^2\colon\ |\mathrm{Re}\,r|<12,\ 2|\nu+\tfrac{1}{12}r|<|\mathrm{Re}(\nu+\tfrac{1}{12}r)|+c\},$$

holomorphic at (r_0,s_0) if $s_0\notin\frac{1}{2}\mathbb{Z}$,
the singularity at $(r_0,-\frac{1}{2}N)$, with $N\in\mathbb{N}$, may be removed by multiplication by $(r,s)\mapsto s+\frac{1}{2}N$.

vi) $$\mu^\nu(r,s;iy)\cdot\frac{\partial}{\partial y}\mu^\nu(r,-s;iy)-\frac{\partial}{\partial y}\mu^\nu(r,s;iy)\cdot\mu^\nu(r,-s;iy)$$
$$=-2s,\ \text{for } s\notin(\tfrac{1}{2}\mathbb{Z}\setminus\{0\}).$$

(2.14) *Remark.* The meromorphy in v) one may show by integrating the series for ${}_1F_1$ term-by-term and by proving uniform convergence for (r,s) in small compact sets.

(2.15) *Notation.* For $f,g\in W^\nu[r,s]$ we write

$$\mathrm{Wr}^\nu(f,g)=f(iy)\frac{\partial}{\partial y}g(iy)-\frac{\partial}{\partial y}f(iy)\cdot g(iy).$$

This *Wronskian* does not depend on y. If it is non-zero, then f and g span $W^\nu[r,s]$.

So vi) in (2.13) reads $\mathrm{Wr}^\nu(\mu^\nu(r,s),\mu^\nu(r,-s))=-2s$, showing that $\mu^\nu(r,s)$ and $\mu^\nu(r,-s)$ form a basis of $W^\nu[r,s]$ if $s\notin\frac{1}{2}\mathbb{Z}$.

(2.16) From iii) in (2.13) we see that $\mathrm{Wr}^\nu(\mu^\nu(r,s),\omega^\nu(r,s))=-2s\,v^\nu(r,-s)$, showing that $\mu^\nu(r,s)$ and $\omega^\nu(r,s)$ span $W^\nu[r,s]$ for (r,s) in an open dense set in $\mathbb{C}^2$.

(2.17) *Remark.* If $f(u_1,\dots,u_m)\colon\mathfrak{h}\to\mathbb{C}$ for $u=(u_1,\dots,u_n)\in\Omega$ for some Ω open in $\mathbb{C}^n$, then f may be holomorphic in u in several ways:

i) As distribution:

ii) Pointwise: $u \mapsto f(u;z)$ is holomorphic for each $z \in \mathfrak{h}$.

iii) In L^2-sense: If $f(u)$ is determined by its values on some set $X \subset \mathfrak{h}$ and $X \ni z \mapsto f(u,z)$ is in $L^2(X, y^{-2}\,dx\,dy)$, then $u \mapsto f(u)$ may determine a holomorphic map $\Omega \to L^2(X, y^{-2}\,dx\,dy)$. In this case weak and strong holomorphy are equivalent if $\|f(u)\|$ is bounded for u in compact sets. (Weak holomorphy: $u \mapsto \langle f(u), h\rangle$ holomorphic for all $h \in L^2(X, y^{-2}\,dx\,dy)$; strong holomorphy: $f(u)$ is locally given by power series in the $(u_i - u_{0i})$ converging in L^2-sense).

In (2.13) v) we see that μ^ν is holomorphic as distribution for $s \notin -\frac{1}{2}\mathbb{N}$; directly from the definition we get pointwise holomorphy.

We always shall understand f to be meromorphic in one of the three ways indicated above, if χf is holomorphic for some nonzero holomorphic function χ. This needs to be valid only locally. So the denominator is not allowed to depend on the test function, or the point. The meromorphy of μ^ν is clearly of this type.

(2.18) Mostly we shall consider holomorphic f on $\Omega \subset \mathbb{C}^2$ such that $L_r f(r,s) = (\frac{1}{4} - s^2) f(r,s)$. It will turn out that holomorphy as distribution and pointwise holomorphy are equivalent in this case.

The main results of this note are:

(2.19) **Proposition.** *Let* $\nu \in \mathbb{Z}$.

i) *There are a neighbourhood* $Y(\nu)$ *of* $(-12, 12) \times \mathbb{C}$ *in* $\{r \in \mathbb{C}: |\mathrm{Re}\, r| < 12\} \times \mathbb{C}$ *and on* $Y(\nu)$ *a unique meromorphic family* E^ν *of functions* $\mathfrak{h} \to \mathbb{C}$ *such that for all* (r,s) *in some nonempty open dense subset of* $Y(\nu)$:

a) $E^\nu(r,s) \in A^{(0,\nu)}[r,s]$

b) *if* $\nu = 0$:

$$F_0 E^0(r,s) = \mu^0(r,s) + C_0^0(r,s)\, \mu^0(r,-s)$$

for some $C_0^0(r,s) \in \mathbb{C}$,
if $\nu \neq 0$:

$$F_0 E^\nu(r,s) = C_0^\nu(r,s)\, \mu^0(r,-s)$$
$$F_\nu E^\nu(r,s) = \mu^\nu(r,s) + C_\nu^\nu(r,s)\, \omega^\nu(r,s)$$

for some $C_0^\nu(r,s)$, $C_\nu^\nu(r,s) \in \mathbb{C}$.

ii) *The* C_λ^ν *with* $\lambda \in \{0, \nu\}$, *in* i) b) *are meromorphic on* $Y(\nu)$ *and the following identities of meromorphic families hold:*

a) $E^0(r,-s) = C_0^0(r,-s)\, E^0(r,s)$

$$E^\nu(r,-s) = C_0^\nu(r,-s)\, E^0(r,s) - \frac{v^\nu(r,s)}{v^\nu(r,-s)}\, E^\nu(r,s) \quad \text{if } \nu \neq 0.$$

b) $J E^\nu(r,s) = E^{-\nu}(-r,s)$

$\overline{E^\nu(\bar r, \bar s)} = E^{-\nu}(-r,s)$.

(2.20) **Proposition.** *Let* $\nu\in\mathbb{Z}$.

i) *There are a neighbourhood* $Y^*(\nu)$ *of* $((-12,12)\setminus\{0\})\times\mathbb{C}$ *in* $\{r\in\mathbb{C}: 0<|\mathrm{Re}\, r|<12\}\times\mathbb{C}$ *and on* $Y^*(\nu)$ *a unique meromorphic family* P^ν *of functions* $\mathfrak{h}\to\mathbb{C}$ *such that*

a) $P^\nu(r,s)\in A^{(\nu)}[r,s]$

b) $F_\nu P^\nu(r,s)=\mu^\nu(r,s)+D_\nu^\nu(r,s)\,\omega^\nu(r,s)$

for some $D_\nu^\nu(r,s)\in\mathbb{C}$.

ii) a) $P^\nu(r,-s)=-\dfrac{v^\nu(r,s)}{v^\nu(r,-s)}P^\nu(r,s)$

b) $JP^\nu(r,s)=P^{-\nu}(-r,s)$

$\overline{P^\nu(\bar r,\bar s)}=P^{-\nu}(-r,s)$.

(2.21) *Remarks.* We may assume that $Y^*(\nu)\subset Y(\nu)$.

On $Y^*(\nu)$ we obtain by unicity:

$$P^0(r,s)=\frac{v^0(r,-s)}{v^0(r,-s)-v^0(r,s)\,C_0^0(r,s)}E^0(r,s)$$

$$P^\nu(r,s)=E^\nu(r,s)+\frac{v^0(r,s)\,C_0^\nu(r,s)}{v^0(r,-s)-v^0(r,s)\,C_0^0(r,s)}E^0(r,s).$$

(2.22) **Definition.** $\Gamma_\infty=\{\gamma\in\Gamma:\ \gamma\infty=\infty\}=\left\{\pm\begin{pmatrix}1&*\\0&1\end{pmatrix}\in\Gamma\right\}$.

(2.23) **Proposition.** *Let* $\nu\in\mathbb{Z}$, $\mathrm{Re}\, s>\frac{1}{2}$, $r\in(-12,12)$.

i) $P_r^\nu(s;z)=\sum_{\gamma\in\Gamma_\infty\setminus\Gamma} v_r(\gamma)^{-1}e^{-ir\arg(c_\gamma z+d_\gamma)}\mu^\nu(r,s;\gamma z)$

converges absolutely, uniformly for (r,s,z) *in compact sets and defines* $P_r^\nu(s)\in A^{(0,\nu)}[r,s]$.

ii) $s\mapsto E^\nu(0,s)$ *is meromorphic on* $\mathbb{C}$, *holomorphic for* $\mathrm{Re}\, s>\frac{1}{2}$ *and*

$$P_0^\nu(s_0)=\lim_{s\to s_0}E^\nu(0,s)\quad \textit{for}\ \ \mathrm{Re}\, s_0>\tfrac{1}{2}.$$

iii) *For* $r\neq 0$ *the function* $s\mapsto P^\nu(r,s)$ *is meromorphic on* $\mathbb{C}$, *and* P^ν *is holomorphic at* (r,s) *with value* $P_r^\nu(s)$ *for each* $s\in\mathbb{C}$ *with* $\mathrm{Re}\, s>\frac{1}{2}$.

Remarks. So E^ν and P^ν give indeed meromorphic continuations in two variables of the Eisenstein and Poincaré series.

The meromorphy along lines $\{r\}\times\mathbb{C}$ in ii) and iii) is non-trivial; there might have been singularities along such lines. E^0 is not holomorphic at points $(0,\frac{1}{2}N)$ with $N\in\mathbb{N}$, $N\geqq 2$; hence the difference between ii) and iii).

(2.24) **Proposition.** *Let* $r\in(-12,12)$, $s\in\mathbb{C}$. *There is a finite set* $B_0\subset\mathbb{Z}$ *such that*

(r,s) and meromorphic functions f_ν on U for each $\nu\in B$ such that

$$H=\sum_{\nu\in B} f_\nu E^\nu$$

is holomorphic on U and $H(r,s)=h$.

Remarks. So the analytically continued Eisenstein and Poincaré series suffice to get all modular forms which do not grow faster than some exponential function.

A sufficient condition on B_0 for the proposition to hold is:

$$S^{B_0}[r,s]=0.$$

Remark that B_0 may depend on (r,s).

3. Holomorphic and Meromorphic Functions

We shall often use holomorphic and meromorphic functions in the two variables r and s. In this section some notations are fixed and some known results mentioned, cf. [5].

(3.1) If $V\subset\mathbb{C}^n$ is an open set, we denote by $\mathcal{O}$ the sheaf of holomorphic functions on V. The space V is always supposed to be clear from the context. If we want to emphasize its complex dimension we write ${}^n\mathcal{O}$.

(3.2) The stalks $\mathcal{O}_v$ are unique factorization domains, [5], p. 22, if $n=1$ even principal ideal domains.

Each finitely generated ${}^1\mathcal{O}_v$-module X is of the form $\bigoplus_{i=1}^{n}{}^1\mathcal{O}_v\cdot x_i$, with suitable $x_1,\dots,x_m\in X$; see [7], § 3.8, p. 181.

(3.3) If $f\in\mathcal{O}(U)$, $U_1\subset U\subset V$, then we denote by f also its restriction to U_1, and its germ in stalks $\mathcal{O}_p$ with $p\in U$. The same convention we use for other sheaves.

(3.4) Let $f\in\mathcal{O}(U)$. Its *zero set* is $N(f)=N_U(f)=\{p\in U\colon f(p)=0\}$. If $f\neq 0$, it consists of isolated points if $n=1$, and is in general an analytic set of dimension $n-1$.

(3.5) In the case $n=2$ each analytic set S of dimension 1 may locally be described by $S\cap U=N_U(\chi_1\cdot\ldots\cdot\chi_t)$, with U a neighbourhood of $p\in S$ and $\chi_1,\dots,\chi_t$ non-equivalent irreducible elements of ${}^2\mathcal{O}_p$, represented by functions on U. Different $N(\chi_i)$ intersect each other in isolated points.

(3.6) Let $p\in V\subset\mathbb{C}^2$, $\chi\in\mathcal{O}(U)$ representing an irreducible element of ${}^2\mathcal{O}_p$. We call a holomorphic map $j\colon W\to V$ a *local curve* through p along $N(\chi)$ if

i) $0\in W$, W open in $\mathbb{C}$

If in coordinates x and y on V the function χ is equivalent at p to $x-x(p)$, then we may take $j: w \mapsto (x(p), y(p)+w)$. In all other cases one may take j of the form $j: w \mapsto (x(p)+w^q, y(p)+h(w))$ with $q \geqq 1$ and h holomorphic, $h(0)=0$, see e.g. [4], III, § 1.5, p. 131.

(3.7) Meromorphic functions are locally given as ϕ/χ with ϕ and χ holomorphic, $\chi \neq 0$; see e.g. [5], p. 24–27 or [6], p. 161. We denote by $\mathcal{M}$ the sheaf of meromorphic functions on V.

(3.8) For $f \in \mathcal{M}(U)$ we define $N(f)$ as the set of all $p \in U$ for which $f=\phi/\chi$ at p implies $\phi(p)=0$.

We define $\mathrm{Sing}(f)$ as the set of all $p \in U$ for which $f=\phi/\chi$ at p implies $\chi(p)=0$.

$N(f)$ and $\mathrm{Sing}(f)$ are, for f not holomorphic, different analytic sets of dimension $n-1$.

We put $\mathrm{Indet}(f)=N(f) \cap \mathrm{Sing}(f)$. If $n=1$ it is empty, if $n=2$ it consists of isolated points.

(3.9) If $\psi: W \to V$ is holomorphic, $W \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^n$ open, there are composition maps $\psi^*: {}^n\mathcal{O}(U) \to {}^m\mathcal{O}(\psi^{-1}(U))$, given by $\psi^* f = f \circ \psi$. In general $\psi^*: {}^n\mathcal{O}_{\psi p} \to {}^m\mathcal{O}_p$ is not surjective for all $p \in W$.

If $m \geqq n$ and ψ is locally surjective, one has also $\psi^*: {}^n\mathcal{M}(U) \to {}^m\mathcal{M}(\psi^{-1}(U))$, otherwise some care is needed. If $m=1$, $n=2$ then $\psi^* f$ makes only sense as meromorphic function if $\psi(W)$ intersects $\mathrm{Sing}(f)$ in isolated points. In this case we call ψ^* a restriction map.

4. Holomorphic Families of Vector Spaces

Spaces like $W^v[r,s]$ vary holomorphically in r and s. We develop some machinery to handle such situations.

(4.1) In this section K denotes a vector space over $\mathbb{C}$, and D its algebraic antidual. The pairing we denote by

$$(d,k) \mapsto \langle d,k \rangle,$$

linear in d, antilinear in k. We provide D with the weak topology, for which all maps $d \mapsto \langle d,k \rangle$ with $k \in K$, are continuous. D is complete in this topology.

(4.2) *Examples*

$K=D=\mathbb{C}^a$, with $a \in \mathbb{N}$.

$K=K_{\mathfrak{h}}=C_c^\infty(\mathfrak{h})$; $D=D_{\mathfrak{h}}$, the space of distributions on $\mathfrak{h}$.

(4.3) **Definition.** Let $V \subset \mathbb{C}^n$, open. A *family of subspaces* B of D on V is a system of linear subspaces $B[v] \subset D$ for each $v \in V$. By $\mathbf{S}(D,V)$ we denote the collection of all such families.

(4.4) *Example.* Let $V \subset \mathbb{C}^n$, $\lambda, r \in \mathcal{O}(V)$. Let $(c)=(a)$ or (w^ν), see (2.4) and (2.6). Define $F((c); V, \lambda, r) \in \mathrm{S}(D_\mathfrak{h}, V)$ by

$$F((c); V, \lambda, r)[v] = \{f\colon \mathfrak{h} \to \mathbb{C}\colon f \text{ satisfies } (c)_{r(v)} \text{ and } L_{r(v)} f = \lambda(v) f\}.$$

In particular, we shall often take $Y = \{r \in \mathbb{C}\colon |\mathrm{Re}\, r| < 12\} \times \mathbb{C}$ and $r(r,s)=r$, $\lambda(r,s) = \frac{1}{4} - s^2$.

Then $F((w^\nu); Y, \frac{1}{4}-s^2, r)[r,s] = W^\nu[r,s]$ and $F((a); Y, \frac{1}{4}-s^2, r)[r,s] = A[r,s]$.

(4.5) **Definition.** Let $j\colon W \to V$ be a holomorphic map. For each $B \in \mathrm{S}(D, V)$ we define $B_j \in \mathrm{S}(D, W)$ by $B_j[w] = B[j\,w]$.

Clearly $F((c); V, \lambda, r)_j = F((c); W, j^* \lambda, j^* r)$.

(4.6) **Definition.** For $B \in \mathrm{S}(D, V)$ we define a sheaf $\mathscr{B}$ on V by:

$$\mathscr{B}(U) = \left\{ f\colon\ U \to D\colon \begin{array}{ll} f(u) \in B[u] & \text{for all } u \in U, \\ u \mapsto \langle f(u), k\rangle & \text{in } \mathcal{O}(U) \text{ for each } k \in K \end{array} \right\}$$

for each $U \subset V$, open.

We denote $u \mapsto \langle f(u), k\rangle$ by $\langle f, k\rangle$.

In general B is not determined by $\mathscr{B}$.

We denote the sheaf associated to an element of $\mathrm{S}(D, V)$ by the corresponding script letter. Exception: to $\mathbb{C}^a \in \mathrm{S}(\mathbb{C}^a, V)$ is associated $\mathcal{O}^a$.

(4.7) $\mathscr{B}$ is an $\mathcal{O}$-module without torsion. We shall write $\mathscr{M} \otimes \mathscr{B}$ for $\mathscr{M} \otimes_{\mathcal{O}} \mathscr{B}$.

If $j\colon W \to V$ is holomorphic we have restriction maps $j^*\colon \mathscr{B}(U) \to \mathscr{B}_j(j^{-1} U)$.

(4.8) **Proposition.** *The sections* $f \in \mathscr{F}((c); V, \lambda, r)(U)$ *are analytic functions on* $U \times \mathfrak{h}$, *pointwise holomorphic on* U.

Proof. f is an eigendistribution of the elliptic differential operator

$$-\frac{\partial^2}{\partial v_1 \partial \bar{v}_1} - \dots - \frac{\partial^2}{\partial v_n \partial \bar{v}_n} + L_r - \lambda,$$

with $v_1, \dots, v_n$ coordinates on V. So f is analytic on $U \times \mathfrak{h}$. The pointwise holomorphy on U may be proved by Cauchy's formula.

(4.9) *Remark.* On the other hand, if f is analytic on $U \times \mathfrak{h}$, pointwise holomorphic on U, then f is holomorphic in distribution sense, so $f \in \mathscr{D}_\mathfrak{h}(U)$.

(4.10) *Notation.* For $f \in \mathscr{B}(U)$ we define $N(f) = \bigcap_{k \in K} N(\langle f, k\rangle)$. If $U \subset V \subset \mathbb{C}^n$, the dimension of $N(f)$ may be much smaller than $n-1$.

$$\begin{aligned}\mathrm{Sing}(f) &= \bigcup_{k\in K} \mathrm{Sing}(\langle f,k\rangle)\\ &= \{p\in U:\ \text{if } \phi\in\mathcal{O}_p,\ \phi\neq 0 \text{ and } \phi f\in\mathcal{B}_p, \text{ then } \phi(p)=0\}\\ N(f) &= \bigcap_{k\in K} N(\langle f,k\rangle)\\ &= \{p\in U:\ \text{if } \phi\in\mathcal{O}_p,\ \phi\neq 0 \text{ and } \phi f\in\mathcal{B}_p, \text{ then } (\phi f)(p)=0\}\\ \mathrm{Indet}(f) &= \mathrm{Sing}(f)\cap N(f)\\ &= \{p\in U:\ \text{if } \phi\in\mathcal{O}_p,\ \phi\neq 0 \text{ and } \phi f\in\mathcal{B}_p, \text{ then } \phi(p)=0\\ &\qquad \text{and } \phi f(p)=0\}\\ &= \{p\in U:\ p\in\mathrm{Sing}(f) \text{ and for each } k\in K:\ \text{if}\\ &\qquad p\in\mathrm{Sing}(\langle f,k\rangle) \text{ then } p\in\mathrm{Indet}(\langle f,k\rangle)\}.\end{aligned}$$

If $n=1$ then $\mathrm{Sing}(f)$ and $N(f)$ consist of isolated points and $\mathrm{Indet}(f)$ is empty. If $n=2$ then $\mathrm{Indet}(f)$ consists at most of isolated points, and $\mathrm{Sing}(f)$ is an analytic set of dimension one, if f is not holomorphic.

(4.11) For $G\in\mathbf{S}(D, V)$ the space $B[v]$ may depend rather wildly on $v\in V$.

(4.12) *Example.* Let $V=\mathbb{C}$, $B[v]=\mathbb{C}$ if $v\in\mathbb{C}\setminus\{0\}$, $B[0]=0$. Then $B\in\mathbf{S}(\mathbb{C}^1,\mathbb{C})$ and $(\mathcal{M}\otimes\mathcal{B})(U)\cap\mathcal{O}^1(U)\neq\mathcal{B}(U)$ if $0\in U$.

(4.13) **Definition.** $B\in\mathbf{S}(D, V)$ is called a *holomorphic family of subspaces* of D on V if for each holomorphic map j: $W\to V$:

if U is open in W and U_1 is open and dense in U then

$$\mathcal{B}_j(U)=\{f\in\mathcal{D}_j(U):\ f\in\mathcal{B}_j(U_1)\}.$$

We denote by $\mathbf{H}(D, V)$ the collection of all such holomorphic families.

(4.14) *Remark.* If $B\in\mathbf{H}(D, V)$ and j: $W\to V$ is holomorphic, then $B_j\in\mathbf{H}(D, W)$.

(4.15) **Lemma.** *Let* $B\in\mathbf{H}(D, V)$.

i) $\mathcal{M}\otimes\mathcal{B}(U)\cap\mathcal{D}(U)=\mathcal{B}(U)$ *for* $U\subset V$ *open.*

ii) *Let U be open in V and U_1 open and dense in U. Suppose $f\in\mathcal{B}(U_1)$ satisfies:*

for each $p\in U$ there are a neighbourhood W of p, $W\subset U$, and non-zero $\phi\in\mathcal{O}(W)$ such that $\phi\langle f,k\rangle\in\mathcal{O}(W\cap U_1)$ extends holomorphically to W for each $k\in K$.

Then f is the restriction of an element of $\mathcal{M}\otimes\mathcal{B}(U)$.

Proof. To get i) let $f\in(\mathcal{M}\otimes\mathcal{B}(U))\cap\mathcal{D}(U)$ and take $U_1=U\setminus\mathrm{Sing}(f)$ in definition (4.13). It is sufficient to prove ii) locally. So we work on W. We see that there is $g\in\mathcal{D}(W)$ such that $g=\phi f$ on $W\cap U_1$. By definition (4.13) we have $g\in\mathcal{B}(W)$, so $\phi^{-1}\cdot g$ is the desired extension of f to W.

(4.16) **Definition.** $B\in\mathbf{S}(D, V)$ is called *closed* if $v_n\to v$ in V, $f_n\in B[v_n]$, $f_n\to f$ in D

Proof. Let $v_n \to v$, $f_n \in F((c); V, \lambda, r)[v_n]$, $f_n \to f$ in $D_{\mathfrak{h}}$. The transformation property $(c)=(a)$ or (w^v) is preserved under limits. For $k \in K_{\mathfrak{h}}$:

$$\begin{aligned}
&\langle (L_{r(v)} - \lambda(v)) f, k\rangle \\
&\quad = \langle f, L_{\overline{r(v)}} k\rangle - \lambda(v) \langle f, k\rangle \\
&\quad = \lim_{n\to\infty} (\langle f_n, L_{\overline{r(v)}} k\rangle - \lambda(v_n) \langle f_n, k\rangle) \\
&\quad = \lim_{n\to\infty} \left(\left\langle f_n, L_{\overline{r(v_n)}} k + i(\overline{r(v)} - \overline{r(v_n)})\, y \frac{\partial k}{\partial x}\right\rangle - \lambda(v_n) \langle f_n, k\rangle \right) \\
&\quad = \lim_{n\to\infty} \left(\langle (L_{r(v_n)} - \lambda(v_n)) f_n, k\rangle - i(r(v) - r(v_n)) \left\langle f_n, y \frac{\partial k}{\partial x}\right\rangle \right) \\
&\quad = 0 - \mathrm{i}.0. \left\langle f, y\frac{\partial k}{\partial x}\right\rangle = 0.
\end{aligned}$$

(4.18) *Remark.* If j: $W \to V$ is holomorphic and $B \in \mathbf{S}(D, V)$, then B closed clearly implies B_j closed.

(4.19) **Lemma.** *If* $B \in \mathbf{S}(D, V)$ *is closed, then* $B \in \mathbf{H}(D, V)$.

Proof. Clear from (4.13), (4.16) and (4.18).

(4.20) *Example.* If $B \in \mathbf{H}(D, V)$ it need not be closed. Take e.g. $V = \mathbb{C}$, $B\left[\frac{1}{n}\right] = \mathbb{C}$ for $n = 1, 2, \ldots$ and $B[v] = 0$ otherwise.

(4.21) *Remark.* By lemma (4.17) and lemma (4.19):

$$F((c); V, \lambda, r) \in \mathbf{H}(D_{\mathfrak{h}}, V).$$

Also $\mathbb{C}^a \in \mathbf{H}(\mathbb{C}^a, V)$.

(4.22) **Definition.** Let $B \in \mathbf{S}(D_1, V)$, $C \in \mathbf{S}(D_2, V)$. A *holomorphic morphism* ϕ: $B \to C$ on V is a family of linear maps

$$\phi[v]\colon\ B[v] \to C[v] \qquad \text{for some } v \in V$$

such that for each holomorphic map j: $W \to V$

defined by

$$\phi_j\colon \mathcal{B}_j \to \mathcal{C}_j$$

$$\phi_j f(u) = \phi[j u] f(u)$$

is a morphism of $\mathcal{O}$-modules on W.

(4.23) *Example.* Take $z \in \mathfrak{h}$. Then $f \mapsto f(z)$ defines a holomorphic morphism $F((c); V, \lambda, r) \to \mathbb{C}^1$ on V. See proposition (4.8).

(4.24) **Proposition.** *Let* $v \in \mathbb{Z}$. *Define* F_v: $F((a); V, \lambda, r) \to F((w^v); V, \lambda, r)$ *as in* (2.6).

Proof. In view of (4.5) we may work on V. For $k\in K_{\mathfrak{h}}$ and $r\in\mathbb{C}$ put

$$k_r(z)=\int_0^1 e^{2\pi i(\nu+\frac{1}{12}\bar{r})x'}\,k(x'+z)\,dx'.$$

Then $k_r\in K_{\mathfrak{h}}$ and $\langle F_v f, k\rangle=\langle f, k_{r(v)}\rangle$ for each $f\in F((a); V,\lambda,r)[v]$.

For $h\in F((a); V,\lambda,r)(U)$ we have to show that $v\mapsto\langle h(v), k_{r(v)}\rangle$ is holomorphic on U.

For a given k we take $C\subset\mathfrak{h}$ compact such that $\operatorname{supp}(k_{r(v)})\subset C$ for all $v\in U$. This is possible if U is small enough. We may also assume that C is the closure of its interior.

The $m\in K_{\mathfrak{h}}$ with $\operatorname{supp}(m)\subset C$ are dense in $H_C=L^2(C, y^{-2}\,dx\,dy)$. If $U_1\subset U$ is compact, h is bounded on $U_1\times C$, see proposition (4.8). So $v\mapsto h(v)$ determines a holomorphic family of elements of H_C. From the definition of k_r follows that $r\mapsto k_r$ is antiholomorphic $\mathbb{C}\to H_C$. So $v\mapsto\langle h(v), k_{r(v)}\rangle$ is holomorphic.

(4.25) **Definition.** If ϕ: $B\to C$ is a holomorphic morphism on V, with $B\in\mathbf{S}(D, V)$, then $\ker\phi\in\mathbf{S}(D, V)$ is defined by

$$(\ker\phi)[v]=\ker(\phi[v])\qquad\text{for each } v\in V.$$

(4.26) **Proposition.** *If* ϕ: $B\to C$ *is a holomorphic morphism on* V *and* $B\in\mathbf{H}(D, V)$, *then* $\ker\phi\in\mathbf{H}(D, V)$.

Proof. Denote $E=\ker\phi\in\mathbf{S}(D, V)$. Let $C\in\mathbf{S}(D_1, V)$. Let j: $W\to V$ be holomorphic and consider $f\in\mathcal{D}_j(U)$, $f\in\mathcal{E}_j(U_1)$, with $U_1\subset U$ as in definition (4.13). So for each $k\in K_1$

$$\langle\phi[j\,u]\,f(u), k\rangle=0\qquad\text{for all } u\in U_1.$$

By holomorphy $\phi f=0$ on U and $f\in\mathcal{E}_j(U)$.

(4.27) *Notation.* If $B\in\mathbf{S}(D, V)$, $v\in V$, we denote by $B(v)$ the image of the natural map $\mathcal{B}_v\to B[v]$.

(4.28) For each holomorphic j: $W\to V$ we have

$$B(j\,w)\subset B_j(w)\subset B[j\,w]=B_j[w].$$

These inclusions need not be equalities.

The kernel of $\mathcal{B}_v\to B(v)$ contains $\mathfrak{m}_v\cdot\mathcal{B}_v$, with $\mathfrak{m}_v$ the maximal ideal of $\mathcal{O}_v$, but it need not be equal to $\mathfrak{m}_v\cdot\mathcal{B}_v$.

(4.29) **Definition.** $B\in\mathbf{S}(D, V)$ is called a *bundle* of rank b on V if

i) $\mathcal{B}$ is locally isomorphic to $\mathcal{O}^b$

ii) $B(v)=B[v]\cong\mathbb{C}^b$ for each $v\in V$.

By $\mathbf{B}^b(D, V)$ we denote the collection of all such B.

By a *local basis* of B on $U\subset V$ we denote $\varepsilon_1,\ldots,\varepsilon_b\in\mathcal{B}(U)$ such that

(4.30) *Example.* $\mathbb{C}^a \in \mathbf{B}^a(\mathbb{C}^a, V)$.

(4.31) **Proposition.** *Let* $B \in \mathbf{B}^b(D, V)$ *and* $j: W \to V$ *a holomorphic map.*

i) $B_j \in \mathbf{B}^b(D, W)$.

ii) *If* $\varepsilon_1, \dots, \varepsilon_b$ *is a local basis of* B *on* $U \subset V$, *then* $j^* \varepsilon_1, \dots, j^* \varepsilon_b$ *is a local basis of* B_j *on* $j^{-1} U$.

iii) *If* $j^*: \mathcal{O}(U) \to \mathcal{O}(j^{-1} U)$ *is surjective for all open* $U \subset V$, *then so is* $j^*: \mathcal{B}(U) \to \mathcal{B}_j(j^{-1} U)$.

Proof. iii) is a direct consequence of ii). Take $\varepsilon_1, \dots, \varepsilon_b$ as in ii). Then $\varepsilon_1(u), \dots, \varepsilon_b(u)$ is a $\mathbb{C}$-basis of $B(u)$ for each $u \in U$. Fix $u_0 \in U$. There are $k_1, \dots, k_b \in K$ such that $\det((\langle \varepsilon_i(u_0), k_m \rangle))$ is nonzero, so we may assume $\langle \varepsilon_i(u_0), k_m \rangle = \delta_{im}$. On a neighbourhood U_1 of u_0 the matrix $((\langle \varepsilon_i(u), k_m \rangle))$ is invertible, with elements holomorphic in u. So we may find $\eta_i^m \in \mathcal{O}(U_1)$ such that

$$f = \sum_{i=1}^{b} \sum_{m=1}^{b} \eta_i^m(u_1) \langle f, k_m \rangle \varepsilon_i(u_1)$$

for each $f \in B(u_1)$, $u_1 \in U_1$. So "taking the coordinates" with respect to $\varepsilon_1, \dots, \varepsilon_b$ is described by $\sum_{m=1}^{b} \eta_i^m \langle \cdot, k_m \rangle$.

For any holomorphic j we now may express $h \in \mathcal{B}_j(U_2)$, $U_2 \subset j^{-1}(U_1)$, as $h = \sum_{i=1}^{b} \alpha_i \cdot j^* \varepsilon_i$, with $\alpha_i \in \mathcal{O}(U_2)$. As the $j^* \varepsilon_i(w)$ are linearly independent for $w \in j^{-1}(U_1)$ we see that $b = \dim B_j(w) \leqq \dim B_j[w] = b$ and $\mathcal{B}_j \cong \mathcal{O}^b$ on $j^{-1} U_1$. So $B_j \in \mathbf{B}^b(D, j^{-1} U_1)$ and $j^* \varepsilon_1, \dots, j^* \varepsilon_b$ is a local basis on $j^{-1} U_1$. As u_0 had been chosen arbitrarily we get i) and ii).

(4.32) **Corollary.** $\mathbf{B}^b(D, V) \subset \mathbf{H}(D, V)$.

Proof. In the proof of proposition (4.31) we saw that the coordinates may locally be written as $\sum_{m=1}^{b} \eta_i^m \langle \cdot, k_m \rangle$. So the corollary is clear from i) of proposition (4.31) and definition (4.13).

(4.33) **Proposition.** *Let* $V \subset \mathbb{C}^n$, $n = 1$ *or* 2; $S \subset V$ *an analytic subset of dimension* $n-1$; $a, b \in \mathbb{N}$; $L \in \mathbf{H}(D, V)$, $\psi: L \to \mathbb{C}^a$ *a holomorphic morphism on* V.

Define $B \in \mathbf{S}(\mathbb{C}^a, V)$ *by* $B[v] = \psi[v] L[v]$ *for each* $v \in V$.

Suppose $\dim B[v] \leqq b$ *for all* $v \in V$. *Suppose there are* $\beta_1, \dots, \beta_b \in \mathcal{M} \otimes \mathcal{L}(V)$ *such that* $\psi \beta_1, \dots, \psi \beta_b$ *are* $\mathcal{M}(V)$*-linearly independent. Suppose* $\ker \psi[v] = 0$ *for* $v \in V \setminus S$.

i) a) $L \in \mathbf{B}^b(D, V \setminus S)$; *there is a holomorphic morphism* $B \to L$ *on* $V \setminus S$ *inverting* $\psi: L \to B$ *on* $V \setminus S$.

b) $\mathcal{M} \otimes \mathcal{L}(U) = \bigoplus_{i=1}^{b} \mathcal{M}(U) \beta_i$ *for each open* $U \subset V$.

ii) *Let* $n = 1$. *For* $v \in V$ *denote by* X_v *the image of the map*

Put $\tilde{S}=\{s\in S\colon X_s\cap\ker\psi[s]=0\}$. *Then for each* $s\in\tilde{S}$:

a) $\psi\colon \mathscr{L}_s\to\mathscr{B}_s$ *is bijective*

b) $\psi[s]\colon L(s)\to B(s)$ *is bijective*; $\dim L(s)=b$.

c) $L[s]=L(s)\oplus\ker\psi[s]$.

d) $B\in\mathbf{B}^b(\mathbb{C}^a, U)$ *for some neighbourhood* U *of* s.

iii) *Let* $n=2$. *Denote by* S_1 *the set of those* $s\in S$ *for which there is a curve* $j_0\colon W_0\to V$ *through* s *such that*

(1) $j_0 W_0\cap S$ *consists of isolated points.*

(2) $j_0^*\beta_1,\dots,j_0^*\beta_b$ *are well defined, and* $j_0^*\psi\beta_1,\dots,j_0^*\psi\beta_b$ *are* ${}^1\mathscr{M}(W_0)$-*linearly independent.*

(3) $L_{j_0}(0)\cap\ker\psi[s]=0$.

Denote by $\tilde{\tilde{S}}$ *the set of those* $s\in S_1$ *such that for each curve* $j\colon W\to V$ *through* s *with* $jW\subset S$, *there exist* $w_m\in W$, $w_m\neq 0$, *with* $\lim_{m\to\infty} w_m=0$ *and* $j\,w_m\in S_1$.

Then for each $s\in\tilde{\tilde{S}}$:

a) $\psi\colon \mathscr{L}_s\to\mathscr{B}_s$ *is bijective.*

If in addition we suppose $a=b$ *or* $\ker\psi[s]=0$, *then:*

b) $\psi[s]\colon L(s)\to B(s)$ *is bijective*; $\dim L(s)=b$.

c) $L[s]=L(s)\oplus\ker\psi[s]$.

d) $B\in\mathbf{B}^b(\mathbb{C}^a, U)$ *for some neighbourhood* U *of* s.

Remark. This complicated proposition will be essential in the proof of proposition (2.24). Think of L as A^B, of ψ as taking the Fourier coefficients for $v\in B$ and of the β_i as something like Poincaré series.

For $n=2$ the condition "$a=b$ or $\ker\psi[s]=0$" in iii) is disappointing. In the formulation of this proposition it is necessary. Whether one could get around it in the situation of proposition (2.24) is not clear to us.

Proof. The proof will need a sequence of lemmas.

We fix the following notations:

$$V_0=V\setminus S,\qquad V_1=V_0\cup\tilde{\tilde{S}},\qquad S_1=\tilde{S}\quad\text{if } n=1.$$

(4.34) **Lemma.** *For each* $v\in V_0$: *if* $\beta\in(\mathscr{M}\otimes\mathscr{L})_v$ *and* $\psi\beta\in\mathscr{O}_v^a$ *then* $\beta\in\mathscr{L}_v$ *and* $\psi\beta\in\mathscr{B}_v$.

Remark. The existence of the β_i is not needed for this lemma.

Proof. We work on a neighbourhood U of v, which is allowed to shrink a finite number of times.

Let $\beta\in\mathscr{M}\otimes\mathscr{L}(U)$, $\psi\beta\in\mathscr{O}^a(U)$. Take a nonzero $\phi\in\mathscr{O}(U)$ such that $\phi\beta\in\mathscr{L}(U)$. If we may take $\phi\in\mathscr{O}(U)^*$ we are done.

As V_0 is open, we may assume $U\subset V_0$. Suppose $N(\phi)\neq\emptyset$. If $u_1\in N(\phi)$ satisfies $\phi\beta(u_1)\neq 0$, then

Take an irreducible factor t of ϕ in $\mathcal{O}_v$ and arrange that $t\in\mathcal{O}(U)$. Then $N(t) \subset N(\phi\beta)$, so t divides $\langle\phi\beta, k\rangle$ for each $k\in K$. So $t^{-1}\phi\beta\in\mathcal{L}(U)$ by i) of lemma (4.15). So we may replace ϕ by $t^{-1}\phi$. This process stops after a finite number of steps, as $\mathcal{O}_v$ has unique factorization.

(4.35) **Lemma.** *There is an analytic set $A\supset S$ with dimension $n-1$ such that*

i) $\psi\beta_i\in\mathcal{B}(V\setminus A)$ *for* $i=1,\dots,b$,

ii) $\psi\beta_1(v),\dots,\psi\beta_b(v)$ *is a* $\mathbb{C}$*-basis of* $B[v]$ *for each* $v\in V\setminus A$.

iii) *The germs of* $\psi\beta_1,\dots,\psi\beta_b$ *at* v *form an* $\mathcal{M}_v$*-basis of* $(\mathcal{M}\otimes\mathcal{B})_v$ *for each* $v\in V$.

Proof. Suppose $\gamma_1,\dots,\gamma_c\in\mathcal{M}\otimes\mathcal{B}(U)$ are $\mathcal{M}(U)$-linearly independent, for some open $U\subset V$. After a choice of coordinates in $\mathbb{C}^a$ we consider $(\gamma_1,\dots,\gamma_c)$ as an $a\times c$-matrix over $\mathcal{M}(U)$. It contains a $c\times c$ submatrix $((c_{i\ell}))$ with determinant $\delta\neq 0$. Take $A_1=N(\delta)\cup\bigcup_{i\ell}\operatorname{Sing}(c_{i\ell})$.

Then for each $v\in U\setminus A_1$ we have $\mathbb{C}$-linearly independent elements $\gamma_1(v),\dots,\gamma_c(v)$ in $B(v)$.

Applying this with $c=b$, $\gamma_i=\psi\beta_i$ and $U=V$ we may take $A=A_1\cup S$ to obtain i) and ii). To see that $B(v)=B[v]$ consider the dimension.

Any $\mathcal{M}(U)$-linear relation between the $\psi\beta_i$ has coefficients vanishing on a dense open subset of $U\setminus A$, so the germs of the $\psi\beta_i$ are $\mathcal{M}_v$-linearly independent. Suppose $\gamma\in\mathcal{M}\otimes\mathcal{B}(U)$ is $\mathcal{M}(U)$-linearly independent of the $\psi\beta_i$. We apply the process given above to $\psi\beta_1,\dots,\psi\beta_b,\gamma$ and get $b+1$ $\mathbb{C}$-linearly independent elements in some $B[v]$, which is impossible.

(4.36) **Lemma.** i) ψ: *$\mathcal{L}\to\mathcal{B}$ is an injective morphism of $\mathcal{O}$-modules.*

ii) ψ: *$\mathcal{M}\otimes\mathcal{L}\to\mathcal{M}\otimes\mathcal{B}$ is an isomorphism of $\mathcal{M}$-modules.*

iii) ψ: *$\mathcal{L}\to\mathcal{B}$ is an isomorphism of $\mathcal{O}$-modules on V_0.*

Proof. These statements may be proved by considering them in stalks.

If $f\in\mathcal{L}(U)$ and $\psi f=0$, then $f(u)=0$ for $u\in V_0$, so $f=0$. This gives i) and implies the injectivity in ii). The surjectivity in ii) follows from iii) of lemma (4.35).

If $v\in V_0$ and $g\in\mathcal{B}_v$ then $g=\psi f$ for some $f\in(\mathcal{M}\otimes\mathcal{L})_v$ by ii). Lemma (4.34) shows that $f\in\mathcal{L}_v$.

(4.37) **Lemma.** *Statement* i b) *in proposition* (4.33) *holds. In the case $n=1$ one has $\tilde S=\{s\in S\colon L(s)\cap\ker\psi[s]=0\}$.*

Proof. The latter assertion follows from the former.

We need to show $(\mathcal{M}\otimes\mathcal{L})_v=\bigoplus_{i=1}^{b}\mathcal{M}_v\cdot\beta_i$ for $v\in V$. By ii) of lemma (4.36) it is sufficient to show $(\mathcal{M}\otimes\mathcal{B})_v=\bigoplus_{i=1}^{b}\mathcal{M}_v\cdot\psi\beta_i$; but this is iii) of lemma (4.35).

(4.38) **Lemma.** *For each $s\in V_1$:*

i) *If $\beta\in(\mathcal{M}\otimes\mathcal{L})$ and $\psi\beta\in\mathcal{O}^a$, then $\beta\in\mathcal{L}$ and $\psi\beta\in\mathcal{B}$*

Proof. By ii) of lemma (4.36) statement ii) follows from i).

Let $s \in V_1$ and U a neighbourhood of s, which may shrink in the course of the proof. As in the proof of lemma (4.34) we consider a nonzero $\phi \in \mathcal{O}(U)$ such that $\phi\beta \in \mathcal{L}(U)$. If $s \in \tilde{S}$ there might be $u_1 \in N(\phi) \cap S$ with $\phi\beta(u_1) \neq 0$.

In the case $n=1$ the set S consists of isolated points, so we may assume $u_1 = s$. Then $\phi\beta(s) \in \ker\psi[s] \cap L(s)$, so $\phi\beta(s)=0$ by lemma (4.37). So the proof of lemma (4.34) works in this case.

In the case $n=2$ we take a local curve $j\colon W \to V$ through s along an irreducible component of $N(\phi)$ on which u_1 lies. So $\gamma = j^*(\phi\beta)$ is a nonzero element of $\mathcal{L}_j(W)$. As $\psi\gamma = j^*\phi \cdot j^*\psi\beta = 0$, we have $jW \subset S$. Take $w_m \in W$ as in the condition in part iii) of proposition (4.33). For almost all m: $\gamma(w_m) \neq 0$.

But $\gamma(w_m) = \phi\beta(jw_m) \in L(jw_m)$, while on the other hand $\gamma(w_m) \in \ker\psi[jw_m]$. This contradicts assumption (3) in part iii) of proposition (4.33) (use (4.28)). So in this case as well the proof of lemma (4.34) works.

(4.39) **Lemma.** *Let* $v \in V_1$. i) *If* $n=1$ *then* $B(v) = B[v]$.

ii) *If* $n=2$ *there exists a curve* $j\colon W \to V$ *through* v *such that* $B_j(0) = B[v]$, $L(0) \cap \ker\psi[v] = 0$ *and* ii) *of lemma* (4.38) *holds for* $\mathcal{B}_j$.

iii) *In both cases* $\dim B[v] = b$.

Proof, case $n=1$. As $\mathcal{B}_v$ is torsion free over the principal ideal domain ${}^1\mathcal{O}_v$ and $(\mathcal{M} \otimes \mathcal{B})_v$ has $\mathcal{M}_v$-dimension b, we have $\mathcal{B}_v = \bigoplus_{i=1}^{b} {}^1\mathcal{O}_v \cdot m_i$ for some $m_1, \ldots, m_b \in \mathcal{B}_v$.

Suppose there are $c_1, \ldots, c_b \in \mathbb{C}$ such that $\sum_{i=1}^{b} c_i m_i(v) = 0$. Then $m = \sum_{i=1}^{b} c_i m_i$ is divisible by a local coordinate t at v, so $t^{-1} m \in \mathcal{B}_v$ by lemma (4.38) ii). This means t divides all c_i, hence $c_1 = \ldots = c_b = 0$. So $m_1(v), \ldots, m_b(v)$ form a basis of $B(v) = B[v]$. This gives i) and iii) for $n=1$.

Proof, case $n=2$. If $v \in \tilde{S}$, choose $j = j_0$ as in the condition in part iii) of proposition (4.33). If $v \in V_0$ we take $jw = v + wd$ for some nonzero $d \in \mathbb{C}^2$ such that jW intersects the set A in lemma (4.35) discretely. In both cases we may restrict all data in proposition (4.33) to W by j^*, and all conditions are satisfied for the restricted data. So everything in ii) follows from the case $n=1$. By i) and iii) we have $B_j(0) = B_j[0]$, with dimension b, hence $B_j(0) = B[v]$. Remark that $L(0) \cap \ker\psi[v] \subset L_j(0) \cap \ker\psi[v] = 0$.

(4.40) **Lemma.** *Let* $v \in V_1$. *There is a basis* $\varepsilon_1, \ldots, \varepsilon_a$ *of* $\mathbb{C}^a$ *such that:*

i) $\varepsilon_1, \ldots, \varepsilon_b$ *is a* $\mathbb{C}$*-basis of* $B[v]$.

ii) *There exist* $m_1, \ldots, m_b \in \mathcal{B}_v$

$d_1, \ldots, d_b \in \mathcal{O}_v$, *all* d_i *nonzero*

$\xi_i^\ell \in \mathfrak{m}_v$, *the maximal ideal of* $\mathcal{O}_v$,

such that

$$m_i = d_i \varepsilon_i + \sum_{\ell=b+1}^{a} \xi_i^\ell \varepsilon_\ell \quad \text{for } i = 1, \ldots, b.$$

Proof, case $n=1$. Take the m_i as in the proof of lemma (4.39). Put $\varepsilon_i=m_i(v)$ for $i=1,\dots,b$, and choose the other ε_j such that $\varepsilon_1,\dots,\varepsilon_a$ span $\mathbb{C}^a$. In coordinates with respect to $\varepsilon_1,\dots,\varepsilon_a$ we get an $a\times b$ matrix $(m_1,\dots,m_b)=\begin{pmatrix}M\\N\end{pmatrix}$ over $\mathcal{O}_v$, such that $M\equiv I_b$, $N\equiv 0$ modulo $\mathfrak{m}_v$. So the columns of $\begin{pmatrix}M\\N\end{pmatrix}M^{-1}$ will serve as the $m_1,\dots,m_b$ in ii) of the lemma.

Proof, case $n=2$. By iii) of lemma (4.39) we may choose a basis $\varepsilon_1,\dots,\varepsilon_a$ of $\mathbb{C}^a$ satisfying i). In coordinates with respect to this basis $\tau(\psi\beta_1,\dots,\psi\beta_b)=\begin{pmatrix}C\\D\end{pmatrix}$, with $\tau\in\mathcal{O}_v$, $\tau\neq 0$, and C and D matrices over $\mathcal{O}_v$, of size $b\times b$, respectively $(a-b)\times b$.

Suppose $\det C=0$. Then $a>b$, and there exists a nonzero $m\in\mathscr{B}_v$, $m\in\bigoplus_{\ell=b+1}^{a}\mathcal{O}_v\cdot\varepsilon_\ell$. By ii) of lemma (3.38) we may arrange that $\mathscr{M}_v m\cap\mathscr{B}_v=\mathcal{O}_v m$. Take $j\colon W\to V$ as in ii) of lemma (3.39), and take $t\in\mathcal{O}_v$, irreducible, such that $N(t)\subset jW$. As t cannot divide all coordinates of m, we find $j^*m\neq 0$. Now ${}^1\mathcal{O}_0^a\cap(\mathscr{M}\otimes\mathscr{B}_j)_0=(\mathscr{B}_j)_0$, by ii) of lemma (4.39). So after dividing by a suitable power of a local coordinate at 0 on W we obtain $n\in(\mathscr{B}_j)_0$ with $n(0)\neq 0$. But $n(0)\in\bigoplus_{\ell=b+1}^{a}\mathbb{C}\,\varepsilon_\ell$, in contradiction with $n(0)\in B_j(0)=B[v]$.

Form $(\det C)\cdot\begin{pmatrix}C\\D\end{pmatrix}C^{-1}$. In each column divide out common factors of the coordinates, and obtain $m_1,\dots,m_b\in\mathscr{B}_v$ such that

$$m_i=d_i\varepsilon_i+\sum_{\ell=b+1}^{a}\xi_i^\ell\varepsilon_\ell,\qquad\text{with } d_i,\ \xi_i^\ell\in\mathcal{O}_v,\ \text{all } d_i \text{ non-zero.}$$

From the choice of $\varepsilon_1,\dots,\varepsilon_a$ follows that $\xi_i^\ell(v)=0$ for all i,ℓ.

If $a=b$ then clearly $d_i(v)\neq 0$ for $i=1,\dots,b$. Suppose $a>b$ and $\ker\psi[v]=0$. Let $d_i(v)=0$. By the lemmas (4.38) and (4.36) there exists $n\in\mathscr{L}_v$ with $\psi n=m_i$. Choose a local curve $j\colon W\to V$ through v such that $jW\subset N(d_i)$. As the coordinates of m_i have no common factor, we know $j^*m_i\neq 0$, and $j^*n\neq 0$. We may divide out a power of a local coordinate at 0 on W and obtain $n_1\in(\mathscr{L}_j)_0$ such that $n_1(0)\neq 0$. As $\psi n_1\in(\mathscr{B}_j)_0$ is a multiple of j^*m_i, we see that $\psi n_1(0)\in\bigoplus_{\ell=b+1}^{a}\mathbb{C}\,\varepsilon_\ell$, and $\psi n_1(0)\in B_j(0)\subset B[v]$. So $\psi n_1(0)=0$, in contradiction with $\ker\psi[v]=0$.

(4.41) **Lemma.** *Let $v\in V_1$; if $n=2$ suppose $a=b$ or $\ker\psi[v]=0$. There exists a neighbourhood U of v such that $B\in\mathbf{B}^b(\mathbb{C}^a,U)$.*

Proof. Take U such that the d_i of lemma (4.40) satisfy $d_i\in\mathcal{O}(U)^*$. Then $m_1(u),\dots,m_b(u)$ form a basis of $B[u]=B(u)$ for each $u\in U$. If $U_1\subset U$ and $f\in\mathscr{B}(U_1)$, then

b

Remark that $\langle \varepsilon_\ell, \hat{\varepsilon}_i \rangle = \delta_{\ell,i}$ for a suitable choice of $\hat{\varepsilon}_1, \ldots, \hat{\varepsilon}_a \in \mathbb{C}^a$. So $\langle f(u), \hat{\varepsilon}_i \rangle = d_i(u)\,\eta_i(u)$ for $u \in U_1$ if $i = 1, \ldots, b$. This shows that $\eta_1, \ldots, \eta_b \in \mathcal{O}(U_1)$.

(4.42) *Proof* of proposition (4.33). i b) has been proved in lemma (4.37). As $V \setminus S \subset V_1$ we see from lemma (4.41) that $B \in \mathbf{B}^b(\mathbb{C}^a, V \setminus S)$. In iii) of lemma (4.36) we see that ψ: $\mathscr{L} \to \mathscr{B}$ is an isomorphism of $\mathcal{O}$-modules on $V \setminus S$. As $\ker \psi = 0$ on $V \setminus S$ we need to show that ψ induces isomorphisms $\mathscr{L}_j \to \mathscr{B}_j$ for each j: $W \to V_0$ holomorphic. This follows from proposition (4.31). So part i) has been proved.

ii a) and iii a) follow from i) and ii) of lemma (4.36) and i) of lemma (4.38).

For $s \in \tilde{S}$ consider the diagram.

$$\begin{array}{ccc} \mathscr{L}_s & \xrightarrow{\ \psi\ } & \mathscr{B}_s \\ \downarrow & & \downarrow \\ L(s) & \xrightarrow{\ \psi[s]\ } & B(s) \\ \cup & & \cup \\ L[s] & \xrightarrow{\ \psi[s]\ } & B[s] \end{array}$$

We have seen that $\mathscr{L}_s \to \mathscr{B}_s$ is bijective. $L[s] \to B[s]$ is surjective by definition; so are $\mathscr{L}_s \to L(s)$ and $\mathscr{B}_s \to B(s)$. From this follows that $\psi[s]$: $L(s) \to B(s)$ is surjective.

In the case $n = 1$ lemma (4.39) shows $B(s) = B[s]$, and $L(s) \cap \ker \psi[s] = 0$ by lemma (4.37). This implies ii c) and the bijectivity of $\psi[s]$: $L(s) \to B(s)$. From lemma (4.41) follows ii d) and $\dim B[s] = \dim B(s) = \dim L(s) = b$.

In the case $n = 2$ we have iii d) by lemma (4.41), and also $B(s) = B[s]$ with dimension b. Take j as in ii) of lemma (4.39); then $L(s) \cap \ker \psi[s] \subset L_j(0) \cap \ker \psi[s] = 0$. So $\psi[s]$: $L(s) \to B(s)$ is bijective, hence iii b) and c) are clear.

(4.43) *Notations*. In the next sections we shall often apply the results of this section on subsets of $\mathbb{C}^2$. We put

$$I = (-12, 12), \qquad I^* = I \setminus \{0\}.$$
$$Y = \{(r, s) \in \mathbb{C}^2 \colon \ \mathrm{Re}\, r \in I\}$$
$$Y^* = \{(r, s) \in \mathbb{C}^2 \colon \ \mathrm{Re}\, r \in I^*\}$$
$$\sigma, \rho \colon \ Y \to Y \text{ are defined by } \begin{cases} \sigma(r, s) = (r, -s) \\ \rho(r, s) = (-r, s). \end{cases}$$

$\sigma^\#$ and $\rho^\#$ act on ${}^2\mathcal{O}$ and ${}^2\mathcal{M}$.

Put $\qquad (a)' = (a), \quad (w^\nu)' = (w^{-\nu})$; let $(c) = (a)$ or (w^ν),

$B = F((c); Y, \frac{1}{4} - s^2, r)$

For $U \subset Y$ open we have

$$\sigma^*: \mathscr{B}(\sigma U) \to \mathscr{B}(U) \text{ by } \sigma^* f = f \circ \sigma$$
$$\rho^*: \mathscr{B}(\rho U) \to \mathscr{B}'(U) \text{ by } \rho^* f(-r, s) = J f(r, s),$$

see (2.3).

(4.44) *Remark.* If all K_i, for i in some index set Q, are linear spaces, and D_i is the antidual of K_i, then $\prod_{i\in Q} D_i$ is the antidual of $\bigoplus_{i\in Q} K_i$.

Let $M_i \in \mathbf{S}(D_i, V)$, then $\prod_{i\in Q} M_i \in \mathbf{S}(\prod_{i\in Q} D_i, V)$ is defined by

$$(\prod_{i\in Q} M_i)[v] = \prod_{i\in Q} M_i[v].$$

If all $M_i \in \mathbf{H}(D_i, V)$ then $\prod_{i\in Q} M_i \in \mathbf{H}(\prod_{i\in Q} D_i, V)$.

If $N \in \mathbf{S}(D, V)$ and all χ_i: $N \to M_i$ are holomorphic morphisms on V, then $\prod_{i\in Q} \chi_i$, defined by

$$((\prod_{i\in Q} \chi_i)[v] f)_\ell = \chi_\ell[v] f_\ell, \quad \text{for } \ell \in Q,$$

is a holomorphic morphism $N \to \prod_{i\in Q} M_i$.

5. Whittaker Families

(5.1) *Notation.* For $\nu \in \mathbb{Z}$ put $W^\nu = F((w^\nu); Y, \frac{1}{4} - s^2, r) \in \mathbf{S}(D_\mathfrak{h}, Y)$. This is in agreement with the notation in (2.6).

(5.2) The results in (2.13) imply that $\mu^\nu \in \mathscr{M} \otimes \mathscr{W}^{\sim\nu}(Y)$, with $\mathrm{Sing}(\mu^\nu) = \bigcup_{\ell \geq 1} \{(r, -\frac{1}{2}\ell)\colon \mathrm{Re}\, r \in I\}$, and $(r, s) \mapsto (s + \frac{1}{2}\ell) \cdot \mu^\nu(r, s)$ is holomorphic on a neighbourhood of $\{(r, -\frac{1}{2}\ell)\colon \mathrm{Re}\, r \in I\}$.

(5.3) **Definition.** For $\nu \in \mathbb{Z}$ define Wr^ν: $W^\nu \times W^\nu \to \mathbb{C}^1$ by

$$\mathrm{Wr}^\nu[r, s](f, g) = f(iy) \frac{\partial}{\partial y} g(iy) - \left(\frac{\partial}{\partial y} f(iy)\right) g(iy).$$

This does not depend on $y > 0$. From proposition (4.8) one sees that Wr^ν is a holomorphic morphism on Y.

If $\mathrm{Wr}^\nu(f, g) \neq 0$ in $\mathscr{O}(U)$ for some $f, g \in \mathscr{W}^{\sim\nu}(U)$, then f and g are $\mathscr{M}(U)$-linearly independent.

(5.4) **Proposition.** i) $W^\nu \in \mathbf{B}^2(D_\mathfrak{h}, Y)$

ii) $\mathscr{M} \otimes \mathscr{W}^{\sim\nu} = \mathscr{M} \cdot \mu^\nu \oplus \mathscr{M} \cdot \sigma^* \mu^\nu$ *on* Y.

Proof. Apply proposition (4.33) with

$\psi[r,s]\,f=\left(f(i\alpha),\dfrac{\partial}{\partial y}f(i\alpha)\right)$ for some $\alpha>0$ (see proposition (4.8)), $\beta_1=\mu^\nu$ and $\beta_2=\sigma^*\mu^\nu$ (see (5.2) and vi) of (2.13)).

As condition (*e*) amounts to a regular second order differential equation on $(0,\infty)$ we see that $\ker\psi[r,s]=0$ and also that $\psi\beta_1$ and $\psi\beta_2$ are $\mathcal{M}(Y)$-linearly independent.

Part i) of proposition (4.33) gives the desired assertions.

(5.5) Put $Y^\nu=Y$ if $\nu\neq 0$ and $Y^0=Y^*$. We see that $\omega^\nu=v^\nu\mu^\nu+\sigma^*(v^\nu\mu^\nu)\in\mathcal{M}\otimes\mathcal{W}^\nu(Y^\nu)$ gives an element of $\mathcal{W}^\nu(Y^\nu)$, see iii) of (2.13).

$f\mapsto \mathrm{Wr}^\nu(f,\omega^\nu)$ gives a holomorphic morphism $W^\nu\to\mathbb{C}^1$ on Y^ν; put ${}^\circ W^\nu=\ker(\mathrm{Wr}^\nu(\cdot,\omega^\nu))$. So ${}^\circ W^\nu\in\mathbf{H}(D_{\mathfrak{h}},Y^\nu)$. In fact ${}^\circ W^\nu\in\mathbf{B}^1(D_{\mathfrak{h}},Y^\nu)$, with ω^ν as global generator.

(5.6) Some formulas:

$$\rho^*\mu^\nu=\mu^{-\nu},\qquad \rho^*\omega^\nu=\omega^{-\nu},\qquad \sigma^*\mu^\nu=\omega^\nu.$$

(5.7) For several regions we give a local basis of W^ν. Taking a coordinate with respect to a local basis is always a holomorphic morphism $W^\nu\to\mathbb{C}^1$, given by

$$f\mapsto\frac{\mathrm{Wr}^\nu(f,\varepsilon_2)}{\mathrm{Wr}^\nu(\varepsilon_1,\varepsilon_2)}$$

i) $\mu^\nu,\sigma^*\mu^\nu$ on $\{(r,s)\in Y\colon 2s\notin\mathbb{Z}\}$, $\mathrm{Wr}^\nu(\mu^\nu,\sigma^*\mu^\nu)=-2s$.

ii) μ^ν,ω^ν on $\{(r,s)\in Y^\nu\colon s\notin-\frac{1}{2}\mathbb{N},\ (r,s)\notin N(\sigma^* v^\nu)\cup\mathrm{Sing}(\sigma^* v^\nu)\}$,
$\mathrm{Wr}^\nu(\mu^\nu,\omega^\nu)(r,s)=-2s\,v^\nu(r,-s)$.

iii) For $\ell\in\mathbb{N}$, $\ell\neq 0$, put $w_\ell^\nu(r,s)=\dfrac{\Gamma(-2s)\,\Gamma(\frac{1}{2}+s-\frac{1}{2}r)}{\Gamma(2s)\,\Gamma(\frac{1}{2}-s-\frac{1}{2}r)}(4\pi(\nu+\frac{1}{12}r))^\ell$ and $\zeta_\ell^\nu=\sigma^*\mu^\nu+w_\ell^\nu\mu^\nu$. Then ζ_ℓ^ν,μ^ν form a local basis near $(r,\frac{1}{2}\ell)$, $\mathrm{Wr}^\nu(\zeta_\ell^\nu,\mu^\nu)=2s$.

6. Eisenstein and Poincaré Series

(6.1) In this section $r\in I$ is fixed.

(6.2) **Definition.** The classes of locally square integrable functions f on $\mathfrak{h}$, satisfying $(a)_r$ in (2.4) and

$$\int_{\Gamma\backslash\mathfrak{h}}|f|^2y^{-2}\,dx\,dy<\infty$$

form a Hilbert space H_r. By $\langle\cdot,\cdot\rangle$ we denote its scalar product.

(6.3) L_r induces in H_r a selfadjoint operator A_r, compare e.g. [2], § 5.

If $r=0$ the spectrum of A_r has a continuous and a discrete part; if $r\in I^*$ only the discrete part is present.

(6.4) Eigenfunctions f of A_r with eigenvalue λ are characterized by

This implies that

$$\ker(A_r - \tfrac{1}{4} + s^2) = S^0[r,s] \quad \text{if } r \in I^*.$$
$$\ker(A_0 - \tfrac{1}{4} + s^2) \supset S^{(0)}[0,s]$$

Actually it is known that the inclusion is an equality, except that $\ker(A_0)$ consists of the constant functions, see [8], thm. (5.2.3).

(6.5) **Proposition.** *Let* $\nu \in \mathbb{Z}$, $\operatorname{Re} s > \frac{1}{2}$, $r \in I$.

i) $P_r^\nu(s;z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} v_r(\gamma)^{-1} e^{-ir\arg(c_\gamma z + d_\gamma)} \mu^\nu(r,s;\gamma z)$

converges absolutely, uniformly for (r,s,z) *in compact sets.*

ii) *The sum in* i) *defines* $P_r^\nu(s) \in A[r,s]$; $z \mapsto P_r^\nu(s;z) - \mu^\nu(r,s;z)$ *is bounded on* F.

iii) *For* $\frac{1}{4} - s^2$ *not in the spectrum of* A_r *the properties in* ii) *determine* $P_r^\nu(s)$ *uniquely.*

iv) $P_r^\nu(s;z)$ *is continuous in* $(r,s;z)$, *holomorphic in* s.

Remark. These facts are well known.

$$P_0^0(s;z) = \frac{1}{2\zeta(2s+1)} \sum_{n,m}' \left(\frac{y}{|nz+m|^2}\right)^{\frac{1}{2}+s},$$

the well known Eisenstein series for weight 0. The Poincaré series $P_0^\nu(s)$, $\nu \neq 0$, have been studied in [11].

Proof. Well known. For $\sigma > \frac{1}{2}$ the series

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma, \gamma \notin \Gamma_\infty} \operatorname{Im}(\gamma z)^{\frac{1}{2}+\sigma}$$

converges with bounded sum for $z \in F$. As

$$\mu^\nu(r,s;z) \ll y^{\frac{1}{2}+\operatorname{Re} s} \quad \text{for } y \downarrow 0,$$

the proposition follows easily.

(6.6) **Lemma.** *Let* $a > 1$. *Define* ${}^aP_r^\nu(s)$: $F \to \mathbb{C}$, *with* $r \in I$, $\nu \in \mathbb{Z}$, $\operatorname{Re} s > \frac{1}{2}$, *by*

$${}^aP_r^\nu(s;z) = \begin{cases} P_r^\nu(s;z) & \text{if } y \leqq a \\ P_r^\nu(s;z) - \mu^\nu(r,s;z) & \text{if } y > a. \end{cases}$$

Let $f \in C^\infty(\mathfrak{h})$ *satisfy* $(a)_r$ *and determine an element of* H_r. *Then*

$$\int_{\Gamma \backslash \mathfrak{h}} {}^aP_r^\nu(s;z) \overline{f(z)} y^{-2} dx\, dy = \int_0^a \mu^\nu(r,s;iy) \overline{F_\nu f(iy)} y^{-2} dy.$$

Proof. ${}^aP_r^\nu(s;z)$ is given by a series like the one in i) of proposition (6.5), with μ^ν truncated. As ${}^aP_r^\nu(s) \in H_r$, the first integral converges. Interchange the order of

7. Meromorphic Continuation

(7.1) *Notation.* $A = F((a); Y, \frac{1}{4} - s^2, r) \in \mathbf{H}(D_\mathfrak{h}, Y)$, see (4.21).

(7.2) **Definition.** For $\nu \in \mathbb{Z} \setminus \{0\}$ the holomorphic morphism $\chi_\nu: A \to \mathbb{C}^1$ is defined by $\chi_\nu f = \mathrm{Wr}^\nu(F_\nu f, \omega^\nu)$, see proposition (4.24) and definition (5.3).

For $0 \in B \subset \mathbb{Z}$, B finite, define $A^B \in \mathbf{H}(D_\mathfrak{h}, Y)$ by

$$A^B = \ker\Big(\prod_{\nu \in \mathbb{Z} \setminus B} \chi_\nu: A \to \prod_{\nu \in \mathbb{Z} \setminus B} \mathbb{C}^1\Big);$$

see prop. (4.26) and remark (4.44).

Put $F_B = \prod_{\nu \in B} F_\nu: A^B \to \prod_{\nu \in B} W^\nu$.

(7.3) **Definition.** For $0 \in B \subset \mathbb{Z}$, B finite, define $S^B \in \mathbf{H}(D_\mathfrak{h}, Y)$ by

$$S^B = \ker\Big(F_B: A^B \to \prod_{\nu \in B} W^\nu\Big).$$

Define $S^0 \in \mathbf{H}(D_\mathfrak{h}, Y^*)$ by $S^0 = \ker\big(\prod_{\nu \in \mathbb{Z}} \chi_\nu: A \to \prod_{\nu \in \mathbb{Z}} \mathbb{C}^1\big)$.

(7.4) **Proposition.** *Let $0 \in B \subset \mathbb{Z}$, B finite, $a > 5$. There exist an open subset Y_B of Y and an analytic subset S_B of Y_B with dimension 1 with the following properties:*

i) *$Y_B = W_B \times \mathbb{C}$ with $I \subset W_B$, W_B open in $\mathbb{C}$.*
ii) *If $(r, s) \in Y_B \setminus S_B$, then $S^B[r, s] = 0$.*
iii) *S_B intersects each line $\{r\} \times \mathbb{C}$, with $r \in W_B$, discretely.*
iv) *Put $b = |B|$. There is a $\mathbb{C}$-linear map $Q: \mathbb{C}^b \to \mathcal{M} \otimes \mathcal{A}^B(Y_B)$ such that*

a) *$Q(\xi) \in \mathcal{A}^B(Y_B \setminus S_B)$ for each $\xi \in \mathbb{C}^b$.*
b) *If $\xi = (\xi_\nu) \in \mathbb{C}^b$, $(r, s) \in Y_B \setminus S_B$, then $F_\nu Q(\xi; r, s; i a) = \xi_\nu \cdot \mu^\nu(r, s; i a)$ for $\nu \in B$.*
c) $F \ni z \mapsto \begin{cases} Q(\xi; r, s; z) & \text{if } y \leqq a \\ Q(\xi; r, s; z) - \sum_{\nu \in B} F_\nu Q(\xi; r, s; z) & \text{if } y > a \end{cases}$

defines for each $\xi \in \mathbb{C}^b$ a family of elements of $L^2(F, y^{-2}\,dx\,dy)$ holomorphic on $Y_B \setminus S_B$, meromorphic on Y_B.

Remark. This is part of propositions (2.6) and (2.8) in [2]. There the $Q(\xi)$ are described as distributions on $Y_B \setminus S_B$; we have used lemma (4.15) to obtain meromorphy on Y_B.

Remark. The statement in iv b) is only meaningful if $s \notin -\frac{1}{2}\mathbb{N}$. We suppose $W_B \times (-\frac{1}{2}\mathbb{N}) \subset S_B$.

Remark. Q, S_B and Y_B depend on B and on a; this is not fully shown in the notation.

Remark. We use the elements of B as indices for the coordinates in $\mathbb{C}^b$.

(7.6) **Proposition.** *There exist an open subset* $Y_0^* = W_0^* \times \mathbb{C}$ *of* Y^* *with* $I^* \subset W_0^*$ *and an analytic subset* S_0 *of* Y_0^* *of dimension one, intersecting lines* $\{r\} \times \mathbb{C}$ *discretely and satisfying:*

$$\text{if } (r,s) \in Y_0^* \setminus S_0^*, \text{ then } S^0[r,s] = 0.$$

Proof. This has not been explicitly stated in proposition (2.5) of [2], but follows immediately from the proof.

(7.7) *Notation.* Define $C^B \in \mathrm{S}(\prod_{v\in B} W^v, Y)$ by $C^B[r,s] = F_B A^B[r,s]$.

(7.8) **Proposition** (Maass-Selberg relation). *If* $f, g \in A[r,s]$, *then*

$$\sum_{v\in\mathbb{Z}} \mathrm{Wr}^v(F_v f, F_v g)$$

converges absolutely and is equal to zero.

Proof. The usual one, compare [12], II, § 9, p. 287.

The idea is to integrate around a cut off fundamental domain the closed Γ-invariant differential form

$$E_r^+ f \cdot J g \cdot y^{-1} dz - f \cdot E_{-r}^- J g \cdot y^{-1} d\bar{z},$$

with

$$E_q^{\pm} = \pm 2 i y \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y} \pm q.$$

(7.9) **Corollary.** $\dim C^B[r,s] \leqq |B|$ *for all* $(r,s) \in Y$.

Proof. If $f, g \in A^B[r,s]$ the sum in proposition (7.8) is finite. So $C^B[r,s]$ is orthogonal to itself for a non-degenerate bilinear antisymmetric form on $\prod_{v\in B} W^v[r,s]$. So $\dim C^B[r,s] \leqq \frac{1}{2} \cdot |B| \cdot 2$.

(7.10) **Proposition.** *Let* $0 \in B \subset \mathbb{Z}$, $b = |B| < \infty$.

i) F_B: $A^B \to C^B$ *is a holomorphic isomorphism on* $Y_B \setminus S_B$.

ii) $A^B \in \mathbf{B}^b(D_b, Y_B \setminus S_B)$.

iii) *Let* $\delta^v \in \mathbb{C}^b$ *be defined by* $(\delta^v)_\lambda = \begin{cases} 0 & \text{if } v \neq \lambda \\ 1 & \text{if } v = \lambda. \end{cases}$

$$\mathscr{A}^B = \bigoplus_{v\in B} \mathscr{M} \cdot Q(\delta^v) \quad \text{on } Y_B.$$

Proof. Apply part i) of proposition (4.33) with: V an open subset of Y_B such that $W^v \cong \mathbb{C}^2$ on V, see i) of proposition (5.4); $n = 2$, $S = V \cap S_B$; $a = 2b$, $b = b$; $L = A^B$, $\psi = F_B$; $\beta_v = Q(\delta^v)$ for $v \in B$.

(7.11) **Lemma.** *Let* $\mathrm{Re}\, r \in I$, *define* j_r: $\mathbb{C} \to Y$ *by* $j_r s = (r,s)$.

i) $\mathscr{S}_{j_r}^0 = 0$ *for each* $r \in W_0^*$.

ii) *If* $f \in (1 \ldots$

Proof. Part i) follows directly from proposition (7.6). If $f \neq 0$ in ii), one would have a nonzero square integrable $f(s) \in A[0,s]$ for all s in an open set with $\operatorname{Re} s > 0$, which is impossible by the selfadjointness of A_0. Remark that

$\sigma^* \mu^0(0,s;z) = y^{\frac{1}{2}-s}$.

(7.12) **Theorem.** *Let* $0 \in B \subset \mathbb{Z}$; *$B$ finite. Put* $b = |B|$.

i) *There are an open subset Y_B of Y, containing I, and a unique $\mathcal{M}$-morphism*

$$E:\ \mathcal{M}^b \to \mathcal{M} \otimes \mathcal{A}^B \quad \text{on } Y_B$$

such that for all sections $\phi \in \mathcal{M}^b(U)$ *with* $U \subset Y_B$:

$$F_0\, E(\phi) \in \phi_0\, \mu^0 + \mathcal{M}(U) \cdot \sigma^* \mu^0$$
$$F_\nu\, E(\phi) \in \phi_\nu\, \mu^\nu + \mathcal{M}(U) \cdot \omega^\nu \quad \text{for } \nu \in B,\ \nu \neq 0.$$

ii) $E:\ \mathcal{M}^b \to \mathcal{M} \otimes \mathcal{A}^B$ *is an isomorphism of $\mathcal{M}$-modules.*

iii) *For each* $\xi \in \mathbb{C}^b$

$$\operatorname{Sing}(E(\xi)) \cap (\{0\} \times \mathbb{C})$$

is a discrete set.

iv) *For* $\kappa \in \mathbb{Z}$ *define the $\mathcal{M}$-morphism*

$$C_\kappa:\ \mathcal{M}^b \to \mathcal{M} \quad \text{on } Y_B$$

by

$$F_0\, E(\phi) = \phi_0\, \mu^0 + C_0(\phi) \cdot \sigma^* \mu^0 \qquad \text{if } \kappa = 0$$

$$F_\kappa\, E(\phi) = \begin{cases} \phi_\kappa\, \mu^\kappa + C_\kappa(\phi)\, \omega^\kappa & \text{if } \kappa \in B \setminus \{0\} \\ C_\kappa(\phi)\, \omega^\kappa & \text{if } \kappa \in \mathbb{Z} \setminus B. \end{cases}$$

Put $Y_B^* = Y_B \cap Y^*$.
Then $1 - v^0(\sigma^* v^0)^{-1}\, C_0(\delta^0) \in \mathcal{M}(Y_B^*)$ *is nonzero, and for each* $r \in I^*$ *and* $\xi \in \mathbb{C}^b$

$$\operatorname{Sing}\left(E\left(\xi + \frac{v^0\, C_0(\xi)}{\sigma^* v^0 - v^0\, C_0(\delta^0)}\, \delta^0\right)\right) \cap (\{r\} \times \mathbb{C})$$

is discrete.

Remark on the notation: We use the elements of B as indices for the coordinates in $\mathbb{C}^b$ and $\mathcal{M}^b$.

Remark. The set Y_B may be chosen equal to the set Y_B in proposition (7.4).

Proof. Consider the diagram:

$$\begin{array}{ccc} \mathcal{M}^b & \xrightarrow{\ Q\ } & \mathcal{M} \otimes \mathcal{A}^B \\ & \searrow^{Z} & \downarrow q \\ & & \mathcal{M}^b \end{array} \qquad \text{on } Y_B.$$

$$q_0 f = \frac{\mathrm{Wr}^0(F_0 f, \sigma^* \mu^0)}{\mathrm{Wr}^0(\mu^0, \sigma^* \mu^0)}$$

$$q_\nu f = \frac{\mathrm{Wr}^\nu(F_\nu f, \omega^\nu)}{\mathrm{Wr}^\nu(\mu^\nu, \omega^\nu)} \quad \text{if } \nu \in B,\ \nu \neq 0.$$

q is a $\mathcal{M}$-morphism. Z is the composition qQ.

We shall show that Z is an isomorphism of $\mathcal{M}$-modules; then $E = QZ^{-1}$ is the unique morphism satisfying i).

After fixing coordinates in $\mathcal{M}^b$ we may describe Z by a $b \times b$ matrix with coefficients in $\mathcal{M}(Y_B)$. Each element e of this matrix satisfies

$$\mathrm{Sing}(e) \cap (\{r\} \times \mathbb{C}) \quad \text{is discrete}$$

for each $r \in W_B$, see proposition (7.4).

To prove that Z is an isomorphism, we show that $j_0^* \det Z \neq 0$.

Suppose $j_0^* \det Z = 0$. Take $\phi_1 \in {}^1\mathcal{M}^b(\mathbb{C})$, $\phi_1 \neq 0$, such that $(j_0^* Z)\phi_1 = 0$. Define $\phi \in \mathcal{M}^b(Y_B)$ by $\phi(r,s) = \phi_1(s)$. Now $j_0^* Q(\phi)$ is well defined and

$$F_0 j_0^* Q(\phi) \in {}^1\mathcal{M}(\mathbb{C}) \cdot j_0^* \sigma^* \mu^0$$
$$F_\nu j_0^* Q(\phi) \in {}^1\mathcal{M}(\mathbb{C}) \cdot j_0^* \omega^\nu \qquad \text{for } \nu \in B,\ \nu \neq 0.$$

So $j_0^* Q(\phi) = 0$ by ii) of lemma (7.11). As $j_0^* \mu^\nu(ia) \neq 0$, this would imply $j_0^* \phi = \phi_1 = 0$. Hence $j_0^* \det Z \neq 0$.

So i) is clear, ii) follows immediately and so does iii), for Z^{-1} does not introduce singularities along $\{0\} \times \mathbb{C}$.

Put $E^0 = E(\delta^0)$. Then

$$F_0 E^0 = (1 - v^0(\sigma^* v^0)^{-1} C_0(\delta^0))\, \mu^0 + (\sigma^* v^0)^{-1} C_0(\delta^0)\, \omega^0$$

on Y_B^*. So i) of lemma (7.11) implies that $1 - v^0(\sigma^* v^0)^{-1} C_0(\delta^0)$ cannot be zero along any line $\{r\} \times \mathbb{C}$ with $r \in W_0^*$. So $1 - v^0(\sigma^* v^0)^{-1} C_0(\delta^0) \neq 0$.

Let the $\mathcal{M}$-morphism $\pi\colon \mathcal{M}^b \to \mathcal{M}^b$ on Y_B^* be defined by

$$\pi \phi = \phi + \frac{v^0 \cdot C_0(\phi)}{\sigma^* v^0 - v^0 C_0(\delta^0)}\, \delta^0,$$

and let $q^*\colon \mathcal{M} \otimes \mathcal{A}^B \to \mathcal{M}^b$ on Y_B^* be defined by

$$(q^* f)_\nu = \frac{\mathrm{Wr}^\nu(F_\nu f, \omega^\nu)}{\mathrm{Wr}^\nu(\mu^\nu, \omega^\nu)} \quad \text{for } \nu \in B.$$

Then a computation shows that $q^* E \pi\colon \mathcal{M}^b \to \mathcal{M}^b$ on Y_B^* is the identity. (Of course π has been chosen to obtain this.)

Suppose $r_0 \in I^*$ and $\mathrm{Sing}(E\pi\xi) \supset \{r_0\} \times \mathbb{C}$ for some $\xi \in \mathbb{C}^b$. Take $n \geqq 1$, $\phi(r,s) = (r - r_0)^n \xi$, such that $E(\pi\phi)$ is not singular, but nonzero, along $\{r_0\} \times \mathbb{C}$. Then $j_{r_0}^* E(\pi\phi)$ contradicts i) of lemma (7.11), for

(7.13) **Corollary** (Functional equations). *In the notations of theorem* (7.12):

i) *Define the morphism of* $\mathcal{M}$*-modules* $\hat{\sigma}: \mathcal{M}^b \to \mathcal{M}^b$ *on* Y_B *by*

$$(\hat{\sigma}\,\phi)_\nu = \begin{cases} \sigma^*(C_0(\sigma^*\,\phi)) & \text{if } \nu=0 \\ -v^\nu(\sigma^*\,v^\nu)^{-1}\,\phi_\nu & \text{if } \nu\in B,\ \nu\neq 0. \end{cases}$$

a) $E\,\hat{\sigma}^* = \sigma^*\,E\,\hat{\sigma}\colon\ \mathcal{M}^b(\sigma U)\to \mathcal{M}\otimes\mathcal{A}^B(U)$ *for* $U\subset Y_B$.

b) $C_0(\delta^0)\cdot\sigma^*\,C_0(\delta^0)=1$.

ii) *Let* $\mathcal{M}^{(b)}$ *be* $\mathcal{M}^b$ *with the coordinates indexed by* $-B$. *Define the* $\mathcal{M}$-*morphism* $\hat{\rho}\colon \mathcal{M}^{(b)}\to\mathcal{M}^b$ *by*

$$(\hat{\rho}\,\phi)_\nu = \phi_{-\nu} \qquad \text{for } \nu\in B.$$

a) $E\,\rho^* = \rho^*\,E\,\hat{\rho}\colon\ \mathcal{M}^{(b)}(\rho U)\to\mathcal{M}\otimes\mathcal{A}^B(U)$ *for* $U\subset Y_B$, $\rho U\subset Y_{-B}$.

b) $C_0\,\rho^* = \rho^*\,C_0\,\hat{\rho}$.

Proof. The parts a) follows from the unicity; use (5.6). For the parts b) one uses the definition of C_0; for instance i b):

$$\begin{aligned} \mu^0 + C_0(\delta^0)\cdot\sigma^*\,\mu^0 &= F_0\,E\,\delta^0 \\ &= F_0\,E\,\sigma^*\,\delta^0 = F_0\,\sigma^*\,E\,\hat{\sigma}\,\delta^0 \\ &= \sigma^*\,F_0\,E(\sigma^*\,C_0(\sigma^*\,\delta^0)\cdot\delta^0) \\ &= C_0(\delta^0)\cdot\sigma^*(\mu^0 + C_0(\delta^0)\,\sigma^*\,\mu^0) \\ &= C_0(\delta^0)\cdot\sigma^*\,C_0(\delta^0)\cdot\mu^0 + C_0(\delta^0)\cdot\sigma^*\,\mu^0. \end{aligned}$$

(7.14) *Proof* of proposition (2.19). Take $B=\{0,\nu\}$, $Y(\nu)=Y_B$, $E^\nu=E(\delta^\nu)$, $C^\nu_\lambda = C_\lambda(\delta^\nu)$. Almost everything may be read off directly from (7.12) and (7.13). Use:

$$\sigma^*\,\delta^\lambda=\delta^\lambda,\qquad \hat{\sigma}\,\delta^0=\sigma^*\,C^0_0\cdot\delta^0,\qquad \hat{\sigma}\,\delta^\nu=\sigma^*\,C^\nu_0\cdot\delta^0 - v^\nu(\sigma^*\,v^\nu)^{-1}\,\delta^\nu \qquad \text{if } \nu\neq 0;$$

for ρ similarly.

For the last assertion in ii b) use unicity and ii) of (2.13), ii) of (2.8).

(7.15) *Proof* of proposition (2.20). Take $Y^*(\nu)=Y(\nu)\cap Y^*$ and $P^\nu=E\,\pi\,\delta^\nu$, with π as in the proof of theorem (7.12). As $q^*\,E\,\pi$ in the proof of (7.12) is the identity, i) of proposition (2.20) follows. Part ii) is mainly a consequence of corollary (7.13), but it is more easily proved directly, with use of unicity.

(7.16) *Proof* of proposition (2.23). Let $r\in I$. By iii) and iv) of theorem (7.12), the restrictions $j_0^\#\,E^\nu$, if $r=0$, and $j_r^\#\,P^\nu$, if $r\in I^*$, are meromorphic on $\mathbb{C}$, and are uniquely determined by the not square integrable part of their Fourier expansion. So they coincide with the P^ν_r defined by a series, at first on a open set, and then by meromorphy on $\mathrm{Re}\,s>\frac{1}{2}$. So in view of proposition (6.5), the major part of proposition (2.23) has been proved. Only the holomorphy of P^ν at (r_0, s_0), $r_0\in I^*$, $\mathrm{Re}\,s_0>\frac{1}{2}$, still needs to be established.

(4.10). If $(r_0, s_0) \in \mathrm{Indet}(P^v)$, then there is a local curve $j: W \to Y^*(v)$ through (r_0, s_0) with $jW \subset \mathrm{Sing}(P^v)$. As $jW \subset \{r_0\} \times \mathbb{C}$ is impossible, j has the form $jw = (r_0 + w^q, s_0 + h(w))$, see (3.6). There are nonzero $w \in W$ such that $r = r_0 + w^q \in I^*$ and $\mathrm{Re}(s_0 + h(w)) > \frac{1}{2}$. This means $(r, s_0 + h(w)) \in \mathrm{Sing}(P^v)$, but $(r, s_0 + h(w)) \notin \mathrm{Indet}(P^v)$ if W is small enough, for $\mathrm{Indet}(P^v)$ consists of isolated points. So P^v has to be holomorphic at (r_0, s_0).

(7.17) **Lemma.** *Let $s_0 \in \mathbb{C}$ and suppose $(0, s_0) \in \mathrm{Sing}(E^v)$. Then there is a local curve $j: W \to Y(v): w \to (w^q, s_0 + h(w))$ through $(0, s_0)$ such that for all $w \in W$, $w \neq 0$:*

$$j\,w \in \mathrm{Sing}(E^v) \setminus \mathrm{Indet}(E^v).$$

Proof. As in (7.16).

(7.18) **Lemma.** *Let $v \in \mathbb{Z}$, $\varepsilon \in (0, \frac{1}{2})$ and let $r \mapsto s(r)$ be a continuous map $(-\varepsilon, \varepsilon) \to \mathbb{C}$.*

i) $\lim_{r \to 0} (\sigma^* v^0 \cdot D_0^v)(r, s(r)) = \lim_{s \to s(0)} C_0^v(0, s)$ *if* $s(0) > \frac{1}{2}$.

ii) *If* $s(0) \notin \frac{1}{2}\mathbb{Z}$, $s(0) > 0$, *then* $\lim_{r \to 0} (v^0 \cdot (\sigma^* v^0)^{-1})(r, s(r)) = 0$.

iii) *If* $s(0) = \frac{1}{2}\ell$, $\ell \in \mathbb{N}$, $\ell \neq 0$, *then* $\lim_{r \to 0} (v^0 \cdot (\sigma^* v^0)^{-1} - w_\ell^0)(r, s(r)) = 0$.

Remark. In i) we cannot write $C_0^v(0, s(0))$, as C_0^v might be indeterminate at $(0, s(0))$.

Proof. From proposition (6.5) follows that $F_0 P_r^v(s, r)$ is continuous in $(r, s; z)$, for $\mathrm{Re}\, s > \frac{1}{2}$. So we obtain continuity of

$$(r,s) \mapsto \begin{cases} y^{\frac{1}{2}-s} \lim_{s_1 \to s} C_0^v(0, s_1) & \text{if } r = 0 \\ D_0^v(r, s)\, \omega^0(r, s; i\,y) & \text{if } r \neq 0. \end{cases}$$

We have, see e.g. (6.7.5) in [1]:

$$\left(\frac{\pi |r|\, y}{3}\right)^{-\frac{1}{2}+s} \omega^0(r, s; i\,y)$$
$$= \pi^{-1} 2^{2s-1} \Gamma(\tfrac{1}{2} + s + \tfrac{1}{2}|r|) \int_{-\infty}^{+\infty} e^{-\frac{1}{3}\pi i |r| y x} (1 + x^2)^{-\frac{1}{2}-s} \cdot e^{i|r| \arctan x}\, dx.$$

This may be extended continuously in (r, s) to $(0, s)$ by giving it there the value

$$\Gamma(2s)/\Gamma(\tfrac{1}{2} + s),$$

see p. 9 of [10]. So we get continuity of

$$(r,s) \mapsto \begin{cases} \dfrac{\Gamma(\frac{1}{2}+s)}{\Gamma(2s)} \cdot \lim_{s_1 \to s} C_0^v(0, s_1) & \text{if } r = 0 \\ D_0^v(r, s) \left(\dfrac{\pi |r|}{3}\right)^{\frac{1}{2}-s} & \text{if } r \neq 0. \end{cases}$$

Part ii) follows directly from iii) in (2.13). To get iii) we take $s=\frac{1}{2}\ell+h$ and $\varepsilon = \operatorname{sign} r$:

$$\left(\frac{v^0}{\sigma^{\#} v^0}-w_\ell^0\right)(r,s)=\frac{\Gamma(-2s)}{\Gamma(2s)}\,\Gamma(\tfrac{1}{2}+s+\tfrac{1}{2}r)\,\Gamma(\tfrac{1}{2}+s-\tfrac{1}{2}r)\left(\frac{\pi|r|}{3}\right)^{\ell}$$

$$\cdot\left[\frac{\left(\frac{\pi|r|}{3}\right)^{2s-\ell}}{\Gamma(\frac{1}{2}-s-\frac{1}{2}|r|)\,\Gamma(\frac{1}{2}+s+\frac{1}{2}|r|)}-\frac{\varepsilon^{\ell}}{\Gamma(\frac{1}{2}-s-\frac{1}{2}r)\,\Gamma(\frac{1}{2}+s+\frac{1}{2}r)}\right]$$

$$=\frac{\Gamma(-\ell-2h)}{\Gamma(\ell+2h)}\,\Gamma\left(\frac{1+\ell+r}{2}+h\right)\Gamma\left(\frac{1+\ell-r}{2}+h\right)\left(\frac{\pi|r|}{3}\right)^{\ell}$$

$$\cdot\left[\left(\frac{\pi|r|}{3}\right)^{2h}\pi^{-1}\sin\pi(\tfrac{1}{2}+s+\tfrac{1}{2}|r|)-\varepsilon^{\ell}\pi^{-1}\sin\pi(\tfrac{1}{2}+s+\tfrac{1}{2}r)\right]$$

$$=\mathcal{O}(|h|^{-1}|r|^{\ell})\cdot\Big[\mathcal{O}(|h|\,|\ln|r||)$$

$$+\pi^{-1}\cos\pi s\left(\cos\frac{\pi|r|}{2}-\varepsilon^{\ell}\cos\frac{\pi r}{2}\right)-\pi^{-1}\sin\pi s\left(\sin\frac{\pi|r|}{2}-\varepsilon^{\ell}\sin\frac{\pi r}{2}\right)\Big]$$

$$=\mathcal{O}(|r|^{\ell}|\ln|r||)+\mathcal{O}(|h|^{-1}|r|^{\ell})$$

$$\cdot\,[\cos\tfrac{1}{2}\pi\ell\cdot\cos\pi h\cdot(1-\varepsilon^{\ell})\cdot\cos\tfrac{1}{2}\pi r$$

$$-\sin\tfrac{1}{2}\pi\ell\cdot\sin\pi h\cdot(1-\varepsilon^{\ell})\cdot\cos\tfrac{1}{2}\pi r$$

$$-\sin\tfrac{1}{2}\pi\ell\cdot\cos\pi h\cdot(\varepsilon-\varepsilon^{\ell})\cdot\sin\tfrac{1}{2}\pi r$$

$$-\cos\tfrac{1}{2}\pi\ell\cdot\sin\pi h\cdot(\varepsilon-\varepsilon^{\ell})\cdot\sin\tfrac{1}{2}\pi r]$$

$$=\mathcal{O}(|r|^{\ell}|\ln|r||)+\mathcal{O}(|h|^{-1}|r|^{\ell})$$

$$\cdot\begin{cases}-\cos\frac{1}{2}\pi\ell\cdot\sin\pi h\cdot(\varepsilon-1)\cdot\sin\frac{1}{2}\pi r & \text{if } \ell \text{ even}\\ -\sin\frac{1}{2}\pi\ell\cdot\sin\pi h\cdot(1-\varepsilon)\cdot\cos\frac{1}{2}\pi r & \text{if } \ell \text{ odd}\end{cases}$$

$$=\mathcal{O}(|r|^{\ell}|\ln|r||+|r|^{\ell}).$$

This gives iii) of the lemma.

(7.19) **Proposition.** *Let* $v\in\mathbb{Z}$, $s_0\in\mathbb{C}$, $\operatorname{Re} s_0>\frac{1}{2}$.

i) *If* $s_0\notin\frac{1}{2}\mathbb{Z}$, *then* E^v *is holomorphic at* $(0,s_0)$, *and* C_0^v *is holomorphic at* $(0,s_0)$.

ii) *Let* $s_0=\frac{1}{2}\ell$, $\ell\in\mathbb{N}$. *Put*

$${}^{\ell}E^v=E^v+\frac{w_\ell^0\,C_0^v}{1-w_\ell^0\,C_0^0}\,E^0.$$

Then a) $F_0\,{}^{\ell}E^v\in\delta_{v,0}\cdot\mu^0+\mathcal{M}(Y^*(v))\cdot\zeta_\ell^0$.

b) ${}^{\ell}E^v$ *is holomorphic at* $(0,\frac{1}{2}\ell)$.

Proof. Put $f=E^v$ *if* $s_0\notin\frac{1}{2}\mathbb{Z}$, $f={}^{\ell}E^v$ *if* $s_0=\frac{1}{2}\ell$. Then $f=P^v-g\,P^0$ on $Y^*(v)$, with

$$g=\frac{v^0\,D_0^v}{1+v^0\,D_0^0}\qquad\text{if } s_0\notin\tfrac{1}{2}\mathbb{Z}$$

this is obtained by expressing C_0^0 in D_0^0 and inverting (2.21). On the line $r=0$ the function f restricts to P_0^ν, for $\operatorname{Re} s>\frac{1}{2}$. Suppose $(0,s_0)\in\operatorname{Sing}(f)$, then $(0,s_0)\in\operatorname{Indet}(f)$, and $(r,s(r))\in\operatorname{Sing}(f)\setminus\operatorname{Indet}(f)$ for all $r\in(-\varepsilon,\varepsilon)$, $r\neq 0$, for some continuous function $r\mapsto s(r)$, compare lemma (7.17). As we are in the region $\operatorname{Re} s>\frac{1}{2}$, the only way f can be nonholomorphic is from g. So $(r,s(r))\in\operatorname{Sing}(g)$ for all small $r\neq 0$. The factor D_0^ν inherits the holomorphy from P^ν. So lemma (7.18) shows that the denominator of g cannot be zero for all $(r,s(r))$ near $(0,s(0))$, and that the numerator of g is not singular at those $(r,s(r))$. This proves i) and ii b); part ii a) is easily checked by a computation.

8. Proof of Proposition (2.24)

(8.1) For non-empty $B\subset\mathbb{Z}$ we define $F_B=\prod_{\nu\in B} F_\nu\colon A^B\to\prod_{\nu\in B} W^\nu$ as in (7.2). We define $C^B\in S(\prod_{\nu\in B} W^\nu, Y)$ by $C^B=F_B A^B$ as in (7.7). For $r\in I$ we write A_r^B and C_r^B instead of $A_{j_r}^B$ and $C_{j_r}^B$.

(8.2) *Notation.* Let $a>1$; $B\subset\mathbb{Z}$ finite, $0\in B$. For each section h of ${}^1\mathcal{M}\otimes\mathcal{A}_r^B$ we may form the pointwise meromorphic family of functions ${}^{(a)}h$ on F by

$$ {}^{(a)}h(s;z)=\begin{cases} h(s;z) & \text{if } y<a\\ h(s;z)-\sum_{\nu\in B} F_\nu h(s;z) & \text{if } y\geqq a.\end{cases} $$

If h is a holomorphic section, then ${}^{(a)}h$ is holomorphic.

(8.3) **Lemma.** *Let a, B as above. Let $h\in{}^1\mathcal{M}\otimes\mathcal{A}_r^B(U)$ for some $U\subset\mathbb{C}$; $r\in I$.*

i) *${}^{(a)}h$ determines a meromorphic family of elements of H_r.*

ii) *If $f\in S^B[r,s]$, $s\in\mathbb{C}$, then*

$$ \langle {}^{(a)}h, f\rangle=0. $$

Proof. From proposition (4.33) follows that $h=\sum_{\nu\in B} m_\nu\cdot j_r^* P^\nu$ if $r\neq 0$, $h=\sum_{\nu\in B} m_\nu\cdot j_0^* E^\nu$ if $r=0$, with $m_\nu\in{}^1\mathcal{M}(U)$. So we may consider only $h=j_r^* P^\nu$, $r\neq 0$, or $h=j_0^* E^\nu$. Proposition (7.4) and the construction of P^ν and E^ν from Q imply that ${}^{(a)}h$ is meromorphic in L^2-sense.

So we consider

$$ s\mapsto\langle {}^{(a)}j_r^* P^\nu(s), f\rangle \quad\text{or}\quad s\mapsto\langle {}^{(a)}j_0^* E^\nu(s), f\rangle $$

as meromorphic function on $\mathbb{C}$. It is sufficient to prove that it vanishes for $\operatorname{Re} s>\frac{1}{2}$. But for those s

$$ \langle {}^{(a)}j_r^*(P^\nu \text{ or } E^\nu)(s), f\rangle=\int_F {}^aP_r^\nu(s;z)\,\overline{f(z)}\,y^{-2}\,dx\,dy, $$

(8.4) **Lemma.** *Let* $r\in I$, $s_0\in\mathbb{C}$, $0\in B\subset\mathbb{Z}$, *B finite. Then*

$$A_r^B(s_0)\cap S^B[r,s_0]=0.$$

Proof. Let h be a section of $\mathscr{A}_r^B$ such that $h(s_0)\in S^B[r,s_0]$. Take $a>1$ and remark that ${}^{(a)}h(s_0)=h(s_0)$. Lemma (8.3) shows that $h(s_0)$ is orthogonal to itself.

Now we apply the full power of proposition (4.33), at first in the one-dimensional case.

(8.5) **Proposition.** *Let* $r\in I$, $0\in B\subset\mathbb{Z}$, *B finite; put* $b=|B|$.

a) $C_r^B\in\mathbf{B}^b(\prod_{v\in B} W_{j_r}^v,\mathbb{C})$

For each $s\in\mathbb{C}$:

b) $A^B[r,s]=A_r^B(s)\oplus S^B[r,s]$.

c) $F_B\colon A_r^B(s)\to C_r^B[s]$ *is bijective.*

d) $F_B\colon (\mathscr{A}_r^B)_s\to(\mathscr{C}_r^B)_s$ *is an isomorphism of* ${}^1\mathcal{O}_s$*-modules.*

Proof. This is part ii) of proposition (4.33); lemma (8.4) gives the last condition to be satisfied.

(8.6) **Proposition.** *Let* $r\in I$, $s\in\mathbb{C}$, $0\in B\subset\mathbb{Z}$, *B finite; put* $b=|B|$.

i) $F_B\colon \mathscr{A}_{(r,s)}^B\to\mathscr{C}_{(r,s)}^B$ *is an isomorphism of* ${}^2\mathcal{O}_{(r,s)}$*-modules.*

ii) *If* $S^B[r,s]=0$, *then*

a) $A^B[r,s]=A^B(r,s)$, *with dimension b.*

b) $F_B\colon A^B[r,s]\to C^B[r,s]$ *is bijective.*

c) $C^B\in\mathbf{B}^b(\prod_{v\in B} W^v, U)$ *for some neighbourhood U of* (r,s).

Remark. Proposition (2.24) follows directly from ii a).

Proof. This is part iii) of proposition (4.33). If $(q,s_1)\in S_B$ and $q\in I$, we may take j_0 in iii) of proposition (4.33) equal to j_q. Proposition (8.5) shows that conditions (1), (2) and (3) are satisfied. So $S_1\supset\{(q,s_1)\in S_B\colon q\in I\}$. Now S_B is an analytic set of dimension one which intersects lines $\{q\}\times\mathbb{C}$ discretely. So we may approach $(r,s)\in S_1$ by elements $(q,s_1)\in S_1$ with $q\in I$. So $(r,s)\in\tilde{S}$ if $(r,s)\in S_B$ and the statements in the proposition follow. If $(r,s)\notin S_B$ we already get the proposition from i) of proposition (4.33).

(8.7) *Remark.* In the proof of lemma (8.4) we needed a truncation device. If we work with B of the form

$$B=\{v\in\mathbb{Z}\colon |v+\tfrac{1}{12}r|<t\}$$

for some $t>\frac{1}{12}|r|$, we can do without:

Let U be a neighbourhood of (r,s_1), with $r\in I$. If $f\in S^B[r,s_0]$ and $h\in\mathscr{A}^B(U)$, then

$$\langle h(s),f\rangle=\int_{\Gamma\backslash\mathfrak{H}} h(s,z)\,\overline{f(z)}\,y^{-2}\,dx\,dy$$

Notations and Terminology

Latin, capital

A	(2.4), (7.1)
A_r	(6.3)
A^B	(2.10), (7.2)
A_r^B	(8.1)
$\mathbf{B}$	(4.29)
C^B	(7.7)
C_r^B	(8.1)
C_λ^ν	(2.19), (7.14)
C_κ	(7.12)
D_λ^ν	(2.20)
D, $D_{\mathfrak{h}}$	(4.2)
E	(7.12)
E^ν	(2.19), (7.14)
${}^{\ell}E^\nu$	(7.19)
F	(2.1)
$F(\cdot;\cdot,\cdot,\cdot)$	(4.4)
F_ν	(2.6)
F_B	(7.2)
G	(2.1)
H_r	(6.2)
$\mathbf{H}$	(4.13)
I, I^*	(4.43)
Indet	(3.8), (4.10)
J	(2.3)
K, $K_{\mathfrak{h}}$	(4.2)
L_r	(2.2)
N	(3.4), (3.8), (4.10)
P^ν	(2.20), (7.15)
P_r^ν	(2.33), (6.5)
${}^{a}P_r^\nu$	(6.6)
Q	(7.4)
S^B	(2.10), (7.3)
$\mathbf{S}$	(4.3)
S_0^*	(7.6)
$\tilde{S}$, S_1	(4.33)
S_B	(7.4)
Sing	(3.8), (4.10)
V_0, V_1	(4.33)
W^ν	(2.6), (5.1)
${}^{\circ}W^\nu$	(2.9), (5.5)
W_B	(7.4)
Wr^ν	(2.15), (5.3), (5.5)
Y_B^*	(7.6), (7.12)
$Y(\nu)$	(2.19)
$Y^*(\nu)$	(2.20)

Latin, lower case

(a)	(2.4)
(e)	(2.4)
j_r	(7.11)
ker	(4.25)
v_r	(2.1)
v^ν	(2.13)
(w^ν)	(2.6)
w_ℓ^ν	(5.7)

Script

$\mathcal{M}$	(3.7)
$\mathcal{O}$, ${}^{n}\mathcal{O}$	(3.1)
other script capitals	(4.6)

Gothic

$\mathfrak{h}$	(2.1)
$\mathfrak{m}_\nu$	(4.28)

Greek, capital

Γ	(2.1), in formulas also gamma function
Γ_∞	(2.22)

Greek, lower case

δ^ν	(7.10)
ζ_ℓ^ν	(5.7)
η_r	(2.12)
μ^ν	(2.13), (5.2)
π	(7.12) (proof), in formulas also 3.1415...
ρ	(4.43)
$\hat{\rho}$	(7.13)
σ	(4.43)
$\hat{\sigma}$	(7.13)
χ_ν	(7.2)
ω^ν	(2.7)

Other symbols and notations

References

1. Bruggeman, R.W.: Fourier coefficients of automorphic forms; Lect. Notes Math. **865**. Berlin-Heidelberg-New York: Springer 1981
2. Bruggeman, R.W.: Modular Forms of Varying Weight. I. Math. Z. **190**, 477–495 (1985)
3. Colin de Verdière, Y.: Pseudo-Laplaciens II; Ann. Inst. Fourier 33, 87–113 (1983)
4. Eichler, M.: Einführung in die Theorie der algebraischen Zahlen und Funktionen. Basel-Stuttgart: Birkhäuser 1963
5. Hervé, M.: Several complex variables. Oxford: University Press 1963
6. Hörmander, L.: An introduction to complex analysis in several variables. Princeton, N.J.: D. van Nostrand Company 1966
7. Jacobson, N.: Basic Algebra I. San Francisco: Freeman and Co. 1974
8. Kubota, T.: Elementary Theory of Eisenstein Series. Tokyo: Kodansha Ltd., and New York-London-Sydney-Toronto: John Wiley and Sons 1973
9. Lang, S.: Introduction to modular forms; Grundl. Math. Wiss. **222**. Berlin-Heidelberg-New York: Springer 1976
10. Magnus, W., Oberhettinger, F., Soni, R.P.: Formulas and Theorems for the Special Functions of Mathematical Physics. Berlin-Heidelberg-New York: Springer 1966
11. Neunhöffer, H.: Über die analytische Fortsetzung von Poincaréreihen; Sitzungsb. der Heidelberger Akad. der Wiss., Mathematisch-naturwiss. Klasse, 2. Abhandlung, 1973
12. Roelcke, W.: Das Eigenwertproblem der automorphen Formen in der hyperbolischen Ebene. I: Math. Ann. **167**, 292–337 (1966); II: Math. Ann. **168**, 216–324 (1967)
13. Slater, L.J.: Confluent hypergeometric functions. University Press: Cambridge 1960

Received November 12, 1984