Math. Z. 202, 181–198 (1989)

Mathematische Zeitschrift

Eisenstein Series and the Distribution of Dedekind Sums

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1. Introduction

1.1. In [2] a meromorphic family $(r, s) \mapsto E(r, s)$ of real analytic modular forms has been studied. At $r=0$ it is the usual Eisenstein series, for $\operatorname{Re} s > \frac{1}{2}$ given by

$$y^{\frac{1}{2}+s} + \sum_{c=1}^{\infty} \sum_{d \in \mathbb{Z}, (d,c)=1} (y/|cz+d|^2)^{\frac{1}{2}+s}.$$

The variable r denotes the weight and s determines the eigenvalue. The Fourier coefficients C_κ of E are meromorphic functions in (r, s). In this paper we consider

$$\left(\frac{d}{dr}\right)^n C_\kappa(r, s)\bigg|_{r=0}.$$

These are meromorphic functions of s which for $\operatorname{Re} s$ large may be expressed in terms of the Dirichlet series

$$\Delta_\kappa(k, s) = \sum_{c=1}^{\infty} \sum_{d(c)}^{*} c^{-s} \left(\frac{S(d, c)}{c}\right)^k e^{2\pi i \kappa d/c},$$

see 2.6, 3.1 and Proposition 3.5. By $\sum_{d(c)}^{*}$ we denote summation over d modulo c, $(d, c) = 1$; $S(d, c)$ is the Dedekind sum

$$S(d, c) = \sum_{m=1}^{c} ((dm/c)) \cdot ((m/c))$$

$$((x)) = \begin{cases} 0 & \text{if } x \in \mathbb{Z} \\ x - [x] - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \end{cases}$$

1.2. The relation of $\Delta_\kappa(k, s)$ with the Fourier coefficients of extended Eisenstein series implies that $\Delta_\kappa(k, s)$ has a meromorphic continuation, and gives information on its poles. This leads to results on the distribution of $\frac{S(d,c)}{c}$:

1.3. **Proposition.** *Let $f: [-1, 1]\times(\mathbb{R}/\mathbb{Z})\to\mathbb{C}$ be continuous. Then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{c=1}^{N} c^{-1}\sum_{d(c)}^{*} f\left(\frac{S(d,c)}{c},\frac{d}{c}\right)=\frac{6}{\pi^2}\int_0^1 f(0,\tau)\,d\tau.$$

So the majority of $\frac{S(d,c)}{c}$ are concentrated near 0; the value of $\frac{d}{c}$ does not influence the distribution. For the minority staying away from 0 we obtain:

1.4. **Proposition.** *For each continuous function f on $[-1, 1]\times(\mathbb{R}/\mathbb{Z})$:*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{c=1}^{N}\sum_{d \bmod c}^{*}\left(\frac{S(c,d)}{c}\right)^2 f\left(\frac{S(c,d)}{c},\frac{d}{c}\right)$$
$$=\sum_{n=1}^{\infty}\frac{1}{(12n)^2}\sum_{k=1}^{n}\frac{\phi((k,n))}{n}\sum_{\pm} f\left(\frac{\pm 1}{12n},\frac{k}{n}\right).$$

Here $\phi(c)$ denotes the number of $d \bmod c$ with $(d,c)=1$ and $\phi((k,n))=\phi(\gcd(k,n))$.

This formulation of the proposition is simpler than my original one; I thank D. Zagier for pointing out this simplification.

1.5. **Corollary.** *For each $\alpha\notin\left(\frac{1}{12(m+1)},\frac{1}{12m}\right)$, $m\in\mathbb{Z}$, $m\geqq 1$:*

$$\lim_{N\to\infty}\frac{1}{N}\#\{(c,d)\colon 1\leqq c\leqq N,\, d \bmod c,\, (d,c)=1,\, |S(c,d)|\geqq\alpha c\}$$
$$=2\sum_{n=1}^{m}\sum_{k=1}^{n}\frac{\phi((k,n))}{n}.$$

2. $E(r, s)$ and its Fourier Coefficients

We review from [2] and [3] the facts concerning E we need, and formulate the resulting properties of the Fourier coefficients in a way suitable for our purpose. We also give the computation of the Fourier coefficients of some Poincaré series.

2.1. The modular forms we study are real analytic functions on the upper half plane satisfying

$$\left[-y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iry\frac{\partial}{\partial x}\right]f=(\tfrac{1}{4}-s^2)f$$

and

We take $-\pi<\arg(cz+d)\leqq\pi$. The multipliersystem v_r is determined by:

$$v_r\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}=e^{\frac{1}{6}\pi i r n}$$

$$v_r\begin{pmatrix}a & b\\ c & d\end{pmatrix}=e^{-2\pi i r S(d,c)-\frac{1}{4}\pi i r+\frac{1}{6}\pi i r(a+d)/c} \qquad \text{for } \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z}),\ c\geqq 1.$$

We call $A[r,s]$ the space of all functions with these properties.

2.2. Each $f\in A[r,s]$ has a Fourier expansion

$$f=\sum_{\kappa\in\mathbb{Z}} f_\kappa$$

with

$$f_\kappa(z)=\int_0^1 e^{-2\pi i(\kappa+\frac{1}{12}r)x'} f(z+x')\,dx'.$$

The f_κ are elements of a two-dimensional space, which is for general (r,s) spanned by $\omega^\kappa(r,s)$ and $\mu^\kappa(r,s)$;

$$\omega^\kappa(r,s;z)=e^{2\pi i(\kappa+\frac{1}{12}r)x}\,W_{\frac{1}{2}r\varepsilon,s}(4\pi\varepsilon(\kappa+\tfrac{1}{12}r)y) \qquad \text{for } \operatorname{Re}(\kappa+\tfrac{1}{12}r)\neq 0,$$

$\varepsilon=\operatorname{sign}\operatorname{Re}(\kappa+\frac{1}{12}r)$, $W_{.,.}(\cdot)$ a Whittaker function, see [8], 1.7;

$$\mu^\kappa(r,s;z)=e^{2\pi i(\kappa+\frac{1}{12}r)x}\,y^{\frac{1}{2}+s}\,e^{\mp 2\pi(\kappa+\frac{1}{12}r)y}\,{}_1F_1[\tfrac{1}{2}+s\mp\tfrac{1}{2}r;1+2s;\pm 4\pi(\kappa+\tfrac{1}{12}r)y]$$

${}_1F_1$ a confluent hypergeometric function, see [8], 1.1.1 (and 1.4 to see that the choice of $\pm$ does not matter).

2.3. We shall only consider $A^0[r,s]=\{f\in A[r,s]\colon f_\kappa$ is a multiple of ω^κ for $\kappa\neq 0\}$, and restrict r to $|\operatorname{Re} r|<12$.

2.4. For $\kappa=0$, $s\notin\frac{1}{2}\mathbb{Z}$, it is better to express f_0 in $\mu^0(r,s)$ and $\mu^0(r,-s)$, as this works for $\operatorname{Re} r=0$ as well. Remark that $\mu^0(0,s;z)=y^{\frac{1}{2}+s}$.

For $\operatorname{Re} r>0$:

$$\omega^0(r,s)=v(r,s)\mu^0(r,s)+v(r,-s)\mu^0(r,-s),$$

$$v(r,s)=\left(\frac{\pi r}{3}\right)^{\frac{1}{2}+s}\frac{\Gamma(-2s)}{\Gamma(\frac{1}{2}-s-\frac{1}{2}r)}.$$

2.5. **Proposition** ([2], (2.19)). i) *There are a neighbourhood Y of $(-12,12)\times\mathbb{C}$ in $\{r\in\mathbb{C}\colon|\operatorname{Re} r|<12\}\times\mathbb{C}$ invariant under $(r,s)\mapsto(r,-s)$ and $(r,s)\mapsto(-r,s)$, and a meromorphic family E of functions $H\to\mathbb{C}$ on Y such that for each $(r,s)\in Y$ at which E is holomorphic:*

b) $E(r, s)=\mu^0(r, s)+C_0(r, s)\,\mu^0(r, -s)+\sum_\kappa' C_\kappa(r, s)\,\omega^\kappa(r, s)$

($\sum_\kappa'$ *denotes* $\sum_{\kappa\in\mathbb{Z}, \kappa\neq 0}$ *and* H *denotes the upper half plane*).

ii) a) *and* b) *in* i) *determine* E *uniquely;*

C_κ *is a meromorphic function on* Y *for all* $\kappa\in\mathbb{Z}$.

iii) a) $E(r, -s)=C_0(r, -s)\,E(r, s)$,
b) $E(-r, s; -\bar{z})=E(r, s; z)$.

This E is denoted E^0 in [2].

2.6. We define meromorphic functions Ψ_κ on Y by

$$\Psi_0(r, s)=\frac{1}{2\pi}\,2^{2s}\,\frac{\Gamma(\frac12+s+\frac12 r)\,\Gamma(\frac12+s-\frac12 r)}{\Gamma(2s)}\,C_0(r, s)$$

$$\Psi_\kappa(r, s)=\pi^{-\frac12-s}(\varepsilon(\kappa+\tfrac{1}{12}r))^{\frac12-s}\,\Gamma(\tfrac12+s+\tfrac12\varepsilon r)\,C_\kappa(r, s),\qquad \varepsilon=\operatorname{sign}\kappa,\ \kappa\neq 0;$$

we take $\arg(\varepsilon(\kappa+\frac{1}{12}r))\in(-\frac12\pi, \frac12\pi)$. In view of 2.5 it seems unnatural to use Ψ_κ instead of C_κ, but thus we shall turn out to stay as close as possible to Dirichlet series. The functional equations arising from iii) a) in 2.5 expressed in the $\Delta_k(k, s)$ will be awkward anyhow, and are not considered in this paper.

2.7. **Corollary.** *On* Y *we have the following identities of meromorphic functions:*

i) a) $\Psi_0(r, s)\,\Psi_0(r, -s)=-\dfrac{s}{2\pi}\,\dfrac{\sin 2\pi s}{\cos\pi(s+\frac12 r)\cos\pi(s-\frac12 r)}$,

b) $\Psi_\kappa(r, -s)=2(2\pi\varepsilon(\kappa+\frac{1}{12}r))^{2s}\,\Gamma(-2s)\cos\pi(s+\frac12\varepsilon r)\,\Psi_0(r, -s)\Psi_\kappa(r, s)$
for $\kappa\neq 0$, $\varepsilon=\operatorname{sign}\kappa$.

ii) $\Psi_\kappa(-r, s)=\Psi_{-\kappa}(r, s)$.

Proof. i) is derived by working out iii) a) in Proposition 2.5; one also uses $\omega^\kappa(r, -s)=\omega^\kappa(r, s)$. Similarly ii) is derived from iii) b) in 2.5, with use of $\mu^0(r, s; -\bar{z})=\mu^0(-r, s; z)$ and $\omega^\kappa(r, s; -\bar{z})=\omega^{-\kappa}(-r, s; z)$.

2.8. One might worry that $\{0\}\times\mathbb{C}$ is contained in the singular set of E; in that case $s\mapsto E(0, s)$ would make no sense.

Proposition ([2], (2.23), (7.19)). i) *Let* $s\in\mathbb{C}$, $s\notin\frac12\mathbb{Z}$, $\operatorname{Re} s>\frac12$.

E is holomorphic at $(0, s)$ *and*

ii) *For $l \in \mathbb{Z}$ put*

$$w_l(r,s)=\left(\frac{\pi r}{3}\right)^l \frac{\Gamma(-2s)\,\Gamma(\frac{1}{2}+s-\frac{1}{2}r)}{\Gamma(2s)\,\Gamma(\frac{1}{2}-s-\frac{1}{2}r)}.$$

If $l \geq 2$ *then* ${}^lE(r,s)=\dfrac{1}{1-w_l(r,s)\,C_0(r,s)}\,E(r,s)$ *is holomorphic at* $(0, \frac{1}{2}l)$, *and* ${}^lE(0, \frac{1}{2}l; z)$ *is represented by the Eisenstein series in* i) *with* $s=\frac{1}{2}l$.

2.9. **Corollary.** i) *Let* $s \in \mathbb{C} \setminus \frac{1}{2}\mathbb{Z}$, $\operatorname{Re} s > \frac{1}{2}$, $\kappa \in \mathbb{Z}$. Ψ_κ *is holomorphic at* $(0, s)$ *and*

$$\Psi_\kappa(0,s)=\begin{cases} \zeta(2s)/\zeta(2s+1) & \text{if } \kappa=0 \\ \sigma_{-2s}(\kappa)/\zeta(2s+1) & \text{if } \kappa \neq 0, \end{cases}$$

$$\sigma_a(\kappa)=\sum_{d\mid\kappa} d^a \qquad (d \text{ positive divisor of } \kappa).$$

ii) *Let* $l \in \mathbb{Z}$, $l \geq 2$

a) $(r,s) \mapsto \dfrac{1}{\Psi_0(r,s)} - 2^{1-2s}\,\Gamma(-2s)\left(\dfrac{\pi r}{3}\right)^l \cos \pi(s+\frac{1}{2}r)$

is holomorphic at $(0, \frac{1}{2}l)$ *with value* $\zeta(l+1)/\zeta(l)$.

b) *For* $\kappa \neq 0$

$$\Psi_\kappa/\Psi_0 \text{ is holomorphic at } (0, \tfrac{1}{2}l) \text{ with value } \sigma_{-l}(\kappa)/\zeta(l).$$

Proof. The Fourier expansion for the Eisenstein series in weight zero

$$y^{\frac{1}{2}+s}+\pi^{+\frac{1}{2}}\,\frac{\Gamma(s)\,\zeta(2s)}{\Gamma(s+\frac{1}{2})\,\zeta(2s+1)}\,y^{\frac{1}{2}-s}$$
$$+\sum_\kappa{}' \frac{|\kappa|^{-\frac{1}{2}+s}\,\sigma_{-2s}(\kappa)}{\pi^{-\frac{1}{2}-s}\,\Gamma(s+\frac{1}{2})\,\zeta(2s+1)}\,W_{0,s}(4\pi|\kappa|y)\,e^{2\pi i\kappa x}$$

is well known. So i) follows easily from i) in Proposition 2.8. For ii) a) we express the Fourier coefficient of order zero of ${}^lE(r,s)$ as

$$(1-w_l(r,s)\,C_0(r,s))^{-1}(\mu^0(r,s)+C_0(r,s)\,\mu^0(r,-s))$$
$$=\mu^0(r,s)+C_0(r,s)(1-w_l(r,s)\,C_0(r,s))^{-1}\,\zeta_l^0,$$

see [2], (5.7). This gives the holomorphy of $C_0/(1-w_l\,C_0)$ with value

$$\pi^{-\frac{1}{2}}\,\frac{\Gamma(\frac{1}{2}l)\,\zeta(l)}{\Gamma(\frac{1}{2}l+\frac{1}{2})\,\zeta(l+1)} \quad \text{at} \quad (0, \tfrac{1}{2}l).$$

A computation gives ii) a). Finally ii) b) follows from the fact that $C_0/(1-w_l\,C_0)$ and $C_\kappa/(1-w_l\,C_0)$ are holomorphic at $(0, \frac{1}{2}l)$ with known values.

2.10. The Ψ_κ restricted to $\{0\} \times \mathbb{C}$ give well known meromorphic functions. We investigate the position of the singularities.

2.11. *Notation.* $\mathscr{S}_\kappa = \{s \in \mathbb{C}\colon \Psi_\kappa$ is singular at $(0,s)\}$.

All points where $s \mapsto \Psi_\kappa(0,s)$ is singular are contained in $\mathscr{S}_\kappa$. This is the

But there may also be points s_0 such that $s \mapsto \Psi_\kappa(0, s)$ is holomorphic at s_0, but still Ψ_κ indeterminate at $(0, s_0)$. Examples of such points are those from the following set, provided it is non-empty:

Notation. $\mathcal{M} = \{s \in i\mathbb{R}: C_0 \text{ is singular at } (0, s)\}$.

In [3], Proposition 2.19, we see that $s \in \mathcal{M}$ if and only if the space of real analytic cusp forms of weight zero and eigenvalue $\frac{1}{4} - s^2$ is not spanned by values at 0 of families of cusp forms of varying weight.

Notation. $\mathcal{H} = \frac{1}{2}\mathbb{Z} \setminus \{0\}$.

2.12. **Proposition.** *Let* $\kappa \in \mathbb{Z}$.

i) $\mathcal{S}_\kappa \subset \mathcal{R} \cup \mathcal{M} \cup \mathcal{H}$.

ii) a) $(r, s) \mapsto \left\{ \dfrac{1}{\Psi_0(r, s)} - 2^{1-2s} \Gamma(-2s) \dfrac{\pi r}{3} \cos \pi(s + \frac{1}{2} r) \right\} (\frac{1}{2} - s - \frac{1}{2} r)^{-1}$

is holomorphic at $(0, \frac{1}{2})$ *with value* $-\frac{1}{3}\pi^2$.

b) *For* $\kappa \neq 0$

$$\Psi_\kappa / \Psi_0 \text{ is holomorphic at } (0, \tfrac{1}{2}) \text{ with value } 0.$$

Proof. Let us first consider ii).

In [3], 6.5–6.10, we see that $(r, s) \mapsto E(r, -s)$ is holomorphic at $(0, \frac{1}{2})$, and

$$C_0(r, -s) - w_1(r, s) = (\tfrac{1}{2} - s - \tfrac{1}{2} r)\, c_1(r, s)$$

with c_1 holomorphic at $(0, \frac{1}{2})$ and $c_1(0, \frac{1}{2}) = -\frac{1}{3}\pi$.

This gives ii) a).

From the Maass-Selberg relation (see e.g. [2], 7.8) follows $E(0, -\frac{1}{2}) = 1$ (constant function). So all C_κ are holomorphic at $(0, -\frac{1}{2})$ with value 0. This gives ii) b); use 2.7 i) b).

To prove i) consider $s_0 \in \mathbb{C}$, $\operatorname{Re} s_0 \geq 0$, $s_0 \notin \mathcal{H}$. Suppose Ψ_κ has a singularity at $(0, s_0)$. From Corollary 2.9 we see that $\operatorname{Re} s_0 \leq \frac{1}{2}$. If $\kappa = 0$ we conclude $s_0 \in \mathcal{M}$ from [3], Propositions 6.5 and 2.19. Now let $\kappa \neq 0$. Then C_κ is singular at $(0, s_0)$, hence E is, for ω^κ is regular and nonzero. [3] 6.5, 6.7 show that C_0 is singular at $(0, s_0)$, hence $s_0 \in \mathcal{M}$, if $s_0 \neq 0$. For $s_0 = 0$ we also use [3] 6.9; we would get a singularity at $(0, 0)$ of $\dfrac{C_\kappa}{1 + C_0}$, hence of $\dfrac{1}{1 + C_0} E$, which leads to a contradiction.

Now suppose $\operatorname{Re} s_0 < 0$, $s_0 \notin \mathcal{H} \cup \mathcal{R}$. Then $s \mapsto \Psi_\kappa(0, s)$ is holomorphic at s_0, so Ψ_κ is at most indeterminate at $(0, s_0)$. If $\kappa = 0$ the functional equation in Corollary 2.7 i) a) is sufficient to conclude that Ψ_κ is also indeterminate at $(0, -s_0)$, which is impossible. If $\kappa \neq 0$ we use Corollary 2.7 i) b) to get either a singularity of Ψ_0 at $(0, s_0)$ or of Ψ_κ at $(0, -s_0)$; both cases have already been considered.

2.13. This concludes our list of results concerning Ψ_κ along $\{0\} \times \mathbb{C}$. To get

2.14. **Proposition** ([2], (2.20)–(2.23)). *Let* $Y^+=\{(r,s)\in Y\colon \operatorname{Re} r>0\}$.

i) $P(r,s)=\dfrac{v(r,-s)}{v(r,-s)-v(r,s)\,C_0(r,s)}\,E(r,s)$

is a meromorphic family of modular forms on Y^+, *characterized by*

$$P(r,s)=\mu^0(r,s)+\sum_\kappa D_\kappa(r,s)\,\omega^\kappa(r,s)$$

with all D_κ *meromorphic on* Y^+.

ii) *Let* $0<r<12$, $\operatorname{Re} s>1/2$. *Then* P *is holomorphic at* (r,s) *and its value is given by a Poincaré series:*

$$P(r,s;z)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} v_r(\gamma)^{-1}\,e^{-ir\arg(cz+d)}\,\mu^0(r,s;\gamma.z).$$

$$\Gamma=SL_2(\mathbb{Z}),\qquad \Gamma_\infty=\left\{\pm\begin{pmatrix}1&*\\0&1\end{pmatrix}\in\Gamma\right\};$$

by c and d in $\arg(cz+d)$ is meant $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}$.

2.15. **Corollary.** *Let* $\kappa\in\mathbb{Z}$.

$$\Phi_\kappa(r,s)=\Psi_\kappa(r,s)\cdot\left[1-2\left(\frac{\pi r}{6}\right)^{2s}\Gamma(-2s)\cos\pi(s+\tfrac{1}{2}r)\,\Psi_0(r,s)\right]^{-1},$$

defines a meromorphic function on Y^+. *Reversely, on* Ψ^+:

$$\Psi_\kappa(r,s)=\Phi_\kappa(r,s)\cdot\left[1+2\left(\frac{\pi r}{6}\right)^{2s}\Gamma(-2s)\cos\pi(s+\tfrac{1}{2}r)\,\Phi_o(r,s)\right]^{-1}.$$

We take $\arg\left(\dfrac{\pi r}{6}\right)\in(-\tfrac{1}{2}\pi,\tfrac{1}{2}\pi)$.

Proof. From Proposition 2.14 i) follows an expression of $D_\kappa(r,s)$ in the $\Psi_\kappa(r,s)$. Write

$$D_\kappa(r,s)=\frac{\pi^{\frac{1}{2}+s}(\varepsilon(\kappa+\frac{1}{12}r))^{-\frac{1}{2}+s}}{\Gamma(\frac{1}{2}+s+\frac{1}{2}r\varepsilon)}\,\Phi_\kappa(r,s)$$

with $\varepsilon=\operatorname{sign}(\kappa+\frac{1}{2})$. The corollary follows by a computation.

2.16. **Proposition.** *Let* $0<r<12$, $\operatorname{Re} s>1/2$.

Φ_κ *is holomorphic at* (r,s) *and*

$$\Phi_\kappa(r,s)=\sum_{c=1}^{\infty}\sum_{d(c)}{}^{*}\,e^{2\pi irS(d,c)+2\pi i\kappa d/c}\,c^{-1-2s}$$
$$\cdot{}_0F_1\,[\,;1+2s;-\tfrac{1}{3}\pi^2 r(\kappa+\tfrac{1}{12}r)c^{-2}]$$

Remark. This formula is not new; it may be dug out of [1] and, I suppose, also out of [4]. As it is central in this paper I sketch a proof, mainly following the ideas in [6].

Proof. The holomorphy of Φ_κ at (r, s) follows from ii) in Proposition 2.14.

$$\int_0^1 e^{-2\pi i(\kappa+\frac{1}{12}r)x} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} v_r(\gamma)^{-1} e^{-ir\arg(cx+ciy+d)} \mu^0(r, s; \gamma\cdot(x+iy))\, dx$$

$$=\delta_{\kappa,0}\, \mu^0(r, s; iy)+\frac{\pi^{\frac{1}{2}+s}|\kappa+\frac{1}{12}r|^{s-\frac{1}{2}}}{\Gamma(\frac{1}{2}+s+\frac{1}{2}\varepsilon r)}\, \Phi_\kappa(r, s)\, \omega^\kappa(r, s; iy),$$

with $\varepsilon=\mathrm{sign}(\kappa+\frac{1}{2})$. The term with $\gamma\in\Gamma_\infty$ leads to the δ-term in the right hand side. The remaining sum over γ may be taken out of the integral by absolute convergence. So

$$\Phi_\lambda(r, s)\, \omega^\kappa(r, s; iy)$$
$$=\sum_{c=1}^{\infty} \sum_{d(c)}^{*} \int_{-\infty}^{+\infty} e^{2\pi irS(d,c)+\frac{1}{4}\pi ir-\frac{1}{6}\pi ir(a+d)/c-ir\arg(cz+d)}$$
$$\cdot\mu^0\left(r, s; \frac{az+b}{cz+d}\right) e^{-2\pi i(\kappa+\frac{1}{12}r)x}\, dx\cdot\pi^{-\frac{1}{2}-s}|\kappa+\tfrac{1}{12}r|^{\frac{1}{2}-s}\, \Gamma(\tfrac{1}{2}+s+\tfrac{1}{2}\varepsilon r).$$

For given c, d we choose $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma$. By z we mean $x+iy$. Each individual integral in the right hand side satisfies the same differential equation in y as does $\omega^\kappa(r, s; iy)$. From its boundedness for $y\to\infty$ follows that it is a multiple of $\omega^\kappa(r, s; iy)$. So

$$\Phi_\kappa(r, s)=\sum_{c=1}^{\infty} \sum_{d(c)}^{*} \varphi(c, d)$$

with

$$\varphi(c, d)\, \omega^\kappa(r, s; iy)=\pi^{-\frac{1}{2}-s}|\kappa+\tfrac{1}{12}r|^{\frac{1}{2}-s}\, \Gamma(\tfrac{1}{2}+s+\tfrac{1}{2}\varepsilon r)\cdot e^{2\pi irS(d,c)}$$
$$\cdot e^{\frac{1}{4}\pi ir}\, I(c, y)\, e^{2\pi i\kappa d/c},$$

$$I(C, y)=\int_{-\infty}^{+\infty} e^{-ir\arg(x+iy)}\, \mu^0\left(r, s; \frac{-1}{c^2(x+iy)}\right) e^{-\frac{1}{2}iqx}\, dx,$$

$$q=4\pi(\kappa+\tfrac{1}{12}r);$$

we have carried out the substitution $x\mapsto x-d/c$.

From the asymptotic behaviour of the Whittaker function W, see [8], 4.1.3, follows

$$\varphi(c, d)=\pi^{-1}\, 2^{2s-1}\, |q|^{\frac{1}{2}-s-\frac{1}{2}\varepsilon r}\, \Gamma(\tfrac{1}{2}+s+\tfrac{1}{2}\varepsilon r)\, e^{+\frac{1}{4}\pi ir+2\pi irS(d,c)+2\pi i\kappa d/c}$$
$$\cdot\lim\, y^{-\frac{1}{2}\varepsilon r}\, e^{\frac{1}{2}|q|y}\, I(c, y).$$

Let $y>1$. We take in the integral $x=(y+u)e^{-\frac{1}{2}\pi i\varepsilon}$ and deform the path of integration into a contour C coming from $+\infty$, passing downward between -1 and 0 and going back to $+\infty$. On C we take $0<\arg u<2\pi$, $|\arg(u+2y)|<\frac{1}{2}\pi$. So we obtain

$$
\begin{aligned}
&y^{-\frac{1}{2}\varepsilon r}e^{\frac{1}{2}|q|y}I(c,y)\\
&\quad=y^{-\frac{1}{2}\varepsilon r}e^{\frac{1}{2}|q|y}\int_{-\infty}^{+\infty}e^{-\frac{1}{6}\pi i r x c^{-2}(x^2+y^2)^{-1}}\cdot y^{\frac{1}{2}+s}c^{-2s-1}(x+iy)^{-\frac{1}{2}-s-\frac{1}{2}r}\\
&\qquad\cdot(x-iy)^{-\frac{1}{2}-s+\frac{1}{2}r}e^{\frac{1}{6}\pi r\varepsilon c^{-2}y(x^2+y^2)^{-1}}\\
&\qquad\cdot{}_1F_1[\tfrac{1}{2}+s+\tfrac{1}{2}\varepsilon r;1+2s;-\tfrac{1}{3}\pi r\varepsilon y c^{-2}(x^2+y^2)^{-1}]\\
&\qquad\cdot e^{-\frac{1}{2}iqx}dx\\
&\quad=-e^{\frac{1}{2}\pi i r(\varepsilon-1)+\pi i s}c^{-1-2s}\int_C e^{\frac{1}{6}\pi r\varepsilon c^{-2}(u+2y)^{-1}}u^{-\frac{1}{2}-s-\frac{1}{2}\varepsilon r}(2+u/y)^{-\frac{1}{2}-s+\frac{1}{2}\varepsilon r}\\
&\qquad\cdot{}_1F_1[\tfrac{1}{2}+s+\tfrac{1}{2}\varepsilon r;1+2s;\tfrac{1}{3}\pi r\varepsilon c^{-2}u^{-1}(2+u/y)^{-1}]e^{-\frac{1}{2}|q|u}du.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\lim_{y\to\infty}y^{-\frac{1}{2}\varepsilon r}e^{\frac{1}{2}|q|y}I(c,y)\\
&\quad=-e^{\frac{1}{2}\pi i r(\varepsilon-1)+\pi i s}c^{-1-2s}\int_C u^{-\frac{1}{2}-s-\frac{1}{2}\varepsilon r}2^{-\frac{1}{2}+s+\frac{1}{2}\varepsilon r}\\
&\qquad\cdot{}_1F_1[\tfrac{1}{2}+s+\tfrac{1}{2}\varepsilon r;1+2s;\tfrac{1}{6}\pi r\varepsilon c^{-2}u^{-1}]e^{-\frac{1}{2}|q|u}du\\
&\quad=e^{-\frac{1}{2}\pi i r}c^{-1-2s}2^{-2s}|q|^{-\frac{1}{2}+s+\frac{1}{2}\varepsilon r}\cos\pi(s+\tfrac{1}{2}\varepsilon r)2\\
&\qquad\cdot\sum_{n=0}^{\infty}\frac{\Gamma(1+2s)\Gamma(\frac{1}{2}+s+\frac{1}{2}\varepsilon r+n)}{\Gamma(1+2s+n)\Gamma(\frac{1}{2}+s+\frac{1}{2}\varepsilon r)n!}\left(\frac{\pi r q}{12c^2}\right)^n\Gamma(\tfrac{1}{2}-s-\tfrac{1}{2}\varepsilon r-n)\\
&\quad=\pi e^{-\frac{1}{2}\pi i r}c^{-1-2s}2^{1-2s}|q|^{-\frac{1}{2}+s+\frac{1}{2}\varepsilon r}\Gamma(\tfrac{1}{2}+s+\tfrac{1}{2}\varepsilon r)^{-1}{}_0F_1\left[;1+2s;-\frac{\pi q r}{12c^2}\right].
\end{aligned}
$$

So $\varphi(c,d)=c^{-1-2s}e^{2\pi i r S(d,c)+2\pi i\kappa d/c}{}_0F_1[;1+2s;-\frac{1}{3}\pi^2 r(\kappa+\frac{1}{12}r)c^{-2}]$, and this completes the proof.

3. Derivatives of Ψ_κ

3.1. Definition. $\Psi_{\kappa,n}(s)=\dfrac{1}{n!}\left(\dfrac{d}{dr}\right)^n\Psi_\kappa(r,s)|_{r=0}$, for $n\geqq 0$.

3.2. Clearly $\Psi_{\kappa,n}$ is a meromorphic function on $\mathbb{C}$, holomorphic at s at least if Ψ_κ is holomorphic at $(0,s)$. In the latter case

$$\Psi_\kappa(r,s)=\sum_{n=0}^{\infty}\Psi_{\kappa,n}(s)r^n\quad\text{for } |r|<\varepsilon_s.$$

3.3. Proposition. *Let* $n, \kappa \in \mathbb{Z}$, $n \geqq 0$.

i) $\Psi_{-\kappa,n} = (-1)^n \Psi_{\kappa,n}$; *in particular* $\Psi_{0,n} = 0$ *for* n odd.

ii) *If* $\Psi_{\kappa,n}$ *is singular at* s, *then*

$$s \in \mathcal{R} \cup \mathcal{M} \cup \{\tfrac{1}{2} l \in \tfrac{1}{2}\mathbb{Z} : l \leqq -1\} \cup \{\tfrac{1}{2}\} \cup \{\tfrac{1}{2} l \in \tfrac{1}{2}\mathbb{Z} : 1 \leqq l \leqq 2[\tfrac{1}{2} n]\}$$

iii) *For* $2 \leqq l \leqq 2[\frac{1}{2} n]$ *the order of the pole of* $\Psi_{\kappa,n}$ *at* $\frac{1}{2} l$ *is at most* $-[-[\frac{1}{2} n]/[\frac{1}{2} l]]$.

(Here $[x]$ denotes the largest integer $\leqq x$)

iv) *For* $n \geqq 2$, n even

$$\operatorname*{res}_{s \to \frac{1}{2} n} \Psi_{\kappa,n}(s) = \frac{(-1)^{\frac{1}{2} n} (\frac{1}{6}\pi)^n \sigma_{-n}(\kappa) \zeta(n)}{n! \, \zeta(n+1)^2},$$

with the convention $\sigma_{-n}(0) = \zeta(n)$.

v) *For* n even

$$\lim_{s \to \frac{1}{2}} (s - \tfrac{1}{2})^{n+1} \Psi_{\kappa,n}(s) = \begin{cases} 3\pi^{-2} 2^{-n} & \text{if } \kappa = 0 \\ 0 & \text{if } \kappa \neq 0. \end{cases}$$

Proof. i) and most of ii) follow from Corollary 2.7 ii) and Proposition 2.12 i). Parts ii) and v) are clear for $n = 0$. So we have to prove iii), iv), v) and the holomorphy at $\frac{1}{2} l > [\frac{1}{2} n]$ for $n \geqq 1$.

Let $\kappa = 0$. Consider

$$G_n(s) = \frac{1}{n!} \left(\frac{d}{dr}\right)^n \frac{1}{\Psi_0(r,s)} \bigg|_{r=0}.$$

It is meromorphic on $\mathbb{C}$ and vanishes for n odd. From Corollary 2.9 ii) a) and Proposition 2.12 ii) a) we see that G_n has at $\frac{1}{2} l \geqq \frac{1}{2}$, $l \in \mathbb{Z}$, at most a first order pole and for $n \geqq 2$:

$$\operatorname*{res}_{s \to \frac{1}{2} l} G_n(s) = \begin{cases} 0 & \text{if } l > n \\ \dfrac{1}{n!} \dbinom{n}{l} (-1)^{1 - \frac{1}{2} n + l} 3^{-l} (\frac{1}{2}\pi)^n & \text{if } l \leqq n. \end{cases}$$

From the definition of G_n follows

$$\Psi_{0,n}(s) = -\Psi_{0,0}(s) \sum_{\substack{k=2 \\ k \text{ even}}}^{n-2} G_{n-k}(s) \Psi_{0,k}(s) - \Psi_{0,0}(s)^2 G_n(s).$$

This expresses the $\Psi_{0,k}$ in the G_n inductively. Remark that $\Psi_{0,0}(s) = \dfrac{\zeta(2s)}{\zeta(2s+1)}$.

Suppose we know already ii)–v) for even $\tilde{n}$ with $2 \leqq \tilde{n} \leqq n-2$. We prove ii)–v) for even n, and $\kappa = 0$.

Let $l \geqq 1$, $l \in \mathbb{Z}$. The order of $\Psi_{0,n}(s)$ at $s = \frac{1}{2} l$ is at least the minimum of:

b) $\underset{s\to\frac{1}{2}l}{\mathrm{order}}\,\zeta(2s)+\underset{s\to\frac{1}{2}l}{\mathrm{order}}\,\Psi_{0,k}(s)$ for $n-l<k\leqq n-2$, k even,

c) $2.\underset{s\to\frac{1}{2}l}{\mathrm{order}}\,\zeta(2s)+\underset{s\to\frac{1}{2}l}{\mathrm{order}}\,G_n(s)$.

Consider first $l>2[\frac{1}{2}n]=n$. Case a) does not occur; cases b) and c) give something $\geqq 0$. So we have proved ii) for $\kappa=0$. Now take $l=1$. Case b) does not occur. Case a) gives

$$-2-(k+1)\quad \text{for } 2\leqq k\leqq n-2,$$

whereas case c) gives

$$-2-1$$

So $\underset{s\to\frac{1}{2}}{\mathrm{order}}\,\Psi_{0,n}(s)\geqq -n-1$, and for $n\geqq 4$

$$\lim_{s\to\frac{1}{2}}(s-\tfrac{1}{2})^{n+1}\,\Psi_{0,n}(s)=\lim_{s\to\frac{1}{2}}-(s-\tfrac{1}{2})^{n+1}\,\Psi_{0,0}(s)\,G_2(s)\,\Psi_{0,n-2}(s)=3\,\pi^{-2}\,2^{-n};$$

and for $n=2$

$$\lim_{s\to\frac{1}{2}}(s-\tfrac{1}{2})^{3}\,\Psi_{0,2}(s)=\lim_{s\to\frac{1}{2}}-(s-\tfrac{1}{2})^{3}\,\Psi_{0,0}(s)^2\,G_2(s)=\tfrac{3}{4}\,\pi^{-2}.$$

This finishes the proof of v).

Now take $2\leqq l\leqq 2[\frac{1}{2}n]=n$. Case a) gives

$$-1+[-[\tfrac{1}{2}k]/[\tfrac{1}{2}l]]\qquad \text{for } 2\leqq k\leqq n-l,\ l\leqq k,\ k \text{ even},$$

and

$$-1\qquad \text{for } 2\leqq k\leqq n-l,\ l>k,\ k \text{ even}.$$

The first is

$$\begin{aligned}-1+[-\tfrac{1}{2}k/[\tfrac{1}{2}l]]&\geqq -1+[(-\tfrac{1}{2}n+\tfrac{1}{2}l)/[\tfrac{1}{2}l]]\\ &\geqq -1+[-\tfrac{1}{2}n/[\tfrac{1}{2}l]]+1=[-[\tfrac{1}{2}n]/[\tfrac{1}{2}l]].\end{aligned}$$

Further also $-1\geqq[-[\frac{1}{2}n]/[\frac{1}{2}l]]=[-\frac{1}{2}n/[\frac{1}{2}l]]$ as $\frac{1}{2}n\geqq\frac{1}{2}l$. Case b) gives:

$$\begin{array}{ll}[-[\tfrac{1}{2}k]/[\tfrac{1}{2}l]] & \text{for } n-l\leqq k\leqq n-2,\ l\leqq k,\ k \text{ even}.\\ 0 & \text{for } n-l\leqq k\leqq n-2,\ l>k,\ k \text{ even}.\end{array}$$

Both clearly are at least $[-[\frac{1}{2}n]/[\frac{1}{2}l]]$. Finally case c) gives -1, which value we have already considered.

Thus iii) has been proved for $\kappa=0$. If $l=n$, then case a) is absent, case b) gives order $\geqq 0$, so

$$\underset{s\to\frac{1}{2}n}{\mathrm{res}}\ \Psi_{\kappa,n}(s)=-\underset{s\to\frac{1}{2}n}{\mathrm{res}}\ \Psi_{0,0}(s)^2\,G_n(s)$$

$$=\ \zeta(n)^2\quad 1\quad (-1)^{\frac{1}{2}n+1}\,2^{-n}(1-n)^n$$

The case $\kappa \neq 0$ we derive from the case $\kappa = 0$. From Corollary 2.9 ii) b) and Proposition 2.12 ii) b) we know that $H_\kappa = \Psi_\kappa/\Psi_0$ is holomorphic at $(0, \frac{1}{2} l)$, $l \in \mathbb{Z}$, $l \geqq 1$ and

$$H_\kappa(0, \tfrac{1}{2} l) = \begin{cases} \sigma_{-l}(\kappa)/\zeta(l) & \text{if } l \geqq 2 \\ 0 & \text{if } l = 1 \end{cases}$$

So

$$\Psi_{\kappa,n}(s) = \sum_{\substack{0 \leqq k \leqq n \\ k \text{ even}}} \Psi_{0,\kappa}(s)\, A_{\kappa,n-k}(s)$$

where for each $l \geqq 1$, $l \in \mathbb{Z}$ the function $A_{\kappa,k}$ is holomorphic at $\frac{1}{2} l$ and $A_{\kappa,0}(\frac{1}{2} l) = H_\kappa(0, \frac{1}{2} l)$.

This gives immediately ii) and iii) for $\kappa \neq 0$; an easy computation gives iv) and v). The restriction to even n is necessary, as we do not know the value of $A_{\kappa,1}(\frac{1}{2} l)$.

3.4. Definition. For $k, \kappa \in \mathbb{Z}$, $k \geqq 0$, Re $s > 2$

$$\Delta_\kappa(k, s) = \sum_{c=1}^{\infty} \sum_{d(c)}{}^{*} c^{-s-k} S(d, c)^k e^{2\pi i \kappa d/c}.$$

This defines $s \mapsto \Delta_\kappa(k, s)$ as holomorphic function on Re $s > 2$, for $|S(d, c)| \leqq c$. From 2.9 i):

$$\Delta_\kappa(0, s) = \Psi_\kappa\left(0, \frac{s-1}{2}\right) = \begin{cases} \zeta(s-1)/\zeta(s) & \text{if } \kappa = 0 \\ \sigma_{1-s}(\kappa)/\zeta(s) & \text{if } \kappa \neq 0. \end{cases}$$

3.5. Proposition. i) *Define for* $k, m, q \geqq 0$, $\kappa \in \mathbb{Z}$:

$$\alpha_\kappa(k, m, q; s) = \frac{(-1)^k (2\pi i)^q \Gamma(s)}{m!(k-m)!\, \Gamma(s+k) q!} (\tfrac{1}{3}\pi^2)^k (\tfrac{1}{12})^m \kappa^{k-m}$$

as meromorphic function of s. Then for $n \geqq 0$ *and* Re $s > \frac{1}{2}(n+1)$

$$\Psi_{\kappa,n}(s) = \sum_{\substack{0 \leqq m \leqq k \\ 0 \leqq q \\ k+m+q=n}} \alpha_\kappa(k, m, q; 2s+1)\, \Delta_\kappa(q, 2s+1+2k-q)$$

ii) *Define for* $k, q \geqq 0$:

$$A(k, q; s) = \alpha_0(k, k, 2q; s)$$

as meromorphic function of s. Then for even $n \geqq 0$ *and* Re $s > \frac{1}{2}(n+1)$:

Proof. Part ii) is a direct consequence of part i) and the fact that $S(-d,c)=S(d,c)$. Part i) follows from the following lemmas. We prove the equality for all s in the open set given by $\frac{1}{2}l<\sigma<\frac{1}{2}(l+1)$, $\operatorname{Re} s=\sigma$, $l\geqq 2$, $l\in\mathbb{Z}$. In the Lemmas 3.6–3.8 we keep κ, s, σ and l fixed.

3.6. **Lemma.** *For* $m\in\mathbb{Z}$, $m\geqq 0$, $r>0$:

$$\sum_{c=1}^{\infty}\sum_{d(c)}{}^{*} e^{2\pi i r S(d,c)+2\pi i\kappa d/c}\, c^{-1-2s-m}$$
$$=\sum_{q=0}^{l-1}\frac{(2\pi i)^q}{q!}\, r^q \Delta_\kappa(q,1+2s+m-q)+\mathcal{O}(r^{2\sigma-1}),\qquad r\downarrow 0.$$

Proof. All terms with $c\geqq r^{-1}$ are estimated by their absolute value, and give a contribution $\mathcal{O}(r^{-1+2\sigma+m})$; so we omit them. In the remaining terms we use a Taylor expansion of $e^{2\pi i r S(d,c)}$; the remainders in this expansion give a contribution estimated by

$$\sum_{1\leqq c<r^{-1}} c\cdot\frac{(2\pi r c)^l}{l!}\, c^{-1-2\sigma-m}=\mathcal{O}(r^{2\sigma-1}).$$

We are left with

$$\sum_{q=0}^{l-1}\frac{(2\pi i)^q}{q!}\, r^q\sum_{1\leqq c<r^{-1}}\sum_{d(c)}{}^{*} e^{2\pi i\kappa d/c} S(d,c)^q\, c^{-1-2s-m}.$$

If we replace the sum over c by $\sum_{c=1}^{\infty}$ we introduce something estimated by

$$\sum_{q=0}^{l-1} r^q\cdot r^{-1+2\sigma+m-q}=\mathcal{O}(r^{2\sigma-1}).$$

3.7. **Lemma.** *For* $0<r<1$;

$$\Phi_\kappa(r,s)=\sum_{\substack{0\leqq m\leqq k\\ 0\leqq q\\ k+m+q\leqq l-1}}\alpha_\kappa(k,m,q;2s+1)\,\Delta_\kappa(q,2s+1+2k-q)\, r^{k+m+q}+\mathcal{O}(r^{2\sigma-1})$$

for $r\downarrow 0$

Proof. In the sum for $\Phi_\kappa(r,s)$ given in Proposition 2.16 we may omit all terms in the power series for ${}_0F_1[;1+2s;-\frac{1}{3}\pi^2 r(\kappa+\frac{1}{12}r)c^{-2}]$ except the first l; the remainder gives a contribution $\mathcal{O}(r^{2\sigma-1})$. So we get with the previous lemma:

$$\Phi_\kappa(r,s)=\sum_{k=0}^{l-1}\frac{\Gamma(1+2s)(-1)^k}{\Gamma(1+2s+k)\,k!}\left(\tfrac{1}{3}\pi^2 r\right)^k\sum_{m=0}^{k}\binom{k}{m}\kappa^{k-m}\left(\tfrac{1}{12}\right)^m r^m$$
$$\cdot\sum_{q=0}^{l-1}\frac{(2\pi i)^q}{q!}\, r^q\Delta_\kappa(q,1+2s+2k-q)+\mathcal{O}(r^{2\sigma-1}).$$

Taking terms with $k+m+q\geqq l$ into the remainder term we obtain the lemma.

3.8. **Lemma.** $\Psi_\kappa(r,s)=\Phi_\kappa(r,s)+\mathcal{O}(r^{2\sigma})$, $r\downarrow 0$.

3.9. Proof of Proposition 3.5. For our fixed s the $\Psi_{k,n}(s)$ with $n \leqq l-1$ are determined by

$$\sum_{n=0}^{l-1} \Psi_{\kappa,n}(s)\, r^n = \Psi_\kappa(r,s) + \mathcal{O}(r^{2\sigma-1}), \qquad r \downarrow 0.$$

Now use the Lemmas 3.8 and 3.7.

4. Properties of Δ_κ

4.1. We may rewrite the relations in Proposition 3.5:

$$\Delta_\kappa(n,s) = (2\pi i)^{-n} n!\, \Psi_{k,n}\left(\frac{s+n-1}{2}\right) - (2\pi i)^{-n} n! \sum_{\substack{0 \leq m \leq k \\ 0 \leq q \leq n-1 \\ m+k+q=n}} \alpha_k(k,m,q;s+n)\, \Delta_\kappa(q, s+n+2k-q),$$

for $\mathrm{Re}\, s > 2$.

This leads inductively to the meromorphic continuation of $\Delta_\kappa(n,s)$.

4.2. **Proposition.** *Let $n \geqq 0$, $n, \kappa \in \mathbb{Z}$.*

i) *$\Delta_\kappa(n,s)$ has a meromorphic continuation to $s \in \mathbb{C}$.*

ii) *If $\Delta_\kappa(n,s)$ has a singularity at $s = s_0$, then either*

$$s_0 \in 2(\mathcal{R} \cup \mathcal{M}) - 2l - n + 1 \quad \text{with} \quad l \in \mathbb{Z},\ 0 \leqq l \leqq n,$$

or

$$s_0 \in \mathbb{Z}, \begin{cases} s_0 < 0 \quad \text{or} \quad s_0 = 2 & \text{if } n = 0 \\ s_0 \leqq 1 & \text{if } n \leqq 2,\ n \text{ even or } n = 1 \\ s_0 \leqq 0 & \text{if } n \text{ odd}, \quad n \geqq 3 \end{cases}$$

iii) *If $n \geqq 1$, $3 - n \leqq l \leqq 2[\frac{1}{2}n] + 1 - n$, then the order of the pole of $\Delta_\kappa(n,s)$ at $s = l$ is at most*

$$-[-[\tfrac{1}{2}n]/[\tfrac{1}{2}(l+n-1)]].$$

iv) *For $n \geqq 2$, n even*

$$\operatorname*{res}_{s \to 1} \Delta_\kappa(n,s) = 2(\tfrac{1}{12})^n \sigma_{-n}(\kappa)\, \zeta(n)\, \zeta(n+1)^{-2},$$

with the convention $\sigma_{-n}(0) = \zeta(n)$.

v) *For $n \geqq 0$, n even*

$$\lim_{s \to 2-n} (s-2+n)^{n+1} \Delta_\kappa(n,s) = \begin{cases} 6\pi^{-2}(2\pi i)^n n! & \text{if } \kappa = 0 \\ 0 & \text{if } \kappa \neq 0. \end{cases}$$

Proof. By induction on n. The case $n=0$ is clear. Suppose i)–v) hold for all $n_1 \leqq n-1$. The formula in 4.1 immediately gives the meromorphic continuation. Any singularity of $\Delta_\kappa(n,s)$ comes from one or more of the terms in the sum in 4.1. The term $(2\pi i)^{-n} n!\, \Psi_{\kappa,n}(\frac{1}{2}(s+n-1))$ may give singularities in $1-n+2(\mathscr{R} \cup \mathscr{M})$ and at $l \in \mathbb{Z}$, $l=2$ or $l<0$ if $n=0$, $l \leqq 1$ if $n=1$, $l \leqq 2[\frac{1}{2}n]+1-n$ if $n>1$. The order of a pole at l, $3-n \leqq l \leqq 2[\frac{1}{2}n]+1-n$, is at most $-[-[\frac{1}{2}n]/[\frac{1}{2}(l+n-1)]]$; see Proposition 3.3. Further if $n \geqq 2$ even, then

$$\frac{n!}{(2\pi i)^n} \operatorname*{res}_{s \to 1} \Psi_{\kappa,n}(\tfrac{1}{2}(s+n-1)) = 2(12)^{-n} \sigma_{-n}(\kappa)\, \zeta(n)\, \zeta(n+1)^{-2},$$

and for $n \geqq 0$, n even

$$\frac{n!}{(2\pi i)^n} \lim_{s \to 2-n} (s-n-2)^{n+1}\, \Psi_{\kappa,n}(\tfrac{1}{2}(s+n-1)) = \begin{cases} \dfrac{6}{\pi^2}(2\pi i)^{-n} n! & \text{if } \kappa=0 \\ 0 & \text{if } \kappa \neq 0. \end{cases}$$

For the other terms in 4.1 we have to check that they satisfy ii) and iii), and that they are holomorphic at $s=1$ and have order $>-n-1$ at $s=2-n$.

The α-factors may only introduce poles at $s_0 \in \mathbb{Z}$, $s_0 \leqq -n$. So we need only look at $\Delta_\kappa(q, s+n+2k-q)$. It may introduce singularities at

$$s_0 \in -2l+1-n-2k+2(\mathscr{R} \cup \mathscr{M}) \quad \text{with} \quad 0 \leqq l \leqq q$$

or at

$$s_0 \in \mathbb{Z}, \begin{cases} s_0 = 2-n-2k \quad \text{or} \quad s_0 < -n-2k & \text{if } q=0 \\ s_0 \leqq 2[\frac{1}{2}q]+1-n-2k & \text{if } q \geqq 1. \end{cases}$$

These points are all admitted in ii).

Consider $s \to 1$ for $n \geqq 2$, n even. If $q=0$, then $\Delta_\kappa(0, s+n+2k)$ is holomorphic at $s=1$ provided $n \geqq 1$. For $1 \leqq q \leqq n-1$ we have $2[\frac{1}{2}q]+1-n-2k \leqq 0$, so $\Delta_\kappa(q, s+n+2k-q)$ is holomorphic at $s=1$ as well. Consider $s \to l$, $3-n \leqq l \leqq 2[\frac{1}{2}n]+1-n$. The case $q=0$ cannot contribute, as $2-n-2k<3-n$. Let $1 \leqq q \leqq n-1$. As $l+n+2k-q \geqq 3-q$ an eventual pole of $\Delta_\kappa(s+n+2k-q)$ has order at most

$$\begin{aligned} &-[-[\tfrac{1}{2}q]/[\tfrac{1}{2}(l+n-1+2k)]] \\ &\quad \leqq -[-[\tfrac{1}{2}n]/(k+[\tfrac{1}{2}(l+n+1)])] \leqq -[-[\tfrac{1}{2}n]/[\tfrac{1}{2}(l+n-1)]]. \end{aligned}$$

Finally consider $s \to 2-n$ for $n \geqq 2$, n even. The case $q=0$ may only contribute if $k=0$; but then n would be zero. For $1 \leqq q \leqq n-1$ the order of $\Delta_\kappa(q, s+n+2k-q)$ at $s=2-n$ is at least:

$$-q-1 > -n-1 \qquad \text{if } \quad k=0$$

$0 \qquad$ if $k \geqq 1$ …

4.3. Proposition

$$\Delta_\kappa(k,s)=\sum_{c=1}^{\infty}\sum_{d(c)}^{*} c^{-k} S(d,c)^k e^{2\pi i\kappa d/c} c^{-s}$$

converges absolutely for $\begin{cases}\operatorname{Re} s>\frac{3}{2} & \text{if } k=1\\ \operatorname{Re} s>1 & \text{if } k\geqq 2.\end{cases}$

Proof. For $s\in\mathbb{R}$ all terms in the series for $\Delta_0(2,s)$ are positive. So we have absolute convergence up to the first singularity, at $s=1$.

Put $A(k,c)=\sum_{d(c)}^{*} c^{-k}|S(d,c)|^k$. As $|S(d,c)|\leqq c$ we may for $k\geqq 2$ estimate $A(k,c)$ by the terms of $\Delta_0(2,s)$. This gives absolute convergence in the case $k\geqq 2$.

For $k=1$ remark that by Schwarz' inequality:

$$A(1,c)\leqq\Big(\sum_{d(c)}^{*} 1\Big)^{\frac{1}{2}} A(2,c)^{\frac{1}{2}}\leqq c^{\frac{1}{2}} A(2,c)^{\frac{1}{2}},$$

so

$$\sum_{c=1}^{\infty} c^{-\frac{1}{2}-2\varepsilon} A(1,c)\leqq\left(\sum_{c=1}^{\infty} c^{-1-2\varepsilon}\right)^{\frac{1}{2}}\left(\sum_{c=1}^{\infty} c^{-1-2\varepsilon} A(2,c)\right)^{\frac{1}{2}}<\infty.$$

5. Distribution of Dedekind Sums

5.1. Now we may derive from the properties of $\Delta_\kappa(k,s)$ the Propositions 1.3 and 1.4 on the distribution of $c^{-1}S(d,c)$. Remark that $c^{-1}S(d,c)\in[-1,1]$. To take the relation with d/c into account we work on $[-1,1]\times(\mathbb{R}/\mathbb{Z})$.

5.2. *Notation.* X is the space of continuous complex valued functions on $[-1,1]\times(\mathbb{R}/\mathbb{Z})$, with supremum-norm $\|\cdot\|$. For $k,\kappa\in\mathbb{Z}$, $k\geqq 0$

$$f_{k,\kappa}(v,\tau)=v^k e^{2\pi i\kappa\tau}.$$

These functions $f_{k,\kappa}$ span a dense linear subspace Y of X.

5.3. *Notation.* Define the following continuous linear forms on X:

$$\mu_N(f)=N^{-1}\sum_{c=1}^{N} c^{-2}\sum_{d(c)}^{*} f\left(\frac{S(d,c)}{c},\frac{d}{c}\right)\qquad \text{for } N\geqq 1,$$

$$\nu_N(f)=N^{-1}\sum_{c=1}^{N}\sum_{d(c)}^{*}\left(\frac{S(d,c)}{c}\right)^2 f\left(\frac{S(d,c)}{c},\frac{d}{c}\right)\qquad \text{for } N\geqq 1,$$

$$\mu(f)=6\pi^2\int_{\mathbb{R}/\mathbb{Z}} f(0,\tau)\,d\tau,$$

$$\nu(f)=\sum_{n=1}^{\infty}\frac{1}{(12n)^2}\sum_{m=1}^{n}\frac{\phi((m,n))}{n}\sum_{\varepsilon=\pm 1} f\left(\frac{\varepsilon}{12n},\frac{m}{n}\right).$$

The proof of Propositions 1.3 and 1.4 consists of showing that $\lim_{N\to\infty}\mu_N(f)=\mu(f)$ and $\lim_{N\to\infty}\nu_N(f)=\nu(f)$ for all $f\in X$.

Proof. It is sufficient to prove it for $f=f_{k,\kappa}$, $k\geqq 0$, $\kappa\in\mathbb{Z}$. Consider first

$$\Delta_\kappa(k,s+1)=\sum_{c=1}^{\infty} c^{-s}\cdot c^{-1}\sum_{d(c)}^{*} f_{k,\kappa}\left(\frac{S(d,c)}{c},\frac{d}{c}\right)\quad \text{for } \operatorname{Re} s>1.$$

From Propositions 4.2 and 4.3 we see that we may apply the Tauberian theorem for Dirichlet series (see e.g. [5], XV, § 3) to conclude

$$\lim_{N\to\infty}\frac{1}{N}\sum_{c=1}^{N} c^{-1}\sum_{d(c)}^{*} f_{k,\kappa}\left(\frac{S(d,c)}{c},\frac{d}{c}\right)=\begin{cases}6/\pi^2 & \text{if } k=\kappa=0\\ 0 & \text{otherwise}\end{cases}$$
$$=\mu(f_{k,\kappa}).$$

Now apply the same Tauberian theorem to

$$\Delta_\kappa(k+2,s)=\sum_{c=1}^{\infty} c^{-s}\sum_{d(c)}^{*}\left(\frac{S(d,c)}{c}\right)^2\cdot f_{k,\kappa}\left(\frac{S(d,c)}{c},\frac{d}{c}\right).$$

The absolute convergence for $\operatorname{Re} s>1$ is obtained by using $\Delta_0(2,\operatorname{Re} s)$ as majorant. So we obtain

$$\lim_{N\to\infty} \nu_N(f_{k,\kappa})=\operatorname*{res}_{s=1}\Delta_\kappa(k+2,s)$$
$$=\begin{cases}2(12)^{-k-2}\sigma_{-k-2}(\kappa)\,\zeta(k+2)\,\zeta(k+3)^{-2} & \text{if } k \text{ even}\\ 0 & \text{if } k \text{ odd.}\end{cases}$$

A computation shows that this is equal to $\nu(f_{k,\kappa})$.

5.5. *Proof of Propositions 1.3 and 1.4.* To extend the limit formulas from Y to X, it is sufficient to check that the norms $\|\mu_N\|$ and $\|\nu_N\|$ are bounded for $N\to\infty$. For μ_N this is easily shown by a counting argument, for ν_N by

$$|\nu_N(f)|\leqq\|f\|\cdot\nu_N(f_{0,0})$$

and

$$\lim_{N\to\infty}\nu_N(f_{0,0}) \text{ exists.}$$

5.6. *Proof of Corollary 1.5.* The corollary states

$$\lim_{N\to\infty}\nu_N(\chi)=\nu(\chi)$$

for

$$\chi(v,\tau)=\begin{cases}v^{-2} & \text{for } |v|\geqq\alpha\\ 0 & \text{for } |v|<\alpha.\end{cases}$$

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Received March 25, 1988