On the distribution of Dedekind sums*

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1 Introduction

The Dedekind sum $S(d,c)$, with $c \in \mathbb{N}$, $d \in \mathbb{Z}$, $\gcd(c,d) = 1$, is defined by

$$S(d,c) = \sum_{m=1}^{c-1} \left(\left(\frac{dm}{c}\right)\right)\left(\left(\frac{m}{c}\right)\right),$$

with

$$((x)) = \begin{cases} 0 & \text{if } x \in \mathbb{Z} \\ x - [x] - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z} \end{cases}$$

A recursive definition is

$$S(0,1) = 0$$
$$S(-d,c) = -S(d,c)$$
$$S(d_1,c) = S(d,c) \quad \text{if } d \equiv d_1 \bmod c$$
$$12S(d,c) + 12S(c,d) = -3 + \frac{d}{c} + \frac{c}{d} + \frac{1}{cd} \quad \text{for } c,d > 0; \qquad (1.0.1)$$

see e.g. [5].

If one puts the graph of

$$\left\{ \frac{d}{c} \in [0, \tfrac{1}{2}] : \gcd(d,c) = 1, 1 \le c \le N \right\} \to \mathbb{Q} : \frac{d}{c} \mapsto \frac{S(d,c)}{c}.$$

onto a computer screen for some value of N, one gets a complicated picture. In this note some results are discussed which explain part of the structure visible in this graph.

At the conference I presented the following proposition.

1.1 Proposition. ([3], prop. 1.4)

*Expanded version of a talk given at the International Conference on Automorphic Functions and Applications, Khabarovsk, 1988

For functions $f : [-1,1] \times (\mathbb{R}/\mathbb{Z}) \to \mathbb{C}$ of the form $f(v,\tau) = v^2 g(v,\tau)$, with g continuous:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{c=1}^{N} \sum_{d \bmod c}{}^{*} f\left(\frac{S(d,c)}{c}, \frac{d}{c}\right) = \sum_{n=1}^{\infty} \sum_{k \bmod n} \frac{\varphi(\gcd(k,n))}{n} \sum_{\pm} f\left(\frac{\pm 1}{12n}, \frac{k}{n}\right).$$

Here $\varphi(c)$ denotes the number of d mod c with $\gcd(d,c) = 1$. In $\sum^{*}_{d \bmod c}$ the d satisfies $\gcd(d,c) = 1$.

The *proof* is discussed in section 2. The proposition implies that the points $\left(\frac{S(d,c)}{c}, \frac{d}{c}\right)$ cluster at the points $\left(\frac{\pm 1}{12n}, \frac{k}{n}\right)$. A more precise description of this clustering is given in the following proposition, proved in section 3.

1.2 Proposition. *Put $X = \{(d,c) \in \mathbb{Z}^2 : c \ge 1, \gcd(d,c) = 1\}$, and for $y = (\mu,\eta) \in [-12,12] \times \mathbb{R}$, $\varepsilon > 0$, $N \ge 1$*

$$X(y;\varepsilon,N) = \left\{ (d,c) \in X : 1 \le c \le N, \left|\frac{12S(d,c)}{c} - \mu\right| \le \varepsilon, \left|\frac{d}{c} - \eta\right| \le \varepsilon \right\}.$$

i) For $y = \left(\frac{\pm 1}{n}, \frac{k}{n}\right)$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$, there exist $\varepsilon > 0$ and $N_0 > 1$ such that for all $N > N_0$

$$X(y;\varepsilon,n) = X(y;\varepsilon,N_0) \cup \{(d,c) \in X : N_0 < c \le N, nd - kc = \pm\gcd(k,n)^2\}$$

ii) For $y = (\mu,\eta)$, $\mu \ne 0$, y not of the form discussed in i), there exist $\varepsilon > 0$ and $N_0 > 2$ such that

$$X(y;\varepsilon,N) = X(y;\varepsilon,N_0) \quad \text{for all } N > N_0.$$

Remark. So the set of $(12S(d,c)/c, d/c)$ is discrete outside $\{(0,\eta)\} \cup \{(\pm 1/n, k/n)\}$. Moreover, the density (measured in c) of the $(d,c) \in X$ satisfying $nd - kc = \pm\gcd(k,n)^2$ equals $\varphi(\gcd(k,n))/n$, in accordance with proposition 1.1.

1.3 *Acknowledgement.* I thank D. Zagier for many remarks and suggestions. He pointed out a simplification in the formulation of proposition 1.1 and raised the question what can be said on the density away from the clustering points in proposition 1.1.

2 Eisenstein series and Dedekind sums

Proposition 1.1 is proved in [3], together with

$$\lim_{N\to\infty} \frac{1}{N} \sum_{c=1}^{N} \frac{1}{c} \sum_{d \bmod c}{}^{*} f\left(\frac{S(d,c)}{c}, \frac{d}{c}\right) = \frac{6}{\pi^2} \int_0^1 f(0,\tau)\, d\tau$$

for continuous $f : [-1,1] \times (\mathbb{R}/\mathbb{Z}) \to \mathbb{C}$.

So the majority of the $S(d,c)/c$ are concentrated near 0; the value of d/c does not influence the distribution. Proposition 1.1 concerns the minority staying away from 0.

2.1 Basis for the proof is the continuation in two variables of the Eisenstein series

$$e_s(z) = y^{s+1/2} + \sum_{c=1}^{\infty} \sum_{d,\, \gcd(d,c)=1} \left(\frac{y}{|cz+d|^2} \right)^{s+1/2}$$

for the full modular group.

In [1], (2.19), the existence is proved of a meromorphic family $E(r,s)$ of functions on the upper half plane, for (r,s) in a neighbourhood of $\{0\} \times \mathbb{C}$ in $\mathbb{C}^2$, such that

i) $E(r,s;\gamma z) = v_r(\gamma) e^{ir \arg(cz+d)} E(r,s;z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$. v_r is the $2r$-th power of the multiplier system of the Dedekind eta function, see [5], Ch.4.

ii) $E(r,s;z)$ has a Fourier expansion

$$\sum_{n \in \mathbb{Z}} e^{2\pi i(n+r/12)x} f_n(r,s;y)$$

where each f_n with $n \neq 0$ decreases quickly for $y \to \infty$.

The meromorphy is in (r,s) jointly. The equations in i) and ii) are understood to hold for (r,s) at which $E(r,s)$ is holomorphic. $E(r,s)$ is uniquely determined by the normalization

$$f_0(r,s) = \mu(r,s) + C_0(r,s)\mu(r,-s)$$

with C_0 meromorphic and

$$\mu(r,s;y) = y^{s+1/2} e^{-\pi r y/6} {}_1F_1 \left[\frac{1}{2} + s - \frac{1}{2}r; 1+2s; \pi r y/3 \right].$$

The line $\{0\} \times \mathbb{C}$ is not contained in the singular set of E; the restriction $E(0,s)$ equals e_s.

The proof of the existence of E follows ideas of Colin de Verdière in [4].

The Fourier terms with $n \neq 0$ are of the form $f_n(r,s;y) = C_n(r,s)\omega_n(r,s;y)$, with ω_n expressed in Whittaker functions and C_n meromorphic.

2.2 It is well known that the coefficients in the Fourier series of e_s may be expressed in terms of divisor sums and the zeta function of Riemann. The paper [3] arose from an attempt to extend these formulas to other values of r. Actually, I did not succeed in that aim, but could get expressions for

$$\Psi_n^m(s) = \left(\left(\frac{d}{dr}\right)^m C_n(r,s) \right)\Big|_{r=0}$$

This is done by considering, for small positive values of r, the Poincaré series

$$P_r(s;z) = \sum_{\gamma \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} v_r(\gamma)^{-1} e^{-ir \arg(cz+d)} \mu(r,s;\gamma z),$$

which converges for $\mathrm{Re}\, s > \frac{1}{2}$. One may compute its Fourier coefficients, and on the other hand express it in terms of $E(r,s)$.

This suffices to get, complicated, formulas expressing $\Psi_n^m(s)$ in the Dirichlet series

$$\Delta_n(k,u) = \sum_{c=1}^{\infty} \sum_{d \bmod c}^{*} c^{-u} \left(\frac{S(d,c)}{c} \right)^k e^{2\pi i n d/c}.$$

Inverting this relation gives the meromorphic continuation of $\Delta(k,u)$ to $u \in \mathbb{C}$ and provides information on the right-most singularities. E.g. for the case $n = 0$:

2.3 Proposition. [3] (4.2)

For $k \geq 2$, k even:

i) $\Delta_0(k,.)$ extends to a meromorphic function on $\mathbb{C}$ and its Dirichlet series converges absolutely for $\mathrm{Re}\, s > 1$.

ii) The only singularity of $\Delta_0(k,u)$ with $\mathrm{Re}\, s > 0$ is at $s = 1$; the order of the pole is 1 and the residue equals $2(12)^{-n}\zeta(n)^2\zeta(n+1)^{-2}$.

2.4 Proposition 1.1 and the result (2) follow from the properties of the $\Delta_n(k,.)$ by standard methods.

3 Proof of proposition 1.2

3.1 Let $\Gamma = SL_2(\mathbb{R})$, $\Gamma_\infty = \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \Gamma\}$, $\Gamma_0 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \neq 0\}$. The set $X = \{(d,c) \in \mathbb{Z}^2 : c \geq 1, \gcd(d,c) = 1\}$ may be identified with $\Gamma_\infty \backslash \Gamma_0$ by $\Gamma_\infty \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (d\,\mathrm{sign}(c), |c|)$.

The action of Γ on $\Gamma_\infty \backslash \Gamma$ by right translation gives for each $\gamma \in \Gamma$ a map $X \to X$ defined almost everywhere. For the generators $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $W = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of Γ:

$$(d,c)T = (d+c,c)$$
$$(d,c)W = (-c\,\mathrm{sign}\, d, |d|) \quad \text{for } (d,c) \neq (0,1)$$

We are interested in the image of the map

$$\beta : X \to [-12,12] \times \mathbb{R} : (d,c) \mapsto \left(\frac{12S(d,c)}{c}, \frac{d}{c} \right).$$

3.2 For $\xi_1 = (d_1, c_1)$, $\xi_2 = (d_2, c_2) \in X$ and $N > 1$ we define subsets $A^+(\xi_1, \xi_2; N)$ and $A^-(\xi_1, \xi_2; N)$ of X by

$$A^\pm(\xi_1, \xi_2; N) = \{\xi_1\gamma \in X \,:\, \gamma \in \xi_2,\ \pm c(\xi_1\gamma) > N\}$$
$$= \{(d,c) \in X : \begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix} = \pm \begin{pmatrix} \cdot & \cdot \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix},$$
$$r \in \mathbb{Z},\ c > N\},$$

here a_2 and b_2 have been chosen such that $\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \Gamma$. The sets $A^\pm$ are non-empty and correspond to $r \in \mathbb{Z}$, $\pm r \geq r_0$ for some r_0.

3.3 **Lemma.** *For* $(d_j, c_j) \in X$, $\gamma_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in \Gamma$, $j = 1, 2$:

$$\beta(\Gamma_\infty \gamma_1 T^r \gamma_2) =$$
$$\left(\frac{\mp 1}{c_1 c_2}, \frac{d_2}{c_2}\right) \pm \frac{1}{r}\left(\frac{12S(d_1, c_1) + 12S(d_2, c_2) \pm 3}{c_1 c_2}, \frac{\mp 1}{c_2^2}\right) + O\left(\frac{1}{r^2}\right)$$

for $\pm r \to \infty$.

Remarks. So $\beta(A^\pm(\xi_1, \xi_2, N))$ has limit point $y = (\frac{\mp 1}{n}, \frac{k}{n})$, with $n = c_1 c_2$, $k = c_1 d_2$. Asymptotically the approach is from a direction determined by c_1, c_2 and $12S(d_1, c_1) + 12S(d_2, c_2)$.

If $y = (\frac{\mp}{n}, \frac{k}{n})$ is given, then ξ_2 is fixed. For ξ_1 the only condition is $c_1 = \gcd(k, n)$. The choice of $d_1 \bmod c_1$, $\gcd(d_1, c_1) = 1$, exhausts the possibilities. So there are at most $\varphi(\gcd(k, n))$ lines converging to y.

This way of looking at the limit set is due to D. Zagier.

The *proof* amounts to a computation based on the following relation, which may be found on p. 49 of [3]:

$$\Phi(\gamma\delta) = \Phi(\gamma) + \Phi(\delta) - 3\operatorname{sign}(c_\gamma c_\delta c_{\gamma\delta})$$

$$\text{for } \begin{cases} \Phi\left(\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \frac{a+d}{c} - 12S(d,c) & \text{if } c > 0 \\ \Phi\left(\pm\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}\right) = r & \text{if } c = 0 \end{cases}$$

3.4 Define for $y \in [-12, 12] \times \mathbb{R}$, $\varepsilon > 0$ and $N > 1$

$$\dot{X}(y; \varepsilon, N) = \{(d,c) \in X \,:\, \|\beta(d,c) - y\| \leq \varepsilon,\ c > N\}$$

with $\|(\mu, \eta)\| = \max(|\mu|, |\eta|)$.

Lemma 3.3 implies that for N large enough

$$A^\pm(\xi_1, \xi_2; N) \subset \dot{X}\left(\left(\frac{\mp 1}{c_1 c_2}, \frac{d_2}{c_2}\right); \varepsilon, N\right).$$

Moreover, the (d,c) arising from

$$\begin{pmatrix} \cdot & \cdot \\ c & d \end{pmatrix} = \pm \begin{pmatrix} \cdot & \cdot \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cdot & \cdot \\ c_2 & d_2 \end{pmatrix}$$

with $\frac{d_2}{c_2} = \frac{k}{n}$, $c_1 = \gcd(k, n)$, $d_1 \bmod c_1$, $(d_1, c_1) = 1$, $c > N$, are precisely the solutions $(d, c) \in X$, $c > N$, of the equation $nd - kc = \mp\gcd(k, n)^2$. So proposition 1.2 follows from:

3.5 **Lemma.**

i) If $y = (\frac{\mp 1}{n}, \frac{k}{n})$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$, *then there exist* $\varepsilon > 0$ *and* $A > 0$ *such that for all* $N > \frac{A}{\varepsilon}$

$$\dot{X}(y; \varepsilon, N) = \bigcup_{d_1} A^\pm(\xi_1, \xi_2; N)$$

where we take $d_1 \bmod \gcd(k, n)$, $(d_1, k, n) = 1$, *and let* $\xi_1 = (d_1, \gcd(k, n))$ *and* $\xi_2 = (k/(\gcd(k, n)), n/\gcd(k, n))$.

ii) If $y = (\mu, \eta)$ *with* $\mu \neq 0$ *is not of the form* $(\frac{\mp 1}{n}, \frac{k}{n})$ *as in i), then there exist* $\varepsilon > 0$ *and* $N \geq 2$ *such that*

$$\dot{X}(y; \varepsilon, N) = \emptyset.$$

Proof of lemma 3.5 in the case $y = (\mu, 0)$.
(1.0.0.1) implies for $(d, c) \in X$, $d \neq 0$

$$\beta(d,c) = \left(-\frac{d}{c}\frac{12S(c, |d|)}{|d|} + \frac{1}{d} + \frac{d}{c^2} + \frac{1}{c^2 d} - \frac{3\operatorname{sign} d}{c}, \frac{d}{c}\right)$$
$$= \left(\frac{1}{d}, 0\right) + O\left(\frac{d}{c} + \frac{1}{c}\right).$$

For $(d, c) \in \dot{X}(y; \varepsilon, N)$ we get $|\mu - \frac{1}{d}| = O(\varepsilon + N^{-1})$. This implies ii).

If $y = (\frac{\mp 1}{n}, \frac{k}{n})$, $(d, c) \in \dot{X}(y; \varepsilon, N)$, then $d = \mp n$ if ε and N^{-1} are sufficiently small. The conditions $c > N$ and $\gcd(d, c) = 1$ imply that $(c, d) = \pm(n, t)\begin{pmatrix} \cdot & \cdot \\ 1 & 0 \end{pmatrix}$ with $t \bmod n$, $\gcd(t, n) = 1$.

3.6 *Action of* Γ. From 1 follows for $(d, c) \in X$:

$$\beta((d,c)T) = \beta(d, c + d)$$
$$\beta((d,c)W) = \left(\frac{c}{|d|}\frac{12S(d,c)}{c} - \frac{c}{d|d|}, -\frac{c}{d}\right) + O\left(\frac{1}{c} + \frac{1}{|d|}\right) \quad \text{for } d \neq 0.$$

This leads us to define the following right action of Γ in $\mathbb{R} \times \mathbb{R}$:

$$(\mu, \eta)T = (\mu, \eta + 1)$$
$$(\mu, \eta)W = \left(\frac{\mu}{|\eta|}, \frac{-1}{\eta}\right)$$

Remark that W is only defined on an open dense subset. One may check that the relations between T and W are respected. So we get an action of Γ for which $y\gamma$ is defined for y in an open dense subset depending on γ.

For $(d,c) \in X$:

$$\beta((d,c)T) = \beta(d,c)T$$
$$\beta((d,c)W) = \beta(d,c)W - \left(\frac{c}{d|d|}, 0\right) + \mathcal{O}\left(\frac{1}{c} + \frac{1}{|d|}\right) \quad \text{for } d \neq 0$$

3.7 *Invariance under* Γ. Consider $y = (\mu, \eta)$ with $\eta \neq 0$, $\mu \neq 0$. For $\gamma = T$ or W:

$$\dot{X}(y;\varepsilon,N)\gamma \subset \dot{X}(y\gamma;\varepsilon_1,N_1)$$

with

$$\begin{cases} \varepsilon_1 = \varepsilon,\ N_1 = N & \text{if } \gamma = T \\ \varepsilon < \frac{1}{2}|\eta|,\ \varepsilon_1 = C_\gamma \varepsilon,\ N_1 = D_\gamma N & \text{if } \gamma = W \end{cases}$$

for some C_γ, $D_\gamma > 0$; take $C_\gamma = D_\gamma = 1$ if $\gamma = T$. Suppose now that lemma 3.5 holds for $y\gamma$.

If $y = (\frac{\pm 1}{n}, \frac{k}{n})$ with $n \in \mathbb{N}$, $k \in \mathbb{Z}$, $k \neq 0$, then $y\gamma = \left(\frac{\pm 1}{n_1}, \frac{k_1}{n_1}\right)$ with

$$\begin{cases} n_1 = n, & k_1 = k + n & \text{if } \gamma = T \\ n_1 = |k|, & k_1 = -n\,\mathrm{sign}(k) & \text{if } \gamma = W \end{cases}$$

So there are $\varepsilon > 0$ and $A_1 > 0$ such that for all $N > A/(D_\gamma C_\gamma \varepsilon)$

$$\dot{X}(y;\varepsilon,N) \subset \bigcup_{d_1} A^{\pm}(\xi_1,\xi_2;D_\gamma N)\gamma^{-1}$$

with $\xi_1 = (d_1, \gcd(k,n))$, $\xi_2 = \left(\frac{-n\,\mathrm{sign}(k)}{\gcd(k,n)}, \frac{|k|}{\gcd(k,n)}\right)$ and d_1 taken $\mathrm{mod}\gcd(k,n)$, $\gcd(d_1,k,n) = 1$. As

$$A^{\pm}(\xi_1,\xi_2;N_1)\gamma^{-1} = A^{\pm}\left(\xi_1, \xi_2\gamma^{-1}; \frac{N_1}{D_\gamma}\right)$$

and we already have $\dot{X}(y;\varepsilon,N) \supset \bigcup_{d_1} A^{\pm}(\xi_1,\xi_2\gamma^{-1};N)$ we conclude that lemma 3.5 holds for y as well.

If y is not of the form $(\frac{\pm 1}{n}, \frac{k}{n})$, then neither is $y\gamma$. Clearly $\dot{X}(y;\varepsilon,N) = \emptyset$ for some $\varepsilon = \varepsilon_1/C_\gamma$ and N large enough.

3.8 *Induction.* Suppose $y = (\mu, \frac{k}{n})$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$, $k \neq 0$, $\gcd(k,n) = 1$, $\mu \neq 0$. Write $k = mn + r$ with $|r| \leq \frac{1}{2}n$, $m \in \mathbb{Z}$. If $r = 0$, then $yT^{-m} = (\mu, 0)$ for which the lemma holds, so it holds for y. If $r \neq 0$, then $yT^{-m}W = \left(\frac{\mu n}{|r|}, \frac{-n}{r}\right)$, with $1 \leq |r| < n$ and $\gcd(n,r) = 1$. So by induction on n lemma 3.5 holds for y.

Finally consider $y = (\mu, \eta)$ with $\mu \neq 0$, $\eta \in \mathbb{R} \setminus \mathbb{Q}$. Write $\eta = m + \theta$ with $m \in \mathbb{Z}$, $|\theta| \leq \frac{1}{2}$. Then $yT^{-m}W = (\mu_1, \frac{-1}{\theta})$ with $|\mu_1| = |\mu|/|\theta| \geq 2|\mu|$. After a finite number of steps we arrive at $|\bar{\mu}| > 12$ for which $\dot{X}((\bar{\mu},\bar{\eta});\varepsilon,N) = 0$ if $\varepsilon < |\bar{\mu}| - 12$.

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