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# On the Distribution of Dedekind Sums

ROELOF W. BRUGGEMAN

ABSTRACT. This paper gives a survey on distribution results for Dedekind sums obtained from the study of modular forms, especially real analytic ones.

## 1. Dedekind sums

The name, *Dedekind sum*, for the expression

$$\begin{aligned} S(d,c) &= \sum_{x \bmod c} \left(\left(\frac{x}{c}\right)\right)\left(\left(\frac{dx}{c}\right)\right) \quad \text{for coprime } c \text{ and } d, \text{ where} \\ ((u)) &= \begin{cases} u-[u]-\frac{1}{2} & \text{if } u \notin \mathbf{Z} \\ 0 & \text{if } u \in \mathbf{Z} \end{cases} \end{aligned}$$

is appropriate, as these sums arose for the first time in R. DEDEKIND's study of

$$\log \eta(z) = \frac{\pi i z}{12} - \sum_{n,m\geq 1} \frac{1}{n} e^{2\pi i n m z} \quad \operatorname{Im} z > 0;$$

see *Erläuterungen zu den Fragmenten XXVIII*, in RIEMANN's collected papers, [18], pp. 466–478.

The eta-function

$$\eta(z) = e^{\pi i z/12} \prod_{n\geq 1} \left(1-e^{2\pi i n z}\right) = e^{\log \eta(z)}, \quad \operatorname{Im} z > 0,$$

may be expressed in terms of theta functions. The transformation of theta functions under modular substitutions had been studied before. Although $\eta$ itself does not occur in it, HERMITE's letter [10] gives, between remarks on good sea air and Spanish cigarettes, a complete study of the quotient of some theta

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functions under modular transformations. This quotient may be expressed in terms of the eta function.

Dedekind uses theta functions to obtain the general form of the transformation behavior under modular substitutions for $\eta$, and for $\log \eta$. The behavior under $z \mapsto z+1$ is easy. For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$, $c > 0$, the transformation formula is

$$\log \eta \left(\frac{az+b}{cz+d}\right) = \log \eta(z) + \frac{1}{2} \log \left(\frac{cz+d}{i}\right) + \frac{\pi i}{12} \Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with some constant $\Phi \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. (Dedekind uses a different notation.) The central part of Dedekind's note is the proof of

$$\Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a+d}{c} - 12S(d,c),$$

with $S(d,c)$ as defined above.

Thus we have two different ways of looking at $S(d,c)$; it is given by an explicit sum of rational numbers, and it occurs in the transformation formula of $\log \eta$. Some of its properties are easy seen from the definition of the sum; for instance, $S(\pm d + c, c) = \pm S(d,c)$. Deeper results are obtained from the latter point of view. For instance, apply the transformation formula for $\gamma_1$, $\gamma_2$ and $\gamma_1\gamma_2$, where $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbf{Z})$. This gives relations for the Dedekind sums. The most famous one is the *reciprocity relation*

$$12S(d,c) + 12S(c,d) = -3 + \frac{d^2+c^2+1}{cd} \quad \text{for coprime, positive } c, d,$$

obtained by taking $\gamma_1 = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\gamma_2 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. It can also be obtained from the former point of view. In [17], RADEMACHER proves it in four different ways, all based on the elementary expression for $S(d,c)$. Nevertheless, if one looks at the sum for $S(d,c)$, the reciprocity relation seems to be a miracle; from the transformation formula of $\log \eta$ it is clear that it is a natural property.

In this note I emphasize the connection between Dedekind sums and functions like $\log \eta$, with automorphic transformation behavior. The next two sections discuss distribution results for Dedekind sums. The uniform distribution results in section 2 are connected to a question posed by RADEMACHER, [17], p. 28. We may view the Dedekind sums as giving a map $S : \mathbf{Q} \to \mathbf{Q}$, by $d/c \mapsto S(d,c)$. This map has period 1. The question is whether its graph is dense in the plane.

The answer "yes" is given by HICKERSON, [11]. His proof uses the reciprocity relation to express $S(d,c)$ in terms of the continued fraction expansion of $d/c$. This density result means that if we fix a rectangle in $\mathbf{R}^2$ and put a dot at

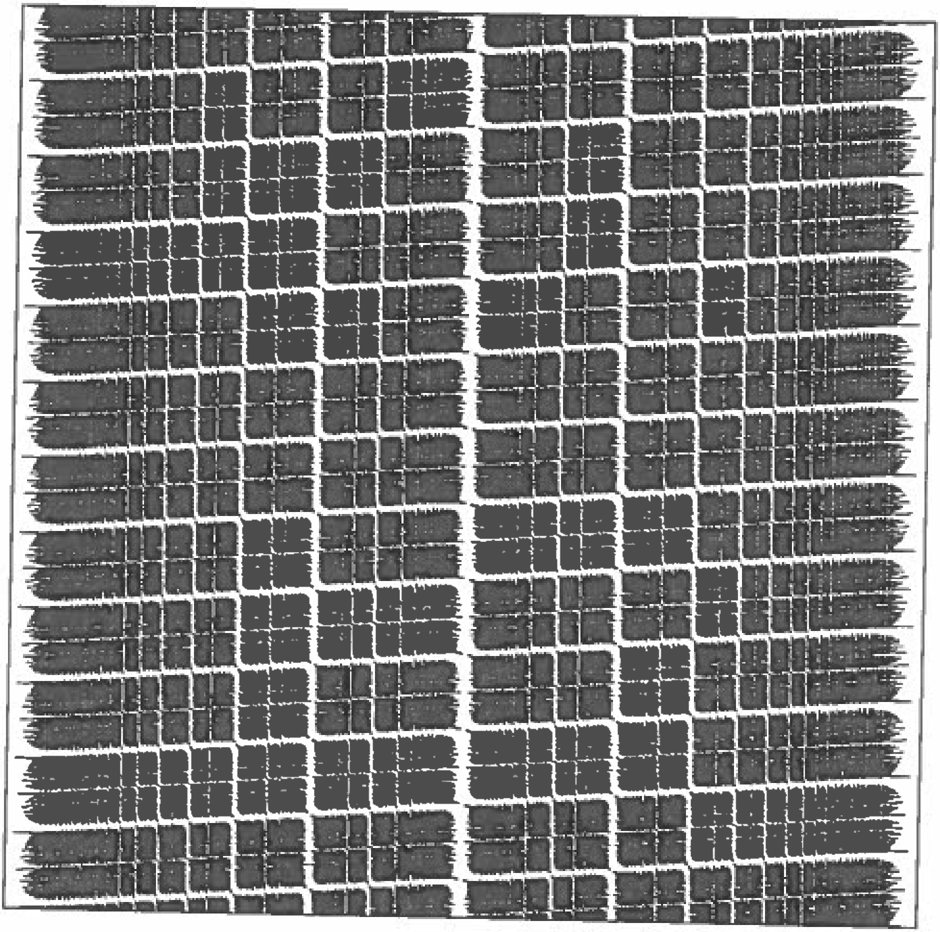

FIGURE 1. $(d/c, S(d,c)) \in [0,1) \times [0,1]$ for coprime $(d,c)$ with $1 \leq c \leq 10000$. Number of points: 7137610.

reader is urged to experiment with varying bounds for $c$. The area of the white spots decreases when the bound increases. In subsection 2.3 I give some remarks on this truncation effect.

In practice, the computation of Dedekind sums as a sum is hopelessly slow. Recursive computation with help of the reciprocity relation and $S(\pm d + nc, c) = \pm S(d,c)$ is much more efficient. In drawing the pictures in this note I have had the computer backtrack in a tree determined by these relations.

this kind of information we look at the growth of the quantity

$$U_X(f) = \sum_{1\le c<X} \sum_{d \bmod c}^{*} f\left(d/c, S(d,c)\right)$$

as $X \to \infty$. The test function $f$ is any bounded function on $(\mathbf{R} \bmod \mathbf{Z}) \times \mathbf{R}$. (As $d/c \mapsto S(d,c)$ has period 1, it is sufficient to consider functions that have this period in the first variable.) By $\sum^{*}_{d \bmod c}$ is meant the sum over $d$ mod $c$ with $(d,c) = 1$. We discuss two results concerning the asymptotic behavior of $U_X(f)$. Clearly $U_X(1) \sim \frac{3}{\pi^2} X^2$.

Myerson, [14], (the essential ideas for this result are already given by Vardi, [22]), showed that

$$U_X(f_{p,qr}) = o(X^2) \tag{1}$$

for all $r > 0$, and all $(p,q) \in \mathbf{Z}^2$, $(p,q) \neq (0,0)$. Here $f_{n,m}(\eta,\sigma) = e^{2\pi i(n\eta + m\sigma)}$.

We view this in the following way: consider, for a given positive $r$, the images of the points $(d/c, S(d,c))$ under the projection $(\mathbf{R} \bmod \mathbf{Z}) \times \mathbf{R} \to (\mathbf{R} \bmod \mathbf{Z}) \times (\mathbf{R} \bmod \frac{1}{r}\mathbf{Z})$. The distribution of these images is measured by $U_X(f)$, where $f$ runs over the continuous functions on $\mathbf{R}^2$ with period 1 in the first variable, and period $\frac{1}{r}$ in the second one. As $f \mapsto U_X(f)$ is continuous with respect to the supnorm on the space of these functions, and as the exponential functions $f_{p,qr}$, $p,q \in \mathbf{Z}$, span a dense subspace, we conclude that

$$U_X(f) \sim X^2 \frac{3}{\pi^2} \int_{\mathbf{R} \bmod \frac{1}{r}\mathbf{Z}} \int_{\mathbf{R} \bmod \mathbf{Z}} f(\eta,\sigma)\, d\eta\, r\, d\sigma,$$

for all continuous functions on $(\mathbf{R} \bmod \mathbf{Z}) \times (\mathbf{R} \bmod \frac{1}{r}\mathbf{Z})$. Thus the points $(\frac{d}{c} \bmod 1, S(d,c) \bmod \frac{1}{r})$ are indeed *uniformly distributed*. (For the transition from the linear combinations of the $f_{p,qr}$ to all continuous functions one uses $|U_X(f)| \le \|f\|_\infty U_X(1)$ and the fact that $U_X(1)/X^2$ has a limit as $X \to \infty$.)

The other distribution result is more in de spirit of Figure 1. It looks at a given region in the plane, and considers how many $(d/c, S(d,c))$ fall into this region: For each continuous compactly supported function $f$ on $(\mathbf{R} \bmod \mathbf{Z}) \times \mathbf{R}$

$$U_X(f) \sim \frac{X^2}{\log X} \frac{6}{\pi^2} \int_{\mathbf{R}} \int_{\mathbf{R} \bmod \mathbf{Z}} f(\eta,\sigma)\, d\eta\, d\sigma. \tag{2}$$

This follows from the main result in [4] by partial summation.

**2.1. Real analytic modular forms.** To indicate the central ideas in the proofs of these results I turn to real analytic modular forms.

Let $\mathcal{H}$ be the upper half plane $\{x + iy \in \mathbf{C} : y > 0\}$. Put

$$L_q = -y^2\partial_x^2 - y^2\partial_y^2 + iqy\partial_x.$$

ii) $f\left(\frac{az+b}{cz+d}\right) = v\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) e^{iq\arg(cz+d)} f(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.
The function $v : \Gamma \to \mathbf{C}^*$ is called the *multiplier system*.

iii) Some growth condition as $y \to \infty$.

Condition ii) is closely related to the more usual transformation behavior

$$h\left(\frac{az+b}{cz+d}\right) = v\begin{pmatrix} a & b \\ c & d \end{pmatrix} (cz+d)^q h(z)$$

of a holomorphic modular form $h$. If $f(z) = y^{q/2} h(z)$, then $f$ satisfies ii).

Condition i) replaces the usual condition of holomorphy. If $f$ corresponds to a holomorphic modular form $h$ as above, of weight $q$, then the eigenvalue is $\frac{q}{2}\left(1 - \frac{q}{2}\right)$. The differential operator $L_q$ is *elliptic*. Hence all its eigenfunctions are real analytic.

MAASS, [13], was the first one who systematically studied these modular forms. (He extended Hecke's theory concerning the relation between modular forms and Dirichlet series.)

For each $r \in \mathbf{C}$ the function $z \mapsto y^{r/2} e^{2r \log \eta(z)}$ is a modular form, with eigenvalue $\lambda = \frac{r}{2}\left(1 - \frac{r}{2}\right)$, weight $q = r$. Its multiplier system $v_r$ is given by

$$\begin{aligned} v_r\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} &= e^{\pi i r k/6} \\ v_r\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= e^{2\pi i r(-S(d,c) - 1/4 + (a+d)/12c)} \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ c > 0. \end{aligned}$$

We remark that $v_r\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a function of $r$ mod 12.

For the modular group all multiplier systems suitable for the weight $q$ are the $v_r$ with $r \equiv q \pmod 2$. Here we see the main rôle of Dedekind sums in the theory of modular forms; they describe the multiplier systems.

In the *spectral theory* of modular forms one fixes a real weight $q$ and a multiplier system $v_r$ suitable for it; hence $q \equiv r \pmod 2$. If a function $f$ satisfies condition ii) above, then $|f|$ is $\Gamma$-invariant. The measure $y^{-2} dx\, dy$ is invariant under the action of $\mathrm{SL}_2(\mathbf{R})$ on $\mathcal{H}$. So it makes sense to talk about the square integrability of $f$ on $\Gamma\backslash\mathcal{H}$. The square integrable $f$ that satisfy ii) form a Hilbert space $H(q, r)$. The differential operator $L_q$ in condition i) has a selfadjoint extension in $H(q, r)$. The eigenfunctions of this extension are square integrable modular forms. For $r = q > 0$ the function $z \mapsto y^{r/2} e^{2r \log \eta(z)}$ is an example.

If $r \in \mathbf{R} \setminus 12\mathbf{Z}$ the extension of $L_q$ has a discrete spectrum. Hence there are infinitely many linearly independent square integrable modular forms of weight $q$, with multiplier system $v_r$. For $0 < r = q < 12$ the only one (up to multiples)

The spectral theory of modular forms is due to SELBERG and ROELCKE. As references I mention ROELCKE, [19, 20], HEJHAL, [9], and VENKOW, [24].

**2.2. Poincaré series.** Most of the square integrable modular forms mentioned above have not been explicitly constructed and are only known to exist. There is a general method of constructing real analytic modular forms, that are in general not square integrable. Take a function $h$ on $\mathcal{H}$ that satisfies the transformation condition ii) for $\gamma \in \Gamma_\infty = \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$, and try to force condition ii) for all $\gamma \in \Gamma$ by taking

$$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} v_r(\gamma)^{-1} e^{-iq \arg(c_\gamma z + d_\gamma)} h(\gamma \cdot z).$$

If $h(z) = \mathcal{O}(y^A)$ $(y \downarrow 0)$, for some $A > 1$, then this sum converges absolutely and defines a function with the desired transformation behavior under $\Gamma$. To satisfy the differential equation in i) we take $h$ to be an eigenfunction of $L_q$. Let $\nu \in \mathbf{Z}$, $q, r \in \mathbf{R}$, $q - r \in 2\mathbf{Z}$, $\operatorname{Re} s > \frac{1}{2}$. Put

$$(3)\quad h^\nu(q,r,s;z) = y^{s+1/2} e^{2\pi i(\nu + r/12)x} {}_1F_1\left[\begin{matrix} 1/2 + s - q/2 \\ 1 + 2s \end{matrix} \middle| 4\pi\left(\nu + \frac{r}{12}\right) y\right],$$

where ${}_1F_1\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\middle| x\right] = \sum_{n=0}^\infty \frac{\Gamma(a+n)\Gamma(c)}{\Gamma(a)\Gamma(c+n)n!} x^n$ is a confluent hypergeometric function. The *Poincaré series*

$$P^\nu(q,r,s;z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} v_r(\gamma)^{-1} e^{-iq \arg(c_\gamma z + d_\gamma)} h^\nu(q,r,s;\gamma \cdot z)$$

converges absolutely, and defines a real analytic modular form $P^\nu(q,r,s)$ of weight $q$, eigenvalue $\frac{1}{4} - s^2$, with the multiplier system $v_r$. For $q \in 2\mathbf{Z}$ the modular form $P^0(q,0,s)$ is the well known *Eisenstein series* of weight $q$.

These modular forms have, in general, exponential growth as $y \to \infty$. Selberg and others do not put the confluent hypergeometric function in the definition of $h^\nu$. They do not get eigenfunctions of $L_q$, but the advantage is that they obtain square integrable Poincaré series. In this note I use the choice indicated above.

Note that I parametrize the eigenvalue as $\frac{1}{4} - s^2$. Often the parametrization $s(1-s)$ is used. This means $s_{\text{here}} = s_{\text{usual}} - \frac{1}{2}$. I think that this choice of the spectral parameter has a notational advantage.

The definition of Poincaré series works fine for $\operatorname{Re} s > \frac{1}{2}$. The resulting families $s \mapsto P^\nu(q,r,s)$ have meromorphic extensions to the complex plane as functions of $s$. In weight 0 this has been proved by NEUNHÖFFER, [15]. Similar methods work for other weights; see [9], Chapter IX. Important in the sequel is the fact plane $\mathrm{Re}\, s > \frac{1}{2} - \varepsilon(q)$. The order of a pole of $P^\nu(q,r,\cdot)$ at these points is at most one. If $q \in 2\mathbf{Z}$, and there is a pole at $s = \frac{1}{2}$, then the residue has to be a square integrable modular form with eigenvalue 0. This has to be a constant function. This occurs in the case $q = \nu + r/12 = 0$. In all other cases $P^\nu(q,r,\cdot)$ is holomorphic on a half plane $\mathrm{Re}\, s > \frac{1}{2} - \varepsilon$.

As $P^\nu(q,r,s;z+1) = e^{\pi i r/6} P^\nu(q,r,s;z)$, there is a *Fourier expansion* in $x$. For $r \notin 12\mathbf{Z}$ it has the form

$$
\begin{aligned}
&P^\nu(q,r,s;z)\\
&\quad = h^\nu(q,r,s;z) + \sum_{\mu\in\mathbf{Z}} e^{2\pi i(\mu+r/12)x} C^\nu_\mu(q,r,s) W_{\varepsilon q/2,s}\left(4\pi\varepsilon\left(\mu+\frac{r}{12}\right)y\right),
\end{aligned}
$$

with $\varepsilon = \mathrm{Sign}\,(\mu + r/12)$. The Whittaker function $W_{\kappa,s}(t)$ can be found in books on special functions, for instance in [21], 1.7. It is exponentially decreasing as $t \to \infty$. If $r \in 12\mathbf{Z}$, there is a term with $\mu + \frac{r}{12} = 0$. In this term we use $h^{-r/12}(q,r,-s;z) = y^{-s+1/2}$ instead of the Whittaker function. The term $h^\nu(q,r,s;z)$ in this expansion is the contribution of $\gamma \in \Gamma_\infty$ to the sum. In the region of convergence one can compute the Fourier expansion of the remaining sum. This leads to the following description of the *Fourier coefficients* $C^\nu_\mu$:

(4)
$$
\begin{aligned}
&C^\nu_\mu(q,r,s)\\
&\quad = G^\nu_\mu(q,r,s)\sum_{c=1}^{\infty} J\left((\mu+r/12)(\nu+r/12),s;c\right) S_r(\mu,\nu;c)\\
&G^\nu_\mu(q,r,s)\\
&\quad = \begin{cases} \pi 2^{1-2s}\Gamma(2s)\Gamma(\frac{1}{2}+s+\frac{q}{2})^{-1}\Gamma(\frac{1}{2}+s-\frac{q}{2})^{-1} \\ \qquad\qquad \text{if } \mu+\frac{r}{12}=0 \\ \pi^{s+1/2}\left|\mu+\frac{r}{12}\right|^{s-1/2}\Gamma(\frac{1}{2}+s+\frac{q}{2}\mathrm{Sign}\,(\mu+r/12))^{-1} \\ \qquad\qquad \text{if } \mu+\frac{r}{12}\neq 0 \end{cases}\\
&J(t,s;c)\\
&\quad = \begin{cases} c^{-1-2s} & \text{if } t=0 \\ \frac{1}{c}J_{2s}\left(4\pi\sqrt{t}/c\right)\Gamma(1+2s)(2\pi)^{2s}|t|^{-s} & \text{if } t>0 \\ \frac{1}{c}I_{2s}(4\pi\sqrt{|t|}/c)\Gamma(1+2s)(2\pi)^{2s}|t|^{-s} & \text{if } t<0 \end{cases}\\
&S_r(\mu,\nu;c)\\
&\quad = \sum_{d \bmod c}^{*} e^{2\pi i r S(d,c)} e^{2\pi i(\nu a+\mu d)/c}.
\end{aligned}
$$

The Bessel function $J_{2s}$ is given by $J_{2s}(u) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(2s+1+n)n!}\left(\frac{u}{2}\right)^{2s+2n}$; omit the $(-1)^n$ to obtain the power series for the modified Bessel function $I_{2s}$. The

$J(t,s;c) = c^{-1-2s}(1+O(1/c))$, as $c$ tends to $\infty$. This gives the meromorphic continuation of

$$\Psi_r(\mu,\nu;s) = \sum_{c=1}^{\infty} c^{-1-2s} \sum_{d \bmod c}^{*} e^{2\pi i r S(d,c)+2\pi i(\nu a+\mu d)/c}, \tag{5}$$

at least to the region $\operatorname{Re} s > 0$. In this region it has singularities at the same points as the series

$$\sum_{c=1}^{\infty} J((\mu + r/12)(\nu + r/12), s; c) S_r(\mu,\nu;c).$$

Suppose we know that $\Psi_r(\mu,\nu)$ is holomorphic on a half plane $\operatorname{Re} s > \frac{1}{2} - \varepsilon$. We apply Ikehara's Tauberian theorem and partial integration to get

$$\sum_{c<X} \sum_{d \bmod c}^{*} e^{2\pi i r S(d,c)} e^{2\pi i(\nu a+\mu d)/c} = o(X^2).$$

The result of Myerson and Vardi, see (1), corresponds to the case $\nu = 0$, $r = qr$, and $\mu = p$.

Hence we consider $\nu = 0$, and either $\mu \neq 0$ or $r \neq 0$. The singularity of $P^0(0,0,\cdot)$ at $s = \frac{1}{2}$ involves only the term with $\mu = 0$ in the Fourier expansion, as the residue is a constant function. So we know that the Fourier coefficient $C^0_\mu(q,r,\cdot)$ has the right behavior in the cases we consider. We can choose $q \equiv r \pmod 2$ such that $G^\nu_\mu(q,r,\frac{1}{2}) \neq 0$. This gives the desired assertion concerning $\Psi_r(\mu,0)$.

The proof of Vardi and Myerson looks a bit different, but the basic ideas are the same. Their proof is based on a result of GOLDFELD and SARNAK, [8], that relates the growth of the sum $\sum_{c<X} c^{-1} S_r(\mu,\nu;c)$ for $X \to \infty$ to some spectral data. That result is a consequence of spectral theory. The proof sketched above avoids the separate treatment of the case $r \in 2\mathbf{Z}$ in section 4 of [14], at the cost of using more results on Poincaré series.

The result of Goldfeld and Sarnak may be derived from a much more general formula relating spectral data to sums of Kloosterman sums, the *sum formula of Kuznetsov;* see, e.g., KUZNETSOV, [12], PROSKURIN, [16], DESHOUILLERS and IWANIEC, [7], BRUGGEMAN, [1]. One may view this result as a further elaboration of the formula (1) that expresses $C^\nu_\mu(q,r,s)$ in terms of Kloosterman sums. If one integrates this equality over $s$ against a rather general test function, one may hope to get a relation that allows one to extract more information. This idea leads to Kuznetsov's sum formula — although the proof is more complicated than this description suggests. (In the literature one often speaks of Kuznetsov's function. A complicated treatment of the resulting equality leads to the distribution result (2). I want to stress that although [4] looks rather complicated, it just extracts information from formula (4) for the Fourier coefficients of Poincaré series. In retrospect, I think that 5.2 of [4] contains the essential step relating Dedekind sums to the sum formula.

**2.3. Truncation effects.** With help of the standard relations for Dedekind sums we can understand the main structure visible in Figure 1. This figure suggests that there are forbidden bands for the points $(\frac{d}{c}, S(d,c))$ with $0 \leq S(d,c) \leq 1$ and $c$ bounded. These bands seem to be specified by certain values of $\frac{d}{c}$ or $12S(d,c) - \frac{d}{c}$.

Let us suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, with $1 < c \leq N$, $0 \leq d < c$, $|a| \leq \frac{1}{2}c$, and $|12S(d,c)| \leq G$. The constant $G$ is fixed, and not too large (12 in Figure 1). The truncation point $N$ is large (10000 in Figure 1). We ignore $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Put $\varepsilon = \left|\frac{a}{c}\right|$, it satisfies $0 < \varepsilon \leq \frac{1}{2}$. Let $\pm 1 = \operatorname{Sign} \frac{a}{c}$. The reciprocity relation gives

$$\begin{aligned} \pm 12S(d,c) + 12S(c,|a|) &= -3 + \frac{|a|}{c} + \frac{c}{|a|} + \frac{1}{c|a|} \\ G + |a| &\geq -3 + \varepsilon + \frac{1}{\varepsilon} + 0 \\ \varepsilon N &\geq -B + \frac{1}{\varepsilon}, \end{aligned}$$

with $B = G + 3$; I used also $S(\pm d, c) = \pm S(d,c)$, $S(a,c) = S(d,c)$, and $|12S(d,c)| \leq c$. Hence

$$\varepsilon \geq \frac{-B + \sqrt{B^2 + 4N}}{2N} \approx \frac{1}{\sqrt{N}}.$$

The assumptions above imply that the quantity

$$12S(d,c) - \frac{d}{c} = \frac{a}{c} - \Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

stays away from integral values by an amount approximately equal to $N^{-1/2}$. This explains the twelve white bands with slope $\frac{1}{12}$ in Figure 1.
*Exercise.* Examine the avoidance of half-integral values that is visible in Figure 1.

Now we turn to the vertical bands, that seem to occur at "small" rational values of $\frac{d}{c}$. Take $G$ and $N$ as above, and fix a rational number $\frac{k}{l}$; take $l > 0$, and $gcd(k,l) = 1$. We assume $\frac{d}{c} = \frac{k}{l} \pm \varepsilon$, with $0 < \varepsilon < \frac{1}{2}$. Take $M, M', M'' \in \Gamma$, This gives

$$\begin{aligned}
\mp 12S(d'',c'') \pm 12S(d,c) \mp 12S(p,l) &= \pm\frac{c''}{cl} \pm \frac{c}{lc''} \pm \frac{l}{cc''} - 3, \\
\varepsilon lc + G + 3 + l &\geq \varepsilon + \frac{1}{\varepsilon l^2} + \frac{1}{\varepsilon c^2}, \\
(lN)\varepsilon^2 + Q\varepsilon - \frac{1}{l^2} &\geq 0,
\end{aligned}$$

with $Q = G + 3 + l$. Hence $\left|\frac{d}{c} - \frac{k}{l}\right| = \varepsilon \geq \left(-Q + \sqrt{Q^2 + 4N/l}\right)/2lN \approx l^{-3/2}N^{-1/2}$.

This explains the presence of the vertical bands. The average width of the bands seems larger than predicted by this computation. I have no explanation of this fact.

## 3. Other distribution results

The previous section discussed the distribution of $S(d,c)$. One has also found results on the distribution of $S(d,c)/\log c$ and $S(d,c)/c$. In [23] VARDI shows that $S(d,c)/\log c$ has a Cauchy distribution. In his proof not only real analytic modular forms play a rôle, but also the relation between Dedekind sums and continued fractions. In this section I consider the distribution of $S(d,c)/c$.

As $|S(d,c)| \leq c$, the points $\left(\frac{d}{c}, \frac{S(d,c)}{c}\right)$ are contained in a compact subset of $(\mathbf{R} \bmod \mathbf{Z}) \times \mathbf{R}$. Plotting them gives more interesting pictures than plotting $\left(\frac{d}{c}, S(d,c)\right)$; see Figure 2. I'll discuss distribution results for $\left(\frac{d}{c}, \frac{S(d,c)}{c}\right)$ that explain part of the structure that one sees in this picture. These distribution results have been proved in [3]. Actually, I state an extension. I hope to publish a full proof of the present version in the future.

We consider the quantity

$$V_X(f) = \sum_{1 \leq c < X} \sum_{d \bmod c}^{*} f\left(\frac{a}{c}, \frac{d}{c}, \frac{12S(d,c)}{c}\right)$$

as $X$ tends to $\infty$. Here $f$ is a continuous function on $(\mathbf{R} \bmod \mathbf{Z})^2 \times \mathbf{R}$. For a given coprime pair $(d,c)$ the choice of $a$ such that $\left(\begin{smallmatrix} a & \cdot \\ c & d \end{smallmatrix}\right) \in \Gamma$ does not influence the definition. The distribution results are

(A) For all continous $f : (\mathbf{R} \bmod \mathbf{Z})^2 \times \mathbf{R} \to \mathbf{R}$

$$V_X(f) \sim \frac{3X^2}{\pi^2} \int_0^1 \int_0^1 f(\xi, \eta, 0)\, d\xi\, d\eta.$$

(B) If $f(\xi, \eta, \sigma) = \sigma^2 g(\xi, \eta, \sigma)$ with $g : (\mathbf{R} \bmod \mathbf{Z})^2 \times \mathbf{R} \to \mathbf{R}$ continuous,

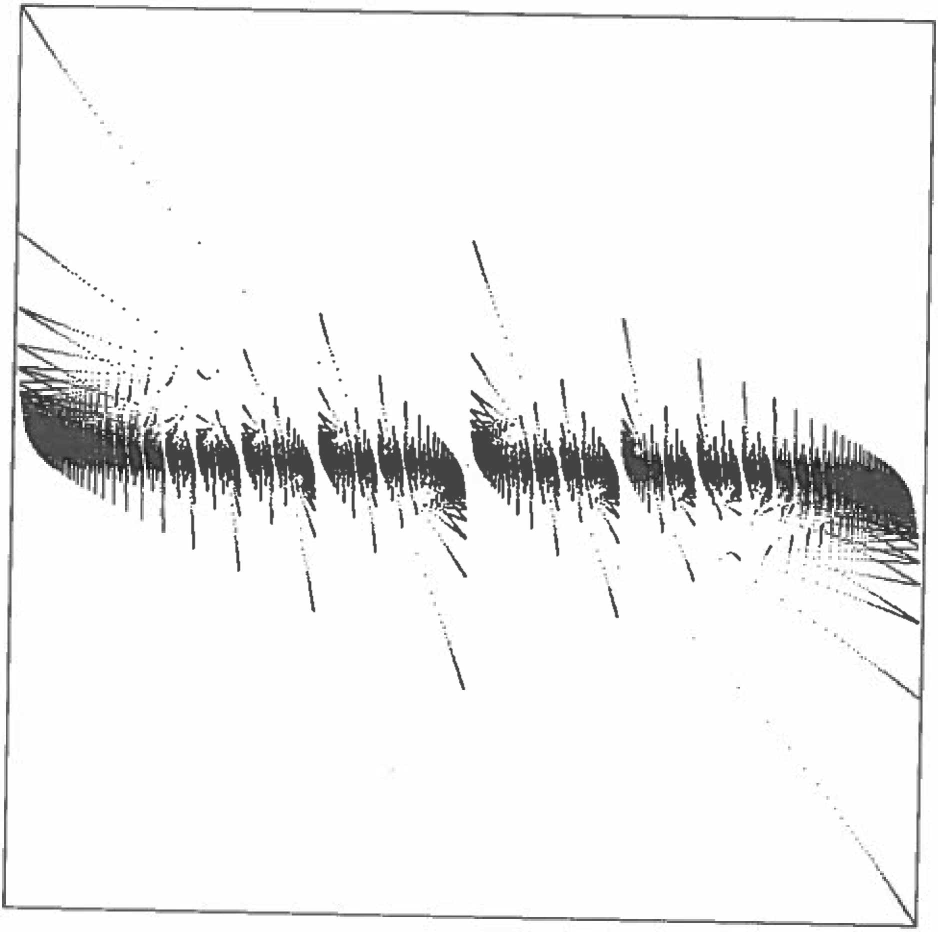

FIGURE 2. $12S(d,c)/c \in [-1,1]$ against $d/c \in [0,1)$, for $1 \leq c \leq 1000$. Number of points: 303792.

where $x$ and $y$ run through the representatives of $\mathbf{Z} \bmod n$, with the additional condition $gcd(x,n) \cdot gcd(y,n) = n$.

In (B) an alternative description of the limiting distribution is

$$\sum_{c_1,c_2\geq 1} \sum_{a_1 \bmod c_1}^{*} \sum_{d_2 \bmod c_2}^{*} \sum_{\pm} \frac{1}{c_1c_2} f\left(\frac{a_1}{c_1}, \frac{d_2}{c_2}, \frac{\pm 1}{c_1c_2}\right).$$

Distribution result (A) states that the majority of the $S(d,c)/c$ are concentrated near 0; their distribution does not depend on $\frac{a}{c}$ and $\frac{d}{c}$. This majority is visible in Figure 2 as the horizontal black band. The gaps in this band are due

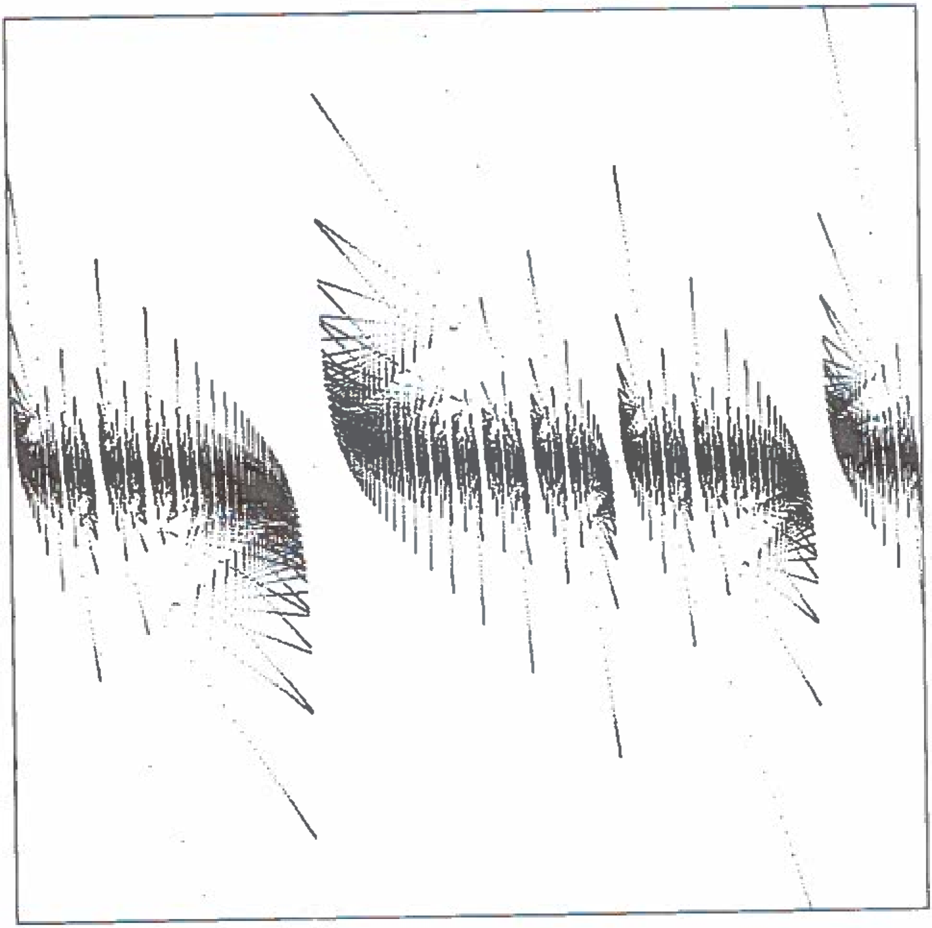

FIGURE 3. $12S(d,c)/c \in [-0.3, 0.3]$ against $d/c \in [0.4, 0.7)$, for $1 \leq c \leq 1000$. Number of points: 89653.

visible in Figure 2, for instance, the slope of the "lines" approaching the limit points, and the self-similarity visible in enlargements like that in Figure 3; see [5].

**3.1. Modular forms of varying weight.** The Poincaré series determine a family of modular forms $(r,s) \mapsto P^\nu(r,r,s)$, where $r \in \mathbf{R}$, and $s \in \mathbf{C}$. In [2] it is shown that this family has meromorphic extensions in $(r,s)$ jointly. The proof is based on ideas of COLIN DE VERDIÈRE, [6]. One of the results is the existence of a neighborhood $U$ of $(0,12)$ in $\mathbf{C}$, contained in $0 < \operatorname{Re} r < 12$, and of a meromorphic family of modular forms $p^\nu$ on $U \times \mathbf{C}$, such that for express it in terms of the Dirichlet series

$$\Delta_k(\mu,\nu;w) = \sum_{c=1}^{\infty} c^{-w} \sum_{d \bmod c}^{*} \left(\frac{S(d,c)}{c}\right)^k e^{2\pi i(\nu a+\mu d)/c}.$$

At this point one might hope that $p^\nu(r,s)$ and $\Psi_r(\mu,\nu;s)$ would have a meromorphic extension in $(r,s)$ to a neighborhood of the complex line $\{0\}\times\mathbf{C}$. This would give directly the meromorphic extension of $\Delta_k(\mu,\nu;w)$ for $w\in\mathbf{C}$. Actually, in situations where this holds, one does not obtain distribution results of the type (A) and (B). So in a sense it is good that the situation is more complicated.

The family $p^\nu(r,s)$ is the unique family of automorphic forms with a Fourier expansion of the type

$$p^\nu(r,s;z) = h^\nu(r,s;z) + \sum_{\mu\in\mathbf{Z}} e^{2\pi i(\mu+r/12)x} c_\mu^\nu(r,s) W_{\varepsilon r/2,s}\left(4\pi\varepsilon\left(\mu+\frac{r}{12}\right)y\right)$$

with $c_\mu^\nu$ meromorphic, and $\varepsilon \mathrm{Re}\,(\mu + r/12) > 0$; see (3) for $h^\nu(r,s) = h^\nu(r,r,s)$. For $\mu = 0$ the Whittaker function $W_{r/2,s}(\pi r y/3)$ has no meromorphic extension across $r=0$. Near $r=0$ we can obtain another meromorphic family, whose Fourier term for $\mu=0$ has a nice form with respect to the basis $h^0(r,s), h^0(r,-s)$. This implies that $p^\nu(r,s) - g^\nu(r,s)p^0(r,s)$ has a meromorphic extension to a neighborhood of $\{0\}\times\mathbf{C}$, if we take $g^\nu = vc_0^\nu/\left(1+vc_0^0\right)$, with

$$v(r,s) = \left(\frac{\pi r}{3}\right)^{s+1/2} \Gamma(-2s)/\Gamma(\tfrac{1}{2}-s-\tfrac{r}{2}).$$

The factor $r^{s+1/2}$ signals that meromorphy at $r=0$ may be problematic. Anyhow, the meromorphic continuation of $c_\mu^\nu - g^\nu c_\mu^0$ in $(r,s)$ leads to the meromorphic continuation of $\Delta_k(\mu,\nu;2s+1-k)$ in $s$. Moreover, the right-most singularity of $\Delta_k(\mu,\nu;w)$ turns out to be situated at $s=1$ if $k\geq 2$ is even, and to the left of 1 if $k$ is odd. The case $k=0$ is known from spectral theory. All these singularities have first order, with known residue. Ikehara's Tauberian theorem leads to the distribution results (A) and (B).

I emphasize that it is essential that $\left(c_\mu^\nu - g^\nu c_\mu^0\right)(r,s)$ has singularities along the lines $\left\{\frac{l}{2}\right\}\times\mathbf{C}$ with $l\in\mathbf{N}$; one can trace this back to the singularities of $h^0(r,s)$ at negative half-integral values of $s$.

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Mathematisch Instituut, Universiteit Utrecht, Postbus 80010, 3508 TA Utrecht, The Netherlands

*E-mail address*: bruggema@math.ruu.nl